FROM PROJECTORS TO 1MP AND MP1 GENERALIZED INVERSES AND THEIR INDUCED PARTIAL ORDERS*

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Abstract. This paper deals with new generalized inverses of rectangular complex matrices, namely 1MP and MP1-inverses. They are constructed from oblique projectors represented by means of inner generalized inverses, by using an adequate equivalence relation, and then passing to the quotient set. We give characterizations and general expressions for 1MP and MP1-inverses. As applications, the binary relations induced for these new generalized inverses are proved to be partial orders.

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1. Introduction and background. Several generalized inverses were defined in the literature, and their duals when possible, from others previously studied. For example, the core inverse and the dual core inverse were introduced for square matrices of index at most 1 [2], two kind of extensions for matrices of arbitrary index, namely, DMP-inverse and its dual [16], and core-EP inverse [19]; and in addition the CMP-inverse for square arbitrary matrices [17], among others.

It is quite usual to present generalized inverses as the solution of a system of matrix equations. In this paper, we will introduce a new class of hybrid generalized inverses of rectangular complex matrices (of arbitrary index) as a representative of an equivalence class for equivalence relations defined on certain set from oblique projectors. The novelty is that the process used here to introduce new generalized inverses is different from the classical one and may serve as starting point for other authors for analyze generalized inverses from another point of view.

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Let $\mathbb{C}^{m\times n}$ be the set of $m\times n$ complex matrices. For $A\in\mathbb{C}^{m\times n}$, let A^*, A^{-1}, $\mathrm{rk}(A)$, and $\mathcal{R}(A)$ denote the conjugate transpose, the inverse ($m=n$), the rank, and the range space of A, respectively. As usual, I_n stands for the $n\times n$ identity matrix and $0_{m\times n}$ denotes the $m\times n$ zero matrix. The subscripts will be omitted when no confusion is caused.

Let $A\in\mathbb{C}^{m\times n}$. A matrix $X\in\mathbb{C}^{n\times m}$ that satisfies the equation (1) $AXA=A$ is called a $\{1\}$-*inverse* of A, and is denoted by A^-. The *Moore-Penrose inverse* of $A\in\mathbb{C}^{m\times n}$ is the unique matrix $X\in\mathbb{C}^{n\times m}$ such that (1) $AXA=A$, (2) $XAX=X$, (3) $(AX)^*=AX$, and (4) $(XA)^*=XA$ hold, and is denoted by $A^\dagger$. In general, for $A\in\mathbb{C}^{m\times n}$, the set of matrices $X\in\mathbb{C}^{n\times m}$ satisfying the equations $(i),(j),\ldots,(t)\in\{(1),(2),(3),(4)\}$ is denoted by $\mathcal{A}\{i,j,\ldots,t\}$. A matrix $X\in\mathcal{A}\{i,j,\ldots,t\}$ is called an $\{i,j,\ldots,t\}$-inverse of A.

The following results are used later.

Theorem 1.1. *[3] Let $A\in\mathbb{C}^{m\times n}$, $B\in\mathbb{C}^{p\times q}$, and $C\in\mathbb{C}^{m\times q}$. The matrix equation $AXB=C$ has a solution if and only if there exist $\{1\}$-inverses A^- and B^- of A and B, respectively, such that $AA^-C=C$ and $CB^-B=C$. In this case, the set of all solutions is given by $X=A^-CB^-+Y-A^-AYBB^-$, for arbitrary $Y\in\mathbb{C}^{n\times p}$.*

Theorem 1.2. *[18] Let $A\in\mathbb{C}^{m\times n}$ and G be a fixed $\{1\}$-inverse of A. The class of all $\{1\}$-inverses of A is given by*

$$\mathcal{A}\{1\}=\{G+U-GAUAG : U \text{ is arbitrary}\}=\{G+(I-GA)V+W(I-AG) : V,W \text{ are arbitrary}\}.$$

For $A\in\mathbb{C}^{n\times n}$, the *index* of A is the smallest nonnegative integer k such that $\mathcal{R}(A^k)=\mathcal{R}(A^{k+1})$, and is denoted by $k=ind(A)$. Let $A\in\mathbb{C}^{n\times n}$ with $k=ind(A)$. The *Drazin inverse* of A is the unique matrix $X\in\mathbb{C}^{n\times n}$ such that $XAX=X$, $AX=XA$, and $A^{k+1}X=A^k$ hold, and is denoted by A^D. If $ind(A)\leq 1$, then the Drazin inverse of A is called the *group inverse* of A and is denoted by $A^{\#}$. A detailed analysis of all these generalized inverses can be found, for example, in [1,3,4,12,13].

