Bounded rank perturbations of regular pencils over arbitrary fields

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Abstract

In this paper we solve the bounded rank perturbation problem for regular pencils over arbitrary fields. The solution is obtained reducing the problem to a row completion problem for matrix pencils. The result generalizes the main result of [1], where a solution to the problem was given requiring a condition on the underlying field.

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1 Introduction

In this paper we study a classical rank perturbation problem for matrix pencils. This problem is well studied for some particular cases and from different points of view (see e.g. [1,2,5,8,15,16]). The solution of the differential algebraic equation $Ex'(t) = Ax(t) + f(t)$ is determined by the Kronecker structure of the associated pencil $A - \lambda E$. Therefore, perturbations of the pencil, apart from theoretical interest, play a strong role in a variety of applications. Just to mention a few, as pointed out in [4], the description of the change of the Kronecker structure under low-rank perturbation is useful when introducing modifications in the system which affect only a small number of parameters. Hence, perturbations involving structured matrices or pencils appear in control design (see, for instance, [3, 6, 13] and the references therein). In [13], the rank-one perturbation of a regular matrix pencil has been related to the pole placement problem for a single-input differential-algebraic equation with feedback.

Let $\mathbb{F}$ be an arbitrary field. $\mathbb{F}[\lambda]$ denotes the ring of polynomials in the indeterminate λ with coefficients in $\mathbb{F}$. Given matrices $A, B \in \mathbb{F}^{n\times m}$, we say that $A + \lambda B \in \mathbb{F}[\lambda]^{n\times m}$ is a *matrix pencil*. Let $E(\lambda), E'(\lambda) \in \mathbb{F}[\lambda]^{n\times m}$ be matrix pencils. We say that they are *strictly equivalent*, denoted by $E(\lambda) \sim E'(\lambda)$, if and only if there exist invertible matrices $P \in \mathbb{F}^{n\times n}$ and $Q \in \mathbb{F}^{m\times m}$ such that

$$E'(\lambda) = PE(\lambda)Q.$$

We say that a pencil $E(\lambda) \in \mathbb{F}[\lambda]^{n\times m}$ is *regular* if and only if $n = m$ and $\det E(\lambda) \not\equiv 0$.

The *normal rank* of a matrix pencil $E(\lambda)$, denoted by $\operatorname{rank} E(\lambda)$, is the order of the largest nonidentically zero minor of $E(\lambda)$, i.e. it is the rank of $E(\lambda)$ considered as a matrix on the field of fractions of $\mathbb{F}[\lambda]$.

The low rank perturbation problem for regular matrix pencils is:

Problem 1 *Let r be a nonnegative integer. Let $B(\lambda), C(\lambda) \in \mathbb{F}[\lambda]^{(n+r)\times(n+r)}$ be two regular matrix pencils. Find necessary and sufficient conditions for the existence of matrix pencils $B'(\lambda)$ and $C'(\lambda)$ strictly equivalent to $B(\lambda)$ and $C(\lambda)$, respectively, such that*

$$\operatorname{rank}(B'(\lambda) - C'(\lambda)) \leq r.$$

We remark that Problem 1 is equivalent to the problem of finding necessary and sufficient conditions for the existence of a matrix pencil $P(\lambda) \in \mathbb{F}[\lambda]^{(n+r)\times(n+r)}$ of $\operatorname{rank}(P(\lambda)) \leq r$ such that $B(\lambda) + P(\lambda) \sim C(\lambda)$.

A solution to Problem 1 is given in [1] for fields $\mathbb{F}$ such that at least one element of the field or the point at infinity is neither an eigenvalue of $B(\lambda)$ nor of $C(\lambda)$. The proof of the necessity of the conditions remains true over arbitrary fields, but the proof of the sufficiency does not work if the restriction is removed.

Recently, a solution to the rank-one perturbation problem for (not necessarily regular) pencils has been obtained independently in [2,8], where the problem has been related to a row pencil completion problem.

A solution to the row pencil completion problem is given in [9, 10]. In this paper, using this result and following the approach of [2, 8], we give a solution to Problem 1. The proof is different from that of [1, Theorem 4.13] and holds for arbitrary fields.

In Section 2 we introduce some basic definitions and preliminary results. In particular, in Theorem 1 we recall the result in [10, Theorem 2] and in Lemma 1 we give a combinatorial result we will need in the solution to Problem 1. In Section 3 we present our solution in Theorem 3.

