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# Extension of the modal superposition method for general damping applied in railway dynamics

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## 1 Introduction

The frequency response function (FRF) permits to characterise in the frequency domain the systems governed by linear dynamics by means of a relationship between an excitation applied at one degree of freedom (dof) and the consequent output response in a particular dof. A modal approach is widely extended in the engineering fields as efficient method of computing the FRF of matrix second-order linear equations of motion derived from the application of the Finite Element Method (FEM) [1]. This approach is based on the truncation of the number of vibration modes that conform the base of the new modal coordinates. The criterion for the truncation is linked to the frequency range of the dynamic study, ordering the vibration modes with respect to the eigenvalues (the square of natural frequencies) associated. The natural frequency associated with the last vibration mode selected establishes the maximum frequency that can describe the time response of the system. The truncation permits to reduce the dimension of the system from $N$ number of dofs in physical coordinates to m truncated vibration modes in modal coordinates.

The fundamental numerical problem derived from the truncation is the resulting non-square vibration modes matrix, used as transformation matrix in the physical to modal change of variable [1, 2]. This change should allow the diagonalisation of the matrices involved in the equation of motion: mass, stiffness and damping matrices. The diagonalisation is essential to decouple the system in $m$ second-order linear differential equations that can be solved analytically in the time domain. Nevertheless, the inverse of vibration modes matrix required for the diagonalisation cannot be applied from its non-square nature and it can only be replaced by the transpose matrix if both mass and stiffness matrices are symmetric. The complexity increases in a case of general damping instead of proportional or spectral ones [3], in which the damping matrix must be included in the eigenproblem in order to diagonalise the whole modal system.

This work proposes a methodology to overcome the issues abovementioned and applies this in the field of railway dynamics in order to compute the modal properties of a railway wheel

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modelled by using FEM [4]. The paper includes a study of the numerical performance of this method and its comparison with other numerical procedures to find the FRF of the wheel.

# 2 Overview of the mathematical approach

The matrix equation of motion of a mechanical system can be formulated as

$$\mathbf{M}\ddot{\boldsymbol{u}} + \mathbf{C}\dot{\boldsymbol{u}} + \mathbf{K}\boldsymbol{u} = \mathbf{F}, \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ correspond with the mass, damping and stiffness matrices, respectively, and $\mathbf{F}$ contains the external force terms and $\boldsymbol{u}$ are the physical coordinates. In this work, it is considered a general case of damping, instead of a proportional or spectral definition.

## 2.1 KC method

The eigenvectors matrix is used as matrix transformation to move to modal coordinates. Only the symmetric part of the stiffness matrix, $\mathbf{K}_{sym}$, is taken in order to be able to use the transpose matrix (instead of the inverse one) to diagonalise the mass and stiffness matrices involved. $N$ is the number of degrees of freedom of the system and $m$ the truncation number selected.

$$eigs(\mathbf{K}_{sym}, \mathbf{M}, m) \rightarrow \mathbf{\Phi} = [\{\phi\}_1, \ldots, \{\phi\}_m]. \tag{2}$$

It is proposed a first variable transformation:

$$\boldsymbol{u} = \mathbf{\Phi}\boldsymbol{q}, \tag{3}$$

where $\boldsymbol{q}$ is the modal coordinates vector. Eq. (1) is reduced to dimension $m$:

$$\ddot{\boldsymbol{q}} + \widetilde{\mathbf{C}}\dot{\boldsymbol{q}} + \widetilde{\mathbf{K}}\boldsymbol{q} = \widetilde{\mathbf{F}}, \tag{4}$$

where the $eigs$ function has normalised the mass matrix $\widetilde{\mathbf{M}} = \mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi} = \mathbf{I}$ and $\widetilde{\mathbf{K}}_{sym} = \mathbf{\Phi}^T\mathbf{K}_{sym}\mathbf{\Phi} = [\omega_r^2]$ is a diagonal matrix that contains the square of the natural frequencies. Nevertheless, the stiffness matrix $\widetilde{\mathbf{K}} = \mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi} = \mathbf{\Phi}^T(\mathbf{K}_{sym} + \mathbf{K}_{antisym})\mathbf{\Phi} = [\omega_r^2] + \widetilde{\mathbf{K}}_{antisym}$ and the damping one $\widetilde{\mathbf{C}} = \mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi}$ are not diagonal. The generalised force is $\widetilde{\mathbf{F}} = \mathbf{\Phi}^T\mathbf{F}$, where $\widetilde{\mathbf{F}}_r = \sum_{k=1}^{N} \phi_{kr} F_k$.