For a matrix $A\in\mathbb{C}^{n\times n}$ of index at most 1, two types of (unique) hybrid generalized inverses, namely $A^{\#}AA^\dagger$ and $A^\dagger AA^{\#}$ were defined in [21, p.97] by using other notation. The same were rediscovered by Baksalary and Trenkler in [2] and since then they were a key point of the study of generalized inverses. The authors introduced the matrix $A^{\#}AA^\dagger$ as the unique matrix $X\in\mathbb{C}^{n\times n}$ such that $AX=AA^\dagger$ and $\mathcal{R}(X)\subseteq\mathcal{R}(A)$. Clearly, the matrix $A^{\#}AA^\dagger$, which is denoted by $A^{\oplus}$, and $A^\dagger AA^{\#}$ denoted by $A_{\oplus}$, exist only when $ind(A)\leq 1$; such matrices are known as the *core* and the *dual core* inverse of A, respectively. In [23, Theorem 2.1], Wang and Liu proved that if $ind(A)\leq 1$ then the core inverse of A is the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying the following three equations:

$$AXA = A, \quad AX^2 = X, \quad \text{and} \quad (AX)^* = AX. \tag{1.1}$$

These inverses were generalized for matrices of arbitrary index by Malik and Thome in [16]. They introduced the *DMP-inverse* and its dual, for a matrix $A \in \mathbb{C}^{n\times n}$ of index k, by $A^{D,\dagger} = A^D AA^\dagger$ and $A^{\dagger,D} = A^\dagger AA^D$, respectively. It was also proved that the matrix $A^{D,\dagger}$ is the unique solution of the matrix equation system

$$XAX = X, \quad XA = A^D A, \quad \text{and} \quad A^k X = A^k A^\dagger. \tag{1.2}$$

In [20], Rakić, Dinčić, and Djordjević generalized the core inverse of a complex matrix to the case of an element in a ring. On the other hand, Chen, Zhu, Patrício, and Zhang analyzed characterizations and representations of core and dual core inverses in a ring in [6]. Other generalizations on the ring environment can be seen in [14, 25–27], and some more applications can be found in [22].

On the other hand, *CMP-inverses* were defined by Mehdipour and Salemi in [17] for a square matrix A as $A^{c,\dagger} = A^\dagger A_1 A^\dagger$, with $A_1 = AA^D A$ as the unique solution of the matricial equations system

$$XAX = X, \quad AXA = A_1 \quad AX = A_1 A^\dagger, \quad \text{and} \quad XA = A^\dagger A_1. \tag{1.3}$$

Some related results, applications, and extensions of these generalized inverses can be found in [5, 7–11, 15–17, 24, 28].

Except for $\{1\}$-inverses, all the aforementioned inverses exist and are unique. Moreover, each of them can be represented as the unique solution to a system of suitable matrix equations.

In this paper, new generalized hybrid inverses (and their duals) are introduced. These new classes of matrices provide not only a generalization of the core inverse to matrices of arbitrary index but also to rectangular matrices.

In [3, Lemma 3, p.45], it was proven that if B and C are $\{1\}$-inverses of A, then the product BAC is a $\{1, 2\}$-inverse of A. This general type of $\{1, 2\}$-inverses play an important role in our definitions due to, in this work, some cases in which B or C are Moore-Penrose inverses are introduced and studied.

The importance of projectors, in all branches of Mathematics, is undoubted. Generalized inverses are a very interesting tool for representing such projectors and operate with them. The main contribution of this paper is the idea of exploiting oblique projectors given by AA^- and A^-A in order to construct new generalized inverses. The novelty is that the process used here to introduce new generalized inverses is different from the classical one. While usually generalized inverses are presented as the solution of a system of matrix equations, we will introduce them as a "nice" representative of an equivalence class of certain quotient sets.

The paper is organized as follows. In Section 2, we introduce a new type of generalized inverses, called 1MP-inverses and denoted by $A^{-\dagger}$, which can be considered as a generalization of core inverses to rectangular matrices. The 1MP-inverses of a matrix A are characterized from a singular value decomposition of A. In Section 3, we give some characterizations of 1MP-inverses of a matrix A as $\{1,2,3\}$-inverses of A, as the solutions of a matrix equation system, and as a 1-parametrized set. As an application, we analyze the restriction of 1MP-inverses to the set of partial isometries. In Section 4, we introduce and study a partial order associated to 1MP-inverses as other application of 1MP-inverses. In Section 5, the dual case (called MP1-inverses) is studied.

2. 1MP-inverses. We notice that aforementioned generalized inverses are placing in some ways the emphasis on the projector A^-A for $A^- \in \mathcal{A}\{1\}$. Firstly, we are going to clarify what information can we extract from this product and then we will use it to define a new class of generalized inverses. This class will provide a generalization of core inverses from square to rectangular matrices.

For any matrix $A \in \mathbb{C}^{m\times n}$ with $\text{rk}(A) = a > 0$, a singular value decomposition (SVD, for short) [3, 4, 18] is given by

$$A = U \begin{pmatrix} D_a & 0 \\ 0 & 0 \end{pmatrix} V^*, \tag{2.1}$$

where $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary matrices, and $D_a \in \mathbb{C}^{a\times a}$ is a positive definite diagonal matrix.

