MODELLING FOR ENGINEERING & HUMAN BEHAVIOUR 2019

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Instituto Universitario de Matemática Multidisciplinar
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Numerical integral transform methods for random hyperbolic models

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1 Introduction

This work deals with the construction of analytic-numerical solutions, in the mean square sense [1], of the random heterogeneous telegraph type problem

$$\begin{aligned} u_{tt}(x,t) &= \left(k(x)\, u_x(x,t)\right)_x + a(x)\, u(x,t) + \psi(x,t)\,, \quad x>0,\ t>0\,, & (1)\\ u(0,t) &= g_1(t)\,, & (2)\\ u_x(0,t) &= g_2(t)\,, & (3)\\ u(x,0) &= g(x)\,, & (4) \end{aligned}$$

where $a(x)$, $k(x)$, $\psi(x,t)$, $g_1(t)$, $g_2(t)$ and $g(x)$ are stochastic processes (s.p.'s) with a finite degree of randomness [1].

Efficient methods for solving numerically deterministic problems such as finite-difference methods become unsuitable for the random case because of the computation of the expectation and the variance of the approximation solution s.p. The drawbacks are essentially of computational complexity such as the handling of big random matrices which appear throughout the iterative levels of the discretization steps and the necessity to store the information of all the previous levels of the iteration process. Then, they motive the search of non iterative alternatives. In this sense, this paper provides an approximation solution s.p. of the problem (1)–(4) which combines the random Fourier sine transform, the Gauss-Laguerre quadrature rule and the Monte Carlo method.

2 Gauss-Laguerre solution of a random hyperbolic model

The construction of an approximated solution s.p. of the problem (1)–(4) will be in two-stages. Firstly, using the Fourier sine transform, an infinite integral form solution of the theoretical

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solution is obtained. Then, using random Gauss-Laguerre quadrature formulae a random numerical solution is represented that is further computer by means of Monte Carlo simulations at appropriated root points of the Laguerre polynomials.
Let $V(x)(\xi) = \mathfrak{F}_s[u(x,\cdot)](\xi)$ be the Fourier sine transform of the unknown $u(x,\cdot)$:

$$V(x)(\xi) = \mathfrak{F}_s[u(x,\cdot)](\xi) = \int_0^{+\infty} u(x,t)\,\sin(\xi\,t)\,\mathrm{dt}\,,\quad \xi>0\,,\ x>0\,. \tag{5}$$

Let us denote

$$\begin{aligned} G_1(\xi) &= \mathfrak{F}_s[u(0,\cdot)](\xi) = \mathfrak{F}_s[g_1(t)](\xi)\,,\quad \xi>0\,, && (6)\\ G_2(\xi) &= \mathfrak{F}_s[u_x(0,\cdot)](\xi) = \mathfrak{F}_s[g_2(t)](\xi)\,,\quad \xi>0\,, && (7)\\ \Psi(x)(\xi) &= \mathfrak{F}_s[\psi(x,\cdot)](\xi)\,,\quad x>0,\ \xi>0\,. && (8)\end{aligned}$$

Let us assume that the s.p.'s $k(x)$, $a(x)$, $\psi(x,t)$, $g_1(t)$, $g_2(t)$ and $g(x)$ of problem (1)–(4) are mean four (m.f.) continuous with a finite degree of randomness. Let $k(x)$ be a positive s.p. 4-differentiable and let $\psi(x,t)$, $g_1(t)$, $g_2(t)$ be m.f. absolutely integrable s.p.'s in $t>0$. By applying random Fourier sine transform to problem (1)–(4) and using the properties of the random Fourier sine transform, [2] , one gets, for $\xi>0$ fixed

$$\frac{d^2}{dx^2}(V(x))(\xi) + \frac{k'(x)}{k(x)}\frac{d}{dx}(V(x))(\xi) + \frac{a(x)+\xi^2}{k(x)}V(x)(\xi) = \frac{\Psi(x)(\xi)-\xi\,g(x)}{k(x)}\,, \tag{9}$$

together with

$$\begin{aligned} &V(0)(\xi) = G_1(\xi)\,,\\ &\frac{d}{dx}(V(0))(\xi) = G_2(\xi)\,. \end{aligned} \tag{10}$$