Considering a harmonic excitation $\widetilde{\mathbf{F}} = \bar{\bar{\widetilde{\mathbf{F}}}} e^{i\omega t}$, it is assumed a harmonic response $\boldsymbol{q} = \bar{\boldsymbol{q}} e^{i\omega t}$. Replacing in Eq. (4):

$$\bar{\boldsymbol{q}} = (-\omega^2\widetilde{\mathbf{I}} + i\omega\widetilde{\mathbf{C}} + \widetilde{\mathbf{K}})^{-1}\bar{\bar{\widetilde{\mathbf{F}}}}. \tag{5}$$

Hence, the receptance can be defined through an inverse matrix:

$$H_{ij}(\omega) = \frac{\bar{\boldsymbol{q}}}{\bar{\bar{\widetilde{\mathbf{F}}}}} = \mathbf{\Phi}_j(-\omega^2\widetilde{\mathbf{I}} + i\omega\widetilde{\mathbf{C}} + \widetilde{\mathbf{K}})^{-1}\mathbf{\Phi}_k^T. \tag{6}$$

## 2.2 AB-decoupling method

Considering the $2m$-extended system through

$$\mathbf{Q} = \begin{Bmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{Bmatrix}, \tag{7}$$

the modal matrix equation results

$$\widetilde{\mathbf{A}}\dot{\mathbf{Q}} + \widetilde{\mathbf{B}}\mathbf{Q} = \begin{Bmatrix} \widetilde{\mathbf{F}} \\ \mathbf{0} \end{Bmatrix}, \tag{8}$$

where

$$\widetilde{\mathbf{A}} = \begin{pmatrix} \widetilde{\mathbf{C}} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}, \quad \widetilde{\mathbf{B}} = \begin{pmatrix} \widetilde{\mathbf{K}} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix}. \tag{9}$$

At this stage, the eigenproblem associated with the linear first-order matrix equation of motion is solved without truncating, obtaining the $2m$-square matrix $\mathbf{\Theta}$,

$$eig(\widetilde{\mathbf{B}}, \widetilde{\mathbf{A}}) \to \mathbf{\Theta} = [\{\theta\}_1 \dots \{\theta\}_{2m}] \tag{10}$$

The $eig$ function has normalised the first matrix $\widetilde{\mathbf{A}} = \mathbf{\Theta}^T\mathbf{A}\mathbf{\Theta} = \mathbf{I}$ and $\widetilde{\mathbf{B}} = \mathbf{\Theta}^T\mathbf{B}\mathbf{\Theta} = [\lambda_s]$ is diagonal, resulting a set of uncoupled first-order linear differential equations:

$$\dot{Q}_s + \lambda_s Q_s = \{\theta\}_s^T \begin{pmatrix} \widetilde{\mathbf{F}} \\ \mathbf{0} \end{pmatrix}. \tag{11}$$

The diagonalisation of the matrix equation of motion permits to compute the receptance for general damping using modal superposition. With just the first variable transformation, non-diagonal modal matrices were found and the application of the inverse in the resulting modal equation was needed to approach the calculation of the receptance. The inversion of a matrix of the dimension for common FE structures (hundreds of thousands of dofs) is not addressable for conventional PCs. The proposed method based on the second variable transformation from the extended $2m$-system drastically reduce the time consumption of the receptance computing through the following expression:

$$H_{jk}(\omega) = \sum_{s=1}^{m} \phi_{js} \sum_{r=1}^{2m} \theta_{sr} \frac{\sum_{l=1}^{2m} \left(\boldsymbol{\theta}^{-1}\right)_{rl} \sum_{t=1}^{m} \left(\tilde{\mathbf{A}}^{-1}\right)_{lt} \Phi_{kt}}{i\omega + \lambda_r} \tag{12}$$

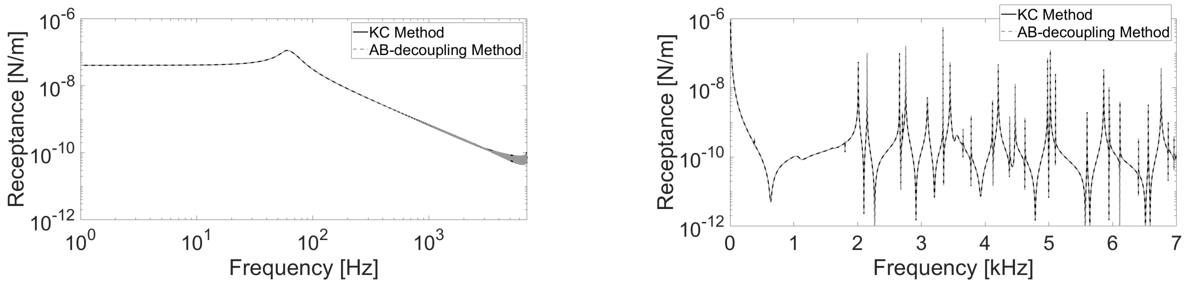
Being $\widetilde{\mathbf{A}}^{-1} = \begin{pmatrix} \widetilde{\mathbf{0}} & \widetilde{\mathbf{I}} \\ \widetilde{\mathbf{I}} & -\widetilde{\mathbf{C}} \end{pmatrix}$:

$$H_{jk}(\omega) = \mathbf{\Phi}_j \sum_{r=1}^{2m} \frac{\mathbf{\Theta}_{(1:m,r)} \left(\mathbf{\Theta}^{-1}\right)_{(r,m+1:2m)}}{i\omega + \lambda_r} \mathbf{\Phi}_k^{\mathrm{T}}. \tag{13}$$