Let $A \in \mathbb{C}^{m\times n}$ be written as in (2.1). In this case, it is well known that the general form for $\{1\}$-inverses of A is given by

$$A^- = V \begin{pmatrix} D_a^{-1} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} U^*, \tag{2.2}$$

partitioned according to the partition of A. Then, it is easy to see that

$$A^-A = V \begin{pmatrix} I_a & 0 \\ A_{21}D_a & 0 \end{pmatrix} V^*.$$

Now, for a given matrix A, we want to analyze conditions under which two of these projectors (of type

A^-A) are different by ranging $A^- \in \mathcal{A}\{1\}$. In order to do that, an equivalence relation is defined on the set $\mathcal{A}\{1\}$.

Let $A \in \mathbb{C}^{m\times n}$. We define the binary relation $\sim$ on the set $\mathcal{A}\{1\}$ as follows. For $A^-, A^= \in \mathcal{A}\{1\}$,

$$A^- \sim_\ell A^= \quad \text{if and only if} \quad A^-A = A^=A.$$

Clearly, $\sim_\ell$ is an equivalence relation on $\mathcal{A}\{1\}$. The equivalence class of $A^- \in \mathcal{A}\{1\}$ is given by $[A^-]_{\sim_\ell} = \{A^= \in \mathcal{A}\{1\} : A^-A = A^=A\}$. It is easy to check that

$$A^= = V \begin{pmatrix} D_a^{-1} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} U^* \in [A^-]_{\sim_\ell} \quad \text{if and only if} \quad A_{21} = B_{21}.$$

That is,

$$[A^-]_{\sim_\ell} = \left\{ V \begin{pmatrix} D_a^{-1} & B_{12} \\ A_{21} & B_{22} \end{pmatrix} U^* \in \mathcal{A}\{1\} : \, B_{12} \in \mathbb{C}^{a\times(m-a)}, \, B_{22} \in \mathbb{C}^{(n-a)\times(m-a)} \right\}.$$

The quotient set of $\mathcal{A}\{1\}$ by $\sim_\ell$ is given by $\mathcal{A}\{1\}/\sim_\ell = \{[A^-]_{\sim_\ell} : A^- \in \mathcal{A}\{1\}\}$. A complete set of representatives of the partition on $\mathcal{A}\{1\}$ induced by $\sim_\ell$ is given by

$$\mathcal{R}_{\sim_\ell} := \left\{ V \begin{pmatrix} D_a^{-1} & 0 \\ A_{21} & 0 \end{pmatrix} U^* : A_{21} \in \mathbb{C}^{(n-a)\times a} \text{ is arbitrary} \right\}.$$

Recalling that, for A written as in (2.1), the Moore-Penrose inverse of A is given by

$$A^\dagger = V \begin{pmatrix} D_a^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^*$$

and by observing that any element in $\mathcal{R}_{\sim_\ell}$ can be written as $A^-AA^\dagger$, we can state the following definition.

Definition 2.1. *Let $A \in \mathbb{C}^{m\times n}$. For each $A^- \in \mathcal{A}\{1\}$, the matrix*

$$A^{-\dagger} := A^-AA^\dagger \in \mathbb{C}^{n\times m}$$

is called a 1MP-inverse of A. That is, $A^{-\dagger}$ is defined as the "most simple" representative of an equivalence class of $\mathcal{A}\{1\}$ by $\sim_\ell$.

The symbol $\mathcal{A}\{-\dagger\}$ stands for the set of all 1MP-inverses of A. Clearly, since $A^\dagger$ is an element of this set, $\mathcal{A}\{-\dagger\} \neq \emptyset$ and, moreover,

$$\mathcal{A}\{-\dagger\} = \{A^-AA^\dagger : A^- \in \mathcal{A}\{1\}\} = \left\{ V \begin{pmatrix} D_a^{-1} & 0 \\ A_{21} & 0 \end{pmatrix} U^* : A_{21} \in \mathbb{C}^{(n-a)\times a} \right\}. \tag{2.3}$$

The existence of $\{1\}$-inverses and the Moore-Penrose inverse of A guarantee that 1MP-inverses of A always exist. It is clear that $\mathcal{A}\{-\dagger\} = \{A^{-1}\}$ whenever $A \in \mathbb{C}^{n\times n}$ is nonsingular and, moreover, $\mathcal{O}\{-\dagger\} = \{O\}$. In general, 1MP-inverses are not unique.

We observe that for a given matrix $A \in \mathbb{C}^{m\times n}$, by using a singular value decomposition of A, we can give a canonical form for 1MP-inverses of A as those matrices in (2.3).

Next result says that the interesting case is that given by matrices $A^-, A^= \in \mathcal{A}\{1\}$ such that $A^-A \neq A^=A$, otherwise, both A^- and $A^=$ provide the same 1MP-inverse $A^-AA^\dagger = A^=AA^\dagger$. This fact is shown in the next result, where the symbol $M \simeq N$ indicates that there exists a bijection between the sets M and N.