Solution of problem (9)–(10) is the first component of the solution of extended random linear differential system, $V(x)(\xi) = [1,\ 0]\,X(x)(\xi)$,

$$\left.\begin{aligned} X'(x)(\xi) &= L(x)(\xi)\,X(x)(\xi) + B(x)(\xi)\,,\quad x>0\,,\\ X(0)(\xi) &= Y_0(\xi)\,, \end{aligned}\right\} \tag{11}$$

where

$$\left.\begin{aligned} &L(x)(\xi) = \begin{bmatrix} 0 & 1\\ -\dfrac{\xi^2+a(x)}{k(x)} & -\dfrac{k'(x)}{k(x)}\end{bmatrix},\ B(x)(\xi) = \begin{bmatrix} 0\\ \dfrac{\Psi(x)(\xi)-\xi\,g(x)}{k(x)}\end{bmatrix},\\ &\qquad\qquad\qquad\qquad Y_0(\xi) = \begin{bmatrix} G_1(\xi)\\ G_2(\xi)\end{bmatrix}. \end{aligned}\right\} \tag{12}$$

Assuming that 4-s.p.'s $a(x)$, $k(x)$ and $k'(x)$ satisfy the moment condition

$$\mathbb{E}\left[|s(x)|^r\right] \le m\,h^r < +\infty\,,\quad \forall\, r\ge 0\,, \tag{13}$$

for every $x>0$, it is guaranteed that the entries of the matrix s.p. $L(x)(\xi)\in L_4^{2\times 2}(\Omega)$, for $\xi>0$ fixed, satisfy condition (13). Condition (13) guarantees that $L(x)(\xi)$ is 4-locally absolutely integrable. Furthermore, it is verifies that vector s.p.'s both $B(x)(\xi)$ and $Y_0(x)(\xi)$ lie in $L_4^{2\times 1}(\Omega)$ and they are absolutely integrables in $x \in [0,+\infty)$. By using random inverse Fourier sine transform to $V(x)(\xi)$ one gets

$$u(x,t) = \frac{2}{\pi}\int_0^\infty V(x)(\xi)\,\sin(\xi\,t)\,\mathrm{d}\xi = \frac{2}{\pi}\int_0^\infty X_1(x)(\xi)\,\sin(\xi\,t)\,\mathrm{d}\xi\,, \tag{14}$$

where $X_1(x)(\xi) = [1,\ 0]\,X(x)(\xi)$. Now, taking advantage of the Gauss-Laguerre quadrature formula of degree N, see page 890 of [3], for a s.p. $\mathcal{J}(\xi) \in L_2(\Omega)$ being m.f.-absolutely integrable respect to $\xi > 0$, we can consider the following numerical approximation for each event $\omega \in \Omega$

$$I_N^{\text{G-L}}[\mathcal{J}](\omega) = \sum_{j=1}^{N} \nu_j\ \mathcal{J}(\vartheta_j;\omega)\,, \qquad \nu_j = \frac{\vartheta_j}{[(N+1)\,L_{N+1}(\vartheta_j)]^2}\,, \tag{15}$$

where ϑ_j is the j-th root of the deterministic Laguerre polynomial, $L_N(\vartheta)$, of degree N and ν_j is the weight. This quadrature formula is going to be applied to the r.v. $u(x,t)$ given by (14) taking

$$\mathcal{J}(\xi) = \mathcal{J}(x,t,\xi) = \frac{2}{\pi}\,X_1(x)(\xi)\,\sin(\xi\,t)\,e^{\xi}\,.$$

Given the degree N, let us denote by $u_N^{\text{G-L}}(x,t)$ the Gauss-Laguerre s.p. approximation of degree N of the exact solution s.p. $u(x,t)$ of the random problem (1)–(4), evaluated at (x,t) and expressed as the r.v.

$$u_N^{\text{G-L}}(x,t) = \frac{2}{\pi}\sum_{j=1}^{N} \nu_j\,\sin(\vartheta_j\,t)\,e^{\vartheta_j}\,X_1(x)(\vartheta_j)\,. \tag{16}$$