2 Notation and auxiliary results

Let $\mathbb{F}[\lambda, \mu]$ be the ring of polynomials in two variables λ and μ, with coefficients in $\mathbb{F}$. All polynomials in the paper are homogeneous from $\mathbb{F}[\lambda, \mu]$, and monic with respect to λ. Also, any homogeneous polynomial $\alpha(\lambda, \mu)$ will be denoted by α. Finally, for any chain of polynomials $\alpha_1 | \cdots | \alpha_n$, we will assume $\alpha_i = 1$ whenever $i < 1$.

We shall deal only with regular and quasi-regular matrix pencils: the complete set of strict equivalence invariants (so called *Kronecker invariants*) of a regular matrix pencil is formed by a chain of homogeneous polynomials $\alpha_1(\lambda, \mu) | \cdots | \alpha_n(\lambda, \mu)$, $\alpha_i(\lambda, \mu) \in \mathbb{F}[\lambda, \mu]$, $i = 1, \dots, n$, called *homogenous invariant factors*, for more details see [1, 12]. We say that a pencil $E(\lambda) \in \mathbb{F}[\lambda]^{n \times (n+r)}$ is *quasi-regular* if and only if $\operatorname{rank} E(\lambda) = n$. The complete set of Kronecker invariants of a quasi-regular matrix pencil is formed by a collection of nonnegative integers $c_1 \geq \cdots \geq c_r$, called the *column minimal indices,* and its homogenous invariant factors. For more details see [7,12,14].

The number of Kronecker invariants of a matrix pencil can be expressed in terms of the size and the rank of the pencil as follows: a regular pencil $E(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ has $n = \operatorname{rank} E(\lambda)$ homogeneous invariant factors. A quasi-regular pencil $E(\lambda) \in \mathbb{F}[\lambda]^{n \times (n+r)}$, has $n = \operatorname{rank} E(\lambda)$ homogeneous invariant factors and r (the number of columns minus the rank of $E(\lambda)$) column minimal indices. The sum of the degrees of the homogeneous invariant factors plus the sum of the colum minimal indices is equal to n. For details on the Kronecker invariants and the Kronecker canonical form see [7, 12].

In the proof of the main result we shall use the Theorem 2 in [10] for row completions up to a regular matrix pencil. We bring it here using the notation appropriate for this paper.

Theorem 1 *Let* $A(\lambda) \in \mathbb{F}[\lambda]^{n\times(n+r)}$ *be a matrix pencil with* $\alpha_1|\cdots|\alpha_n$ *and* $c_1 \geq \cdots \geq c_r$ *as homogeneous invariant factors and column minimal indices, respectively. Let* $C(\lambda) \in \mathbb{F}[\lambda]^{(n+r)\times(n+r)}$ *be a regular matrix pencil with* $\gamma_1|\cdots|\gamma_{n+r}$ *as homogeneous invariant factors.*

There exists a pencil $Y(\lambda) \in \mathbb{F}[\lambda]^{r\times(n+r)}$ *such that*

$$\left[\begin{array}{c} A(\lambda) \\ \hline Y(\lambda) \end{array}\right]$$

is strictly equivalent to $C(\lambda)$ *if and only if the following conditions are satisfied:*

$$\begin{array}{ll} (i) & \gamma_i|\alpha_i|\gamma_{i+r}, \quad i = 1,\dots,n, \\ (ii) & \sum_{i=1}^{j} c_i \leq \sum_{i=1}^{j} a_i, \quad j = 1,\dots,r, \end{array}$$

where $a_j = d(\epsilon_{r-j+1}) - d(\epsilon_{r-j}) - 1$, $j = 1,\dots,r$, *with* $\epsilon_j = \prod_{i=1}^{n+j} \operatorname{lcm}(\alpha_{i-j},\gamma_i)$, $j = 0,\dots,r$.