Proposition 2.2. *Let $A \in \mathbb{C}^{m\times n}$ of rank $a > 0$ written as in (2.1). Then*

$$\mathcal{A}\{1\}/\!\sim_\ell \quad \simeq \quad \mathcal{A}\{-\dagger\} \quad \simeq \quad \mathbb{C}^{(n-a)\times a}.$$

Proof. Let $\varphi : \mathcal{A}\{1\}/\!\sim_\ell \to \mathcal{A}\{-\dagger\}$ be the function defined by $\varphi([A^-]_{\sim_\ell}) = A^-AA^\dagger$. Clearly, φ is well-defined. Let $[A^-]_{\sim_\ell}$ and $[A^=]_{\sim_\ell}$ be in $\mathcal{A}\{1\}/\!\sim_\ell$ such that $\varphi([A^-]_{\sim_\ell}) = \varphi([A^=]_{\sim_\ell})$. Then $A^-AA^\dagger = A^=AA^\dagger$, so $A^-AA^\dagger A = A^=AA^\dagger A$. Hence, $A^-A = A^=A$, i.e., $[A^-]_{\sim_\ell} = [A^=]_{\sim_\ell}$, from where φ is injective. If $Y \in \mathcal{A}\{-\dagger\}$, by (2.3) there exists $A^- \in \mathcal{A}\{1\}$ such that $Y = A^-AA^\dagger$. Thus, $\varphi([A^-]_{\sim_\ell}) = A^-AA^\dagger = Y$. Hence, φ is surjective. In consequence, φ is a one-to-one correspondence between the sets $\mathcal{A}\{1\}/\!\sim_\ell$ and $\mathcal{A}\{-\dagger\}$, i.e., $\mathcal{A}\{1\}/\!\sim_\ell \simeq \mathcal{A}\{-\dagger\}$.

Let $A \in \mathbb{C}^{m\times n}$ of rank $a > 0$ written as in (2.1). We consider $\Gamma : \mathcal{A}\{-\dagger\} \to \mathbb{C}^{(n-a)\times a}$ as the function defined by $\Gamma(A^{-\dagger}) = A_{21}$, where $A^{-\dagger}$ is given as in (2.3). It is easy to see that Γ is a bijective function. Hence, $\mathcal{A}\{-\dagger\} \simeq \mathbb{C}^{(n-a)\times a}$. ∎

By Theorem 1.1, by solving the matrix equation $A^-A = A^=A$ (in $A^=$), its solution set is given by

$$[A^-]_{\sim_\ell} = \{A^= \in \mathcal{A}\{1\} : A^= = A^-AA^\dagger + Y(I - AA^\dagger) \text{ for arbitrary } Y \in \mathbb{C}^{n\times m}\},$$

which allows us to express the solution set as a 1-parametrized set.

Proposition 2.3. *Let $A \in \mathbb{C}^{m\times n}$. For a given $A^- \in \mathcal{A}\{1\}$, the matrix $A^{-\dagger} \in \mathbb{C}^{n\times m}$ satisfies the following properties:*

(a) $A^{-\dagger} \in \mathcal{A}\{1, 2, 3\}$.

(b) $A^{-\dagger}A = A^-A$ *and* $AA^{-\dagger} = AA^\dagger$, *(i.e., $A^{-\dagger}A$ is an oblique projector onto $\mathcal{R}(A^-A)$ along $\mathcal{N}(A)$ and $AA^{-\dagger}$ is an orthogonal projector onto $\mathcal{R}(A)$).*

Proof. (a) Since $A^-, A^\dagger \in \mathcal{A}\{1\}$, it follows that $A^{-\dagger} \in \mathcal{A}\{1,2\}$. Since $AA^{-\dagger} = AA^-AA^\dagger = AA^\dagger$, which is hermitian, we have $A^{-\dagger} \in \mathcal{A}\{3\}$.

(b) $A^{-\dagger}A = A^-AA^\dagger A = A^-A$, which is a projector onto $\mathcal{R}(A^-A)$ along $\mathcal{N}(A^-A) = \mathcal{N}(A)$. The other equality was proved in the previous item. ∎

From Proposicion 2.3 (b) and recalling that two $n \times n$ projectors coincide if and only if they have the same range and the same null space, we can state that

$$[A^-]_{\sim_\ell} = \{A^= \in \mathcal{A}\{1\} : A^-\mathcal{R}(A) = A^=\mathcal{R}(A)\},$$

where, for an adequate set M, the notation AM means $AM := \{Am : m \in M\}$.

We recall that a *complete system of invariants* for an equivalence relation $\approx$ on a nonempty set S is a family $\mathcal{F}$ of functions defined on S that satisfy: $s_1 \approx s_2$ if and only if $f(s_1) = f(s_2)$, for all $f \in \mathcal{F}$. We conclude that, for A written as in (2.1), the family $\mathcal{F}$ containing the sole function $f : \mathcal{A}\{1\} \to \mathbb{C}^{(n-a)\times n}$ defined, for every $X \in \mathcal{A}\{1\}$ written as in (2.2), by

$$f(X) = \begin{bmatrix} 0 & I_a \end{bmatrix} V^* X U \begin{bmatrix} I_a \\ 0 \end{bmatrix},$$

constitutes a complete system of invariants on $\mathcal{A}\{1\}$ for the equivalence relation $\sim_\ell$ because

$$A^- \sim_\ell A^= \iff A_{21} = B_{21} \iff f(A^-) = f(A^=).$$

3. Characterizations of 1MP-inverses. In this section we obtain characterizations from 1MP-inverses from several points of view.