The exact solution $X_1(x)(\vartheta_j)$ is going to be obtained using Monte Carlo simulation because it is not available. We denoted by $\mathbb{E}_{MC}^{K}[\bar{X}_1(x)(\vartheta_j)]$ and $\mathrm{Cov}_{MC}^{K}\left[\bar{X}_1(x)(\vartheta_j)\,,\bar{X}_1(x)(\vartheta_\ell)\right]$ the expectation and the covariance, respectively, of K number of realizations used in the Monte Carlo (MC) simulation and $\bar{X}_1(x)(\vartheta_j)$ the deterministic numerical solution obtained after taking K realizations. Thus the final expressions for the approximations of the expectation and the variance of the solution s.p. take the form

$$\mathbb{E}[u_N^{\text{G-L}}(x,t)] \approx \mathbb{E}[u_{N,K}^{\text{G-L}}(x,t)] = \frac{2}{\pi}\sum_{j=1}^{N} \nu_j\,\sin(\vartheta_j\,t)\,e^{\vartheta_j}\,\mathbb{E}_{MC}^{K}[\bar{X}_1(x)(\vartheta_j)]\,, \tag{17}$$

$$\begin{aligned}\mathrm{Var}\left[u_N^{\text{G-L}}(x,t)\right] &\approx \mathrm{Var}\left[u_{N,K}^{\text{G-L}}(x,t)\right] = \\ &\left(\frac{2}{\pi}\right)^2 \sum_{j=1}^{N}\sum_{\ell=1}^{N} \nu_j\,\nu_\ell\,\sin(\vartheta_j\,t)\,\sin(\vartheta_\ell\,t)\,e^{\vartheta_j+\vartheta_\ell}\,\mathrm{Cov}_{MC}^{K}\left[\bar{X}_1(x)(\vartheta_j)\,,\bar{X}_1(x)(\vartheta_\ell)\right]\,.\end{aligned} \tag{18}$$

3 Numerical example

Consider the random heterogeneous telegraph type problem (1)–(4) with the following input data having a finite degree of randomness

$$\left.\begin{array}{l} k(x) = 1 + b\,\cos(\pi\,x)\,,\ a(x) = e^{-ax}\,,\ \psi(x,t) = e^{-(x+t)} \\ g_1(t) = 0\,,\ g_2(t) = 0\,,\ g(x) = 0 \end{array}\right\}\,, \tag{19}$$

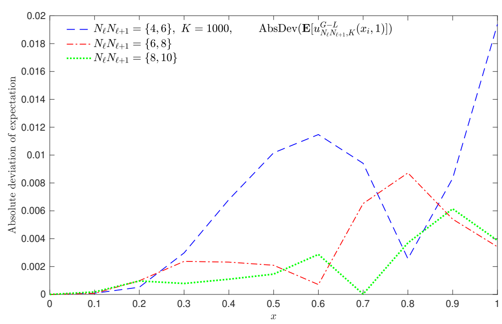
where parameters a and b are assumed to be both independent r.v.'s, specifically, a has a uniform distribution giving values in $[0,1]$, that is, $a \sim Un(0,1)$, and $b > 0$ has an exponential distribution of parameter 2 truncated on the interval $[0.1, 0.2]$, that is, $b \sim Exp_{[0.1,0.2]}(2)$. Then it is verified that s.p.'s $k(x)$, $a(x)$ and the deterministic functions $\psi(x,t)$, $g_1(t)$, $g_2(t)$ and $g(x)$ are 4-continuous and 4-absolutely integrable with respect to the time variable those depending on t. Furthermore, $k(x)$ is positive and 4-differentiable.

To study the numerical convergence of the approximations of both the expectation and the standard deviation we have studied the behaviour of their root mean square deviations (RMSD) and the absolute deviations (AbsDev), that is