The modal static correction [5] is implemented and applied to the previous expression in order to compensate the lack of contribution of the truncated vibration modes on the static response of the system.

# 3 Results

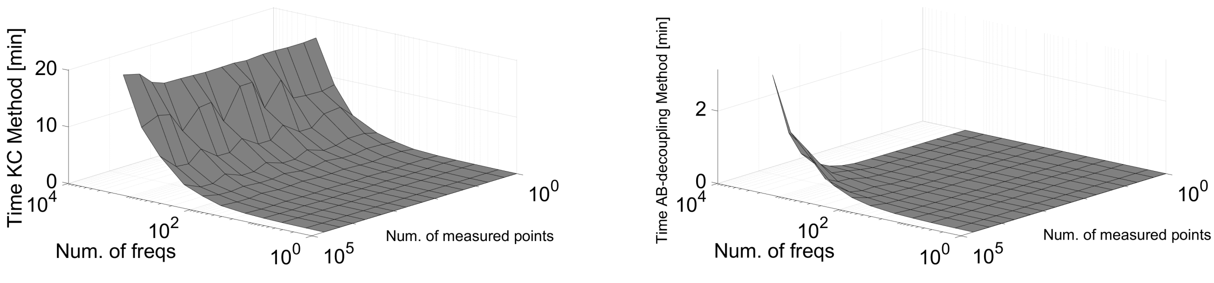
Using both KC and AB-decoupling methods sinthetised in Eqs. (6) and (13), respectively, the receptances for the track and the wheel have been evaluated. Fig. 1(a) shows that both methods give the same receptance for a rail modelled by the Moving Element Method [6] supported by a continuous viscoelastic Winkler bedding. A S1002 undamped Finite Element wheel model [1] has been also computed, obtaining again two overlapped curves for both methods.




(a) (b)

Figure 1: (a) Track receptance; (b) wheel receptance.

In terms of computational performance, the KC method requires the calculation of the $(-\omega^2\tilde{\mathbf{I}}+i\omega\tilde{\mathbf{C}}+\tilde{\mathbf{K}})^{-1}$ inverse, which is the most expensive operation. Hence, the time consumption exponentially grows with the number of frequencies selected to build the receptance. The pre- and post-multiplication of the modal transformation matrix $\mathbf{\Phi}$ barely increases the computational time, as reflected in Fig. 2(a), in which the influence of the number of physical measured points selected to calculate the receptance is almost negligible. Since there is not any inverse to compute for the AB-decoupling method, the influence of the number of frequencies and measured points can be clearly observed in Fig. 2(b).




(a) (b)

Figure 2: Computational time for the calculation of the wheel receptance.

The previous figures show that KC method requires higher computational time, especially when the receptance is evaluated for low number of measured points. The ratio between KC and AB-decoupling times plotted in Fig. 3 is in line with this observation since the first method needs to compute a very large matrix only for evaluating a few terms of the resulting matrix. When increasing the number of measured points, the ratio is reduced asymptotically but always above 1, showing that the AB-decoupling method is a more efficient one to compute the receptance.

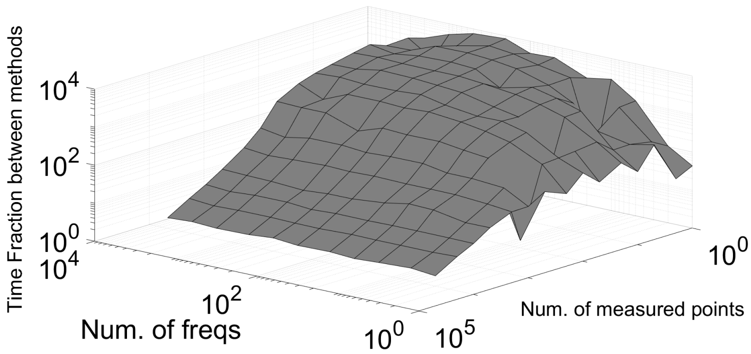



Figure 3: Ratio between the computational time for the KC and AB-decoupling methods for the calculation of the wheel receptance.

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