By Proposition 2.3, we have that $\mathcal{A}\{-\dagger\} \subseteq \mathcal{A}\{1,2,3\}$ holds. As we will see below, these sets are the same.

Also, by Proposition 2.3, every matrix $X \in \mathcal{A}\{-\dagger\}$ satisfies the two equations of the system given by

$$XAX = X \quad \text{and} \quad AX = AA^\dagger. \tag{3.1}$$

We will prove that the set $\mathcal{A}\{-\dagger\}$ is the solution set of the system (3.1) by characterizing the solutions of the system (3.1).

THEOREM 3.1. *Let $A \in \mathbb{C}^{m\times n}$. The following conditions are equivalent:*

(a) $Z \in \mathcal{A}\{-\dagger\}$.

(b) *Z is solution of system (3.1).*

(c) $Z \in \mathcal{A}\{1,2,3\}$.

Proof. (a) $\Rightarrow$ (b) It follows from Proposition 2.3.

(b) $\Rightarrow$ (a) Assume that Z satisfies $XAX = X$ and $AX = AA^\dagger$, and let A be written in its SVD form as in (2.1).

Let $Z := V \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} U^*$ be partitioned accordingly to the sizes of the blocks of A. It is easy to see that

$$AZ = U \begin{pmatrix} D_a Z_{11} & D_a Z_{12} \\ 0 & 0 \end{pmatrix} U^* \quad \text{and} \quad AA^\dagger = U \begin{pmatrix} I_a & 0 \\ 0 & 0 \end{pmatrix} U^*.$$

From $AZ = AA^\dagger$, we have $Z_{11} = D_a^{-1}$ and $Z_{12} = 0$. Moreover,

$$ZAZ = V \begin{pmatrix} D_a^{-1} & 0 \\ Z_{21} & 0 \end{pmatrix} U^*.$$

From $ZAZ = Z$, we obtain $Z_{22} = 0$. Therefore, $Z = V \begin{pmatrix} D_a^{-1} & 0 \\ Z_{21} & 0 \end{pmatrix} U^*$, where $Z_{21} \in \mathbb{C}^{(n-a)\times a}$ is arbitrary. Hence, from (2.3), we get $Z \in \mathcal{A}\{-\dagger\}$.

(b) $\Rightarrow$ (c) It is clear that if Z satisfies $XAX = X$ and $AX = AA^\dagger$ then $Z \in \mathcal{A}\{2,3\}$. Now, $AZA = AA^\dagger A = A$, thus $Z \in \mathcal{A}\{1\}$.

(c) $\Rightarrow$ (b) Suposse $Z \in \mathcal{A}\{1,2,3\}$. Then, Z satisfies $XAX = X$.

Let A be written in its SVD form as in (2.1). From $Z \in \mathcal{A}\{1\}$, by (2.2) we have

$$Z = V \begin{pmatrix} D_a^{-1} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} U^*,$$

which is partitioned accordingly to the sizes of the blocks of A.

From $(AZ)^* = AZ$ we obtain $Z_{12} = 0$. So,

$$AZ = U \begin{pmatrix} I_a & 0 \\ 0 & 0 \end{pmatrix} U^* = AA^\dagger.$$

□

The previous result characterizes the set $\mathcal{A}\{-\dagger\}$ as follows

$$\mathcal{A}\{-\dagger\} = \{X \in \mathbb{C}^{n\times m} : \ XAX = X,\, AX = AA^\dagger\} = \mathcal{A}\{1,2,3\}.$$

Now, we will give another characterization of the set $\mathcal{A}\{-\dagger\}$ from a fixed 1MP of A.

Proposition 3.2. *Let $A \in \mathbb{C}^{m\times n}$ and let $A^{-\dagger}$ be a fixed matrix in $\mathcal{A}\{-\dagger\}$. The set of all 1MP-inverses of A is given by the 1-parametrized set*

$$\mathcal{A}\{-\dagger\} = \{A^{-\dagger} + (I_n - A^{-\dagger}A)WAA^{-\dagger} : W \in \mathbb{C}^{n\times m} \text{ is arbitrary}\}.$$

Proof. If we denote $\mathcal{S} := \{A^{-\dagger} + (I_n - A^{-\dagger}A)WAA^{-\dagger} : \text{for arbitrary } W \in \mathbb{C}^{n\times m}\}$, we have to show that $\mathcal{A}\{-\dagger\} = \mathcal{S}$. In fact, by Proposition 2.3, we can see that $\mathcal{S} \subseteq \mathcal{A}\{-\dagger\}$ because $A[A^{-\dagger} + (I_n - A^{-\dagger}A)WAA^{-\dagger}] = AA^{-\dagger} + A(I_n - A^{-\dagger}A)WAA^{-\dagger} = AA^{-\dagger} = AA^\dagger$, which is hermitian and, moreover, $[A^{-\dagger} + (I_n - A^{-\dagger}A)WAA^{-\dagger}]A[A^{-\dagger} + (I_n - A^{-\dagger}A)WAA^{-\dagger}] = [A^{-\dagger} + (I_n - A^{-\dagger}A)WAA^{-\dagger}]AA^{-\dagger} = A^{-\dagger} + (I_n - A^{-\dagger}A)WAA^{-\dagger}$. Now, the result follows from the equivalences between (a) and (b) in Theorem 3.1.