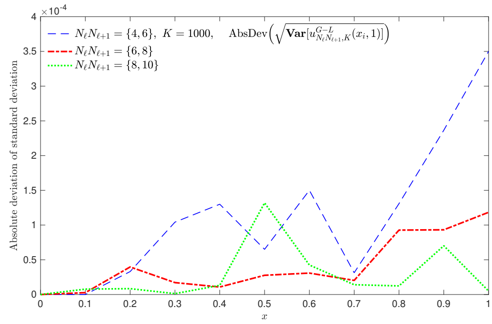
$$
\begin{aligned}
\mathrm{RMSD}\left[\mathbb{E}[u^{\text{G-L}}_{N,K_\ell K_{\ell+1}}(x_i,t)]\right] &= \sqrt{\frac{1}{(n+1)}\sum_{\ell=0}^{n}\left(\mathbb{E}[u^{\text{G-L}}_{N,K_{\ell+1}}(x_i,t)] - \mathbb{E}[u^{\text{G-L}}_{N,K_\ell}(x_i,t)]\right)^2}\,,\\
\mathrm{RMSD}\left[\sqrt{\mathrm{Var}[u^{\text{G-L}}_{N,K_\ell K_{\ell+1}}(x_i,t)]}\right] &= \sqrt{\frac{1}{(n+1)}\sum_{\ell=0}^{n}\left(\sqrt{\mathrm{Var}[u^{\text{G-L}}_{N,K_{\ell+1}}(x_i,t)} - \sqrt{\mathrm{Var}[u^{\text{G-L}}_{N,K_\ell}(x_i,t)}\right)^2}\,,\\
\mathrm{AbsDev}\left(\mathbb{E}[u^{\text{G-L}}_{N_\ell N_{\ell+1},K}(x,t)]\right) &= \left|\mathbb{E}[u^{\text{G-L}}_{N_{\ell+1},K}(x,t)] - \mathbb{E}[u^{\text{G-L}}_{N_\ell,K}(x,t)]\right|\,,\\
\mathrm{AbsDev}\left(\sqrt{\mathrm{Var}[u^{\text{G-L}}_{N_\ell N_{\ell+1},K}(x,t)]}\right) &= \left|\sqrt{\mathrm{Var}[u^{\text{G-L}}_{N_{\ell+1},K}(x,t)]} - \sqrt{\mathrm{Var}[^{\text{G-L}}_{N_\ell,K}(x,t)]}\right|\,,
\end{aligned}
$$

in two stages. Firstly, varying the number K of realizations in the Monte Carlo method but considering fixed N in the Gauss-Laguerre quadrature rule and secondly, varying N but considering the number of realizations K fixed. Table 1 and Figure 1 illustrated the numerical convergence of our approximations.

Figures



(a)



(b)

Figure 1: (a): Comparative graphics of the absolute deviations for successive approximations to the expectation $\mathbb{E}[u^{\text{G-L}}_{N_\ell N_{\ell+1},K}(x_i,1)]$. (b): Comparative graphics of the absolute deviations for successive approximations to the standard deviation $\sqrt{\mathrm{Var}[u^{\text{G-L}}_{N_\ell N_{\ell+1},K}(x_i,1)]}$. Both graphics correspond to the time $t=1$ on the spatial interval $0 \le x \le 1$, $K = 1000$ realizations and the degrees $N = \{4,6,8,10\}$ for the Laguerre polynomials.

Tables

$K_\ell K_{\ell+1}$	$\text{RMSD}\left[\mathbb{E}[u^{\text{G-L}}_{6,K_\ell K_{\ell+1}}(x_i,1)]\right]$	$\text{RMSD}\left[\sqrt{\text{Var}[u^{\text{G-L}}_{6,K_\ell K_{\ell+1}}(x_i,1)]}\right]$
K_0K_1	$2.81922e-05$	$2.91484e-05$
K_1K_2	$1.96565e-05$	$1.18180e-05$
K_2K_3	$1.11618e-05$	$2.49163e-06$
K_3K_4	$1.07918e-05$	$4.96128e-06$
K_4K_5	$3.59937e-06$	$2.83452e-06$

Table 1: Values of the RMSDs for the approximations of the expectation, $\text{RMSD}\left[\mathbb{E}^{\text{G-L}}_{N,K_\ell K_{\ell+1}}(x_i,t)\right]$, and the standard deviation, $\text{RMSD}\left[\sqrt{\text{Var}^{\text{G-L}}_{N,K_\ell K_{\ell+1}}(x_i,t)}\right]$, at $t=1$ on the spatial domain $0 \le x \le 1$, $N=6$ the degree of the Laguerre polynomial and the realizations $K_0 = 2500$, $K_1 = 5000$, $K_2 = 10^4$, $K_3 = 2\times 10^4$, $K_4 = 4\times 10^4$ and $K_5 = 5\times 10^4$.

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