Remark 2 *We note that* $a_1 \geq \cdots \geq a_r$ *(see e.g. [11, Lemma 2]).*

We shall also need the following combinatorial result:

Lemma 1 *Let* $\beta_1|\cdots|\beta_{n+r}$ *and* $\gamma_1|\cdots|\gamma_{n+r}$ *be two chains of homogeneous polynomials in* $\mathbb{F}[\lambda,\mu]$, *such that*

$$\beta_i|\gamma_{i+r} \quad \text{and} \quad \gamma_i|\beta_{i+r}, \quad i = 1,\dots,n, \tag{1}$$

$$\textstyle\sum_{i=1}^{n+r} d(\beta_i) = \sum_{i=1}^{n+r} d(\gamma_i) = n + r. \tag{2}$$

Then

$$\sum_{i=1}^{n} d(\operatorname{lcm}(\beta_i,\gamma_i)) \leq n. \tag{3}$$

Proof: Let $k := \sum_{i=1}^{n+r} d(\gamma_i) - \sum_{i=1}^{n} d(\operatorname{lcm}(\beta_i,\gamma_i))$. From (1) we have $k \geq 0$.

Suppose on the contrary to (3) that $0 \leq k < r$. Let us denote by

$$x_j := \sum_{i=1}^{n+j} d(\operatorname{lcm}(\beta_{i-j},\gamma_i)) - \sum_{i=1}^{n+j-1} d(\operatorname{lcm}(\beta_{i-j+1}\gamma_i)), \quad j = 1,\dots,r.$$

By definition,

$$x_j \geq 0, \quad j = 1,\dots,r, \tag{4}$$

and from (1) and the definition of k, we have

$$x_1 + \cdots + x_r = k. \tag{5}$$

By the convexity property of polynomial chains (see e.g. [11, Lemma 2]),

$$x_1 \leq \cdots \leq x_r. \tag{6}$$

Equations (4), (5), and (6) give

$$x_1 = x_2 = \cdots = x_{r-k} = 0. \tag{7}$$

From (7), $x_1 + \ldots + x_{r-k} = 0$, then

$$\sum_{i=1}^{n+r-k} d(\operatorname{lcm}(\beta_{i-r+k}, \gamma_i)) = \sum_{i=1}^{n} d(\operatorname{lcm}(\beta_i, \gamma_i)).$$

Hence, we have

$$\gamma_{r-k} = 1, \quad \text{and} \quad \gamma_{i+r-k} | \operatorname{lcm}(\beta_i, \gamma_i), \quad i = 1, \ldots, n. \tag{8}$$

Since the conditions are symmetric for $\beta_1 | \cdots | \beta_{n+r}$ and $\gamma_1 | \cdots | \gamma_{n+r}$, completely analogously we also obtain

$$\beta_{r-k} = 1, \quad \text{and} \quad \beta_{i+r-k} | \operatorname{lcm}(\beta_i, \gamma_i), \quad i = 1, \ldots, n. \tag{9}$$

Thus (8) and (9) imply

$$\operatorname{lcm}(\beta_{i+r-k}, \gamma_{i+r-k}) | \operatorname{lcm}(\beta_i, \gamma_i), \quad i = 1, \ldots, n,$$

i.e.

$$\operatorname{lcm}(\beta_{i+r-k}, \gamma_{i+r-k}) = \operatorname{lcm}(\beta_i, \gamma_i), \quad i = 1, \ldots, n. \tag{10}$$

Finally, since $\beta_{r-k} = \gamma_{r-k} = 1$, (10) implies $\operatorname{lcm}(\beta_i, \gamma_i) = 1$, for $i = 1, \ldots, n$, and so by definition of k we get $k = n + r$, contradicting the assumption that $k < r$.

Hence $k \geq r$, i.e. (3) holds. ■

3 Main result - a solution to Problem 1

In this section we prove [1, Theorem 4.13] without any restrictions on the underlying field, and thus solve Problem 1 over arbitrary fields.

Theorem 3 *Let $B(\lambda), C(\lambda) \in \mathbb{F}[\lambda]^{(n+r)\times(n+r)}$ be two regular matrix pencils. Let $\beta_1 | \cdots | \beta_{n+r}$ and $\gamma_1 | \cdots | \gamma_{n+r}$ be homogeneous invariant factors of $B(\lambda)$ and $C(\lambda)$, respectively.*

There exist matrix pencils $B'(\lambda)$ and $C'(\lambda)$ strictly equivalent to $B(\lambda)$ and $C(\lambda)$, respectively, such that

$$\operatorname{rank}(B'(\lambda) - C'(\lambda)) \leq r,$$

if and only if

$$\beta_i | \gamma_{i+r} \quad \textit{and} \quad \gamma_i | \beta_{i+r}, \quad i = 1, \ldots, n. \tag{11}$$

Proof:

Necessity:

It was proven in [1, Proposition 4.3].