Now, if $Z \in \mathcal{A}\{-\dagger\}$, there exists $C \in \mathcal{A}\{1\}$ such that $Z = CAA^\dagger$. On the other hand, the fixed matrix $A^{-\dagger} \in \mathcal{A}\{-\dagger\}$ given in the hypothesis can be written as $A^{-\dagger} = A^{-}AA^\dagger$, for some matrix $A^{-} \in \mathcal{A}\{1\}$. Hence, from Theorem 1.2, there exists $W \in C^{n\times m}$ such that $C = A^{-} + W - A^{-}AWAA^{-}$. Thus, since $A^{-}A = A^{-\dagger}A$ and $AA^\dagger = AA^{-\dagger}$, by applying distributive property in the last equality we have $Z = CAA^\dagger = (A^{-} + W - A^{-}AWAA^{-})AA^\dagger = A^{-\dagger} + (I_n - A^{-\dagger}A)WAA^{-\dagger}$. Thus, $Z \in \mathcal{S}$. Hence, $\mathcal{A}\{-\dagger\} \subseteq \mathcal{S}$. ◻

Remark 3.3. By using that $\mathcal{A}\{-\dagger\} \subseteq \mathcal{A}\{1,2\}$, we notice that, for each $A^{-} \in \mathcal{A}\{1\}$, any 1MP-inverse $A^{-\dagger}$ of A can be seen as an $A^{(1,2)}_{T,S}$ inverse of A, for $T = A^{-}\mathcal{R}(A)$ and $S = \mathcal{N}(A^*)$, because $\mathcal{R}(A^{-\dagger}) = \mathcal{R}(A^{-\dagger}A) = \mathcal{R}(A^{-}A) = T$ and $\mathcal{N}(A^{-\dagger}) = \mathcal{N}(AA^{-\dagger}) = \mathcal{N}(AA^\dagger) = \mathcal{N}(A^\dagger) = \mathcal{N}(A^*) = S$.

3.1. 1MP-inverses for partial isometries. It is well known that a matrix $A \in \mathbb{C}^{m\times n}$ is a *partial isometry* if and only if $A^\dagger = A^*$, which is equivalent to $AA^*A = A$ [4]. In this case, $A^{-\dagger} = A^{-}AA^* \in \mathcal{A}\{-\dagger\}$. The following result analyzes the restriction of 1MP-inverses to the set of partial isometries.

Proposition 3.4. *Let $A \in \mathbb{C}^{m\times n}$ be a partial isometry. The following conditions are equivalent:*

(a) *The matrix equation system*

$$XAX = X \quad \text{and} \quad AX = AA^* \tag{3.2}$$

has a solution.

(b) *There exists* $A^- \in \mathcal{A}\{1\}$ *such that* $X = A^-AA^* + (I - A^-A)WAA^*$, *for some matrix* W *of suitable size.*

Proof. (a) ⇒ (b) Since A is a partial isometry, $A^\dagger = A^*$ and system (3.2) is equivalent to system (3.1). By Theorem 3.1, $X \in \mathcal{A}\{-\dagger\}$. Thus, from Proposition 3.2, by using $A^{-\dagger} = A^-AA^*$, $A^{-\dagger}A = A^-A$, and $AA^{-\dagger} = AA^*$, we conclude that there exists $A^- \in \mathcal{A}\{1\}$ such that $X = A^-AA^* + (I - A^-A)WAA^*$, for some $W \in \mathbb{C}^{n\times m}$.

(b) ⇒ (a) Let $X = A^-AA^* + (I - A^-A)WAA^*$, with $W \in \mathbb{C}^{n\times m}$ for some fixed $A^- \in \mathcal{A}\{1\}$. It is easy to see that $AX = AA^*$. Since $AA^*A = A$, by making some computations, we conclude that $XAX = XAA^* = X$ holds. □

We notice that if we obtain one 1MP-inverse from Proposition 3.4, we can completely solve system (3.2) by using Proposition 3.2.

4. An application: Partial order associated to 1MP-inverses. This section is devoted to develop a partial order associated 1MP-inverses previously introduced.

Definition 4.1. *Let* $A, B \in \mathbb{C}^{m\times n}$. *We will say that* A *is below* B *under the binary relation* $\leq^{-\dagger}$, *and it is denoted by* $A \leq^{-\dagger} B$, *if there exists* $A^{-\dagger} \in \mathcal{A}\{-\dagger\}$ *such that* $A^{-\dagger}A = A^{-\dagger}B$ *and* $AA^{-\dagger} = BA^{-\dagger}$.

For a fixed matrix $A \in \mathbb{C}^{m\times n}$, next result provides all matrices $B \in \mathbb{C}^{m\times n}$ such that $A \leq^{-\dagger} B$ and, in this case, the general form for $B^{-\dagger}$. Recall that, if A is written as in (2.1), from (2.3) we have that every 1MP-inverse can be expressed as

$$V \begin{pmatrix} D_a^{-1} & 0 \\ A_{21} & 0 \end{pmatrix} U^*, \tag{4.1}$$

for arbitrary $A_{21} \in \mathbb{C}^{(n-a)\times a}$.

Theorem 4.2. *Let* $A \in \mathbb{C}^{m\times n}$ *written as in* (2.1).

(a) *For* $B \in \mathbb{C}^{m\times n}$, *the following conditions are equivalent:*
  (i) $A \leq^{-\dagger} B$;
  (ii)
$$B = U \begin{pmatrix} D_a & 0 \\ -B_4A_{21}D_a & B_4 \end{pmatrix} V^* \tag{4.2}$$
  for some $A_{21} \in \mathbb{C}^{(n-a)\times a}$ *and* $B_4 \in \mathbb{C}^{(m-a)\times(n-a)}$.

(b) *Let $B \in \mathbb{C}^{m\times n}$ such that $A \leq^{-\dagger} B$. The following conditions are equivalent:*

(i) $X \in \mathcal{B}\{-\dagger\}$.

(ii) *There are matrices X_3 and X_4 of suitable sizes such that*

$$X = V \begin{pmatrix} D_a^{-1} & 0 \\ X_3 & X_4 \end{pmatrix} U^*, \tag{4.3}$$

where $B_4 X_3 = B_4 A_{21}$ and $X_4 \in \mathcal{B}_4\{-\dagger\}$.

Proof. Let A be written as in (2.1).

(a) Suppose that there exists $B \in \mathbb{C}^{m\times n}$ such that $A \leq^{-\dagger} B$. Then, there exists $A^{-\dagger} \in \mathcal{A}\{-\dagger\}$ such that $A^{-\dagger}A = A^{-\dagger}B$ and $AA^{-\dagger} = BA^{-\dagger}$. From (4.1), we can write $A^{-\dagger} = V \begin{pmatrix} D_a^{-1} & 0 \\ A_{21} & 0 \end{pmatrix} U^*$, for some $A_{21} \in \mathbb{C}^{(n-a)\times a}$. Let $B := U \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} V^*$ be partitioned accordingly to the sizes of the blocks of A. Then,

$$A^{-\dagger}A = V \begin{pmatrix} I_a & 0 \\ A_{21}D_a & 0 \end{pmatrix} V^* \quad \text{and} \quad A^{-\dagger}B = V \begin{pmatrix} D_a^{-1}B_1 & D_a^{-1}B_2 \\ A_{21}B_1 & A_{21}B_2 \end{pmatrix} V^*.$$

From $A^{-\dagger}A = A^{-\dagger}B$ we get $B_1 = D_a$ and $B_2 = 0$. Furthermore,

$$AA^{-\dagger} = U \begin{pmatrix} I_a & 0 \\ 0 & 0 \end{pmatrix} U^* \text{ and } BA^{-\dagger} = U \begin{pmatrix} I_a & 0 \\ B_3 D_a^{-1} + B_4 A_{21} & 0 \end{pmatrix} V^*.$$

From $AA^{-\dagger} = BA^{-\dagger}$ we arrive at $B_3 = -B_4 A_{21} D_a$. Hence, $B = U \begin{pmatrix} D_a & 0 \\ -B_4A_{21}D_a & B_4 \end{pmatrix} V^*$. Thus, (i) $\Longrightarrow$ (ii) is proved.

Conversely, suppose that there exist $A_{21} \in \mathbb{C}^{(n-a)\times a}$ and $B_4 \in \mathbb{C}^{(m-a)\times(n-a)}$ such that B is written as in (4.2). Let

$$A^{-\dagger} := V \begin{pmatrix} D_a^{-1} & 0 \\ A_{21} & 0 \end{pmatrix} U^*.$$

From (2.3), it is clear that $A^{-\dagger} \in \mathcal{A}\{-\dagger\}$. It is easy to see that $A^{-\dagger}A = A^{-\dagger}B$ and $AA^{-\dagger} = BA^{-\dagger}$, i.e., $A \leq^{-\dagger} B$ which proves (ii) $\Longrightarrow$ (i).

(b) Let $B \in \mathbb{C}^{m\times n}$ such that $A \leq^{-\dagger} B$.

(i) $\Longrightarrow$ (ii) From item (a) it is clear that B can be written as in (4.2).