Sufficiency:

Let us suppose that condition (11) holds.

Our aim is to define homogeneous polynomials $\alpha_1|\cdots|\alpha_n$ and nonnegative integers $c_1 \geq \cdots \geq c_r$ satisfying

$$\sum_{i=1}^{n} d(\alpha_i) + \sum_{i=1}^{r} c_i = n, \tag{12}$$

and

$$\gamma_i|\alpha_i|\gamma_{i+r}, \qquad \text{and} \qquad \beta_i|\alpha_i|\beta_{i+r}, \quad i = 1, \ldots, n, \tag{13}$$

$$\sum_{i=1}^{j} c_i \leq \sum_{i=1}^{j} a_i, \qquad \text{and} \qquad \sum_{i=1}^{j} c_i \leq \sum_{i=1}^{j} b_i, \quad j = 1, \ldots, r, \tag{14}$$

where

$$a_j = d(\epsilon_{r-j+1}) - d(\epsilon_{r-j}) - 1, \quad b_j = d(\phi_{r-j+1}) - d(\phi_{r-j}) - 1, \quad j = 1, \ldots, r,$$

$$\epsilon_j = \prod_{i=1}^{n+j} \operatorname{lcm}(\alpha_{i-j}, \gamma_i), \quad \phi_j = \prod_{i=1}^{n+j} \operatorname{lcm}(\alpha_{i-j}, \beta_i), \quad j = 0, \ldots, r.$$

Once this is achieved, from (12) there exists a quasi-regular matrix pencil $A(\lambda) \in \mathbb{F}[\lambda]^{n\times(n+r)}$ having $\alpha_1|\cdots|\alpha_n$ as homogeneous invariant factors and $c_1 \geq \cdots \geq c_r$ as column minimal indices. Then, from (13) and (14), by Theorem 1 there exist pencils $X(\lambda), Y(\lambda) \in \mathbb{F}[\lambda]^{r\times(n+r)}$ such that

$$B(\lambda) \sim \left[\frac{A(\lambda)}{X(\lambda)}\right] \text{ and } C(\lambda) \sim \left[\frac{A(\lambda)}{Y(\lambda)}\right].$$

Since

$$\operatorname{rank}(\left[\frac{A(\lambda)}{X(\lambda)}\right] - \left[\frac{A(\lambda)}{Y(\lambda)}\right]) = \operatorname{rank}(\left[\frac{0}{*}\right]) \leq r,$$

taking $B'(\lambda) = \left[\frac{A(\lambda)}{X(\lambda)}\right]$ and $C'(\lambda) = \left[\frac{A(\lambda)}{Y(\lambda)}\right]$ we shall finish our proof.

Hence, we are left with defining polynomials $\alpha_1|\cdots|\alpha_n$ and nonnegative integers $c_1 \geq \cdots \geq c_r$ satisfying (12)-(14).

Let

$$\alpha_i := \operatorname{lcm}(\beta_i, \gamma_i), \quad i = 1, \ldots, n. \tag{15}$$

Then, (13) follows from (11). Moreover, we can write

$$\epsilon_j = \prod_{i=1}^{n+j} \operatorname{lcm}(\beta_{i-j}, \gamma_i), \quad \phi_j = \prod_{i=1}^{n+j} \operatorname{lcm}(\gamma_{i-j}, \beta_i), \quad j = 0, \dots, r.$$

Furthermore,

$$\sum_{i=1}^{r} a_i = \sum_{i=1}^{n+r} d(\gamma_i) - \sum_{i=1}^{n} d(\operatorname{lcm}(\beta_i, \gamma_i)) - r,$$

$$\sum_{i=1}^{r} b_i = \sum_{i=1}^{n+r} d(\beta_i) - \sum_{i=1}^{n} d(\operatorname{lcm}(\beta_i, \gamma_i)) - r,$$

and since $\sum_{i=1}^{n+r} d(\beta_i) = \sum_{i=1}^{n+r} d(\gamma_i) = n + r$, we have

$$\sum_{i=1}^{r} a_i = \sum_{i=1}^{r} b_i = n - \sum_{i=1}^{n} d(\operatorname{lcm}(\beta_i, \gamma_i)). \tag{16}$$

By Lemma 1, we obtain

$$\sum_{i=1}^{r} a_i = \sum_{i=1}^{r} b_i \geq 0.$$

We shall define $c_1 \geq \ldots \geq c_r$ such that

$$c_r + 1 \geq c_1 \geq \cdots \geq c_r \geq 0, \tag{17}$$

and

$$\sum_{i=1}^{r} c_i = n - \sum_{i=1}^{n} d(\operatorname{lcm}(\beta_i, \gamma_i)) \geq 0, \tag{18}$$

i.e., $c_1, \dots, c_r$ will be the most homogeneous partition of $n - \sum_{i=1}^{n} d(\operatorname{lcm}(\beta_i, \gamma_i))$. Explicitly, let q and w be integers such that $n - \sum_{i=1}^{n} d(\operatorname{lcm}(\beta_i, \gamma_i)) = qr + w$, with $0 \leq w < r$. Then, let

$$c_i := q + 1, \quad i = 1, \dots, w, \tag{19}$$

$$c_i := q, \qquad i = w + 1, \dots, r. \tag{20}$$

The non-negative integers $c_1 \geq \ldots \geq c_r$ defined by (19) and (20), and the polynomials $\alpha_1 | \cdots | \alpha_n$ given by (15) clearly satisfy (12). Moreover, by (16) and (18), the sequences $a_1 \geq \ldots \geq a_r$, $b_1 \geq \ldots \geq b_r$ and $c_1 \geq \ldots \geq c_r$ have the same total sum. Since the sequence $c_1 \geq \ldots \geq c_r$ satisfies (17), we have that (14) holds, as desired. This finishes our proof. ■

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References

[1] I. Baragaña and A. Roca, Weierstrass structure and eigenvalue placement of regular matrix pencils under low rank perturbations, SIAM Journal on Matrix Analysis and Applications 40 (2) (2019) 440-453.

[2] I. Baragaña and A. Roca, Rank-one perturbations of matrix pencils, arXiv:1912.08540v1.

[3] L. Batzke, Generic rank-one perturbations of structured regular matrix pencils, Linear Algebra Appl. 458 (2014), 638–670.

[4] F. De Terán and F.M. Dopico, Generic change of the partial multiplicities of regular matrix pencils under low-rank perturbations, SIAM Journal on Matrix Analysis and Applications 37 (3) (2016) 823–835.

[5] F. De Terán, F. Dopico and J. Moro, Low rank perturbation of Weierstrass structure, SIAM Journal on Matrix Analysis and Applications 30 (2) (2008) 538-547.

[6] F. De Terán, C. Mehl and V. Mehrmann, Low rank perturbation of regular matrix pencils with symmetry structures, arXiv:1902.00444.

[7] J. Dieudonné, Sur la réduction canonique des couples de matrices, Bulletin de la Société Math. France 74 (1946) 130-146.

[8] M. Dodig, M. Stošić, Rank one perturbations of pencils (2019), submitted.

[9] M. Dodig, Explicit solution of the row completion problem for matrix pencils, Linear Algebra Appl. 432 (2010) 1299-1309.

[10] M. Dodig, Completion up to a matrix pencil with column minimal indices as the only nontrivial Kronecker invariants, Linear Algebra Appl. 438 (2013) 3155-3173.

[11] M. Dodig and M. Stošić, On convexity of polynomial paths and generalized majorizations, The Electronic Journal of Combinatorics, Vol. 17, No.1, R61 (2010).

[12] F. Gantmacher, Matrix Theory, Vols I, II. Chelsea, New York, 1974.

[13] H. Gernandt and C. Trunk, Eigenvalue placement for regular matrix pencils with rank one perturbations, SIAM Journal on Matrix Analysis and Applications 38 (1) (2017) 134-154.

[14] A. Roca, Asignación de Invariantes en Sistemas de Control. PhD thesis, 2003.

[15] F. Silva, The Rank of the Difference of Matrices with Prescribed Similarity Classes, Linear and Multilinear Algebra 24 (1988) 51-58.

[16] R. Thompson, Invariant factors under rank one perturbations, Canad. J. Math 32 (1980) 240-245.