Let $X \in \mathcal{B}\{-\dagger\} = \mathcal{B}\{1,2,3\}$ (see Theorem 3.1) partitioned as $X = V \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} U^*$, accordingly to the sizes of the blocks of A. Thus,

$$BXB = U \begin{pmatrix} D_aX_1D_a - D_aX_2B_4A_{21}D_a & D_aX_2B_4 \\ B_4[(X_3 - A_{21}D_aX_1) - \Pi B_4A_{21}]D_a & B_4\Pi B_4 \end{pmatrix} V^*,$$

where $\Pi := X_4 - A_{21}D_aX_2$. By making some computations, from $BXB = B$, we have $X_1 = D_a^{-1}$, $B_4 = B_4X_4B_4$, and $B_4A_{21} = B_4X_3$. Since

$$BX = U \begin{pmatrix} I_a & D_aX_2 \\ 0 & -B_4A_{21}D_aX_2 + B_4X_4 \end{pmatrix} U^*$$

is hermitian, we obtain $X_2 = 0$ and $B_4X_4 = (B_4X_4)^*$. Now, $X = V \begin{pmatrix} D_a^{-1} & 0 \\ X_3 & X_4 \end{pmatrix} U^*$. From $XBX = X$, we get $X_4 = X_4B_4X_4$. Therefore, $X_4 \in \mathcal{B}_4\{1,2,3\} = \mathcal{B}_4\{-\dagger\}$.

(ii) $\Longrightarrow$ (i) It is an easy computation by using Theorem 3.1. $\square$

Now, we are able to state that $\leq^{-\dagger}$ is a partial order on $\mathbb{C}^{m\times n}$.

Theorem 4.3. *The relation $\leq^{-\dagger}$ defined on $\mathbb{C}^{m\times n}$ is a matrix partial order.*

Proof. It is clear that $\leq^{-\dagger}$ is reflexive. Let $A \in \mathbb{C}^{m\times n}$ written as in (2.1).

Let $B \in \mathbb{C}^{m\times n}$ such that $A \leq^{-\dagger} B$ and $B \leq^{-\dagger} A$. From $B \leq^{-\dagger} A$, there exists $B^{-\dagger} \in \mathcal{B}\{-\dagger\}$ such taht $B^{-\dagger}B = B^{-\dagger}A$. Since $A \leq^{-\dagger} B$, by Theorem 4.2, B admits a representation as (4.2) and $B^{-\dagger}$ can be represented as in (4.3), i.e., $B^{-\dagger} = V \begin{pmatrix} D_a^{-1} & 0 \\ X_3 & X_4 \end{pmatrix} U^*$ and satisfies $B_4X_3 = B_4A_{21}$ and $X_4 \in \mathcal{B}_4\{-\dagger\}$. Then,

$$B^{-\dagger}B = V \begin{pmatrix} I_a & 0 \\ (X_3 - X_4B_4A_{21})D_a & X_4B_4 \end{pmatrix} V^* \quad \text{and} \quad B^{-\dagger}A = V \begin{pmatrix} I_a & 0 \\ X_3D_a & 0 \end{pmatrix} V^*.$$

From $B^{-\dagger}B = B^{-\dagger}A$, we have $X_4B_4 = 0$ and, consequently, $B_4 = B_4X_4B_4 = 0$. Then, $B = A$. Hence, $\leq^{-\dagger}$ is antisymmetric.

Let $B, C \in \mathbb{C}^{m\times n}$ such that $A \leq^{-\dagger} B$ and $B \leq^{-\dagger} C$. Since $B \leq^{-\dagger} C$, there exists a 1MP-inverse $B^{-\dagger}$ of B such that $B^{-\dagger}B = B^{-\dagger}C$ and $BB^{-\dagger} = CB^{-\dagger}$. From $A \leq^{-\dagger} B$, by Theorem 4.2, B and $B^{-\dagger}$ can be written as in (4.2) and (4.3), respectively, i.e.,

$$B = U \begin{pmatrix} D_a & 0 \\ -B_4 A_{21} D_a & B_4 \end{pmatrix} V^* \quad \text{and} \quad B^{-\dagger} = V \begin{pmatrix} D_a^{-1} & 0 \\ X_3 & X_4 \end{pmatrix} U^*,$$

and satisfy $B_4 X_3 = B_4 A_{21}$ and $X_4 \in \mathcal{B}_4\{-\dagger\}$.

Let $C := U \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} V^*$ be partitioned accordingly to the sizes of the blocks of A. Then

$$B^{-\dagger} B = V \begin{pmatrix} I_a & 0 \\ X_3 D_a - X_4 B_4 A_{21} D_a & X_4 B_4 \end{pmatrix} V^*$$

and

$$B^{-\dagger} C = V \begin{pmatrix} D_a^{-1} C_1 & D_a^{-1} C_2 \\ X_3 C_1 + X_4 C_3 & X_3 C_2 + X_4 C_4 \end{pmatrix} V^*,$$

From $B^{-\dagger}B = B^{-\dagger}C$ we get $C_1 = D_a$, $C_2 = 0$, $X_4 C_3 = -X_4 B_4 A_{21} D_a$, and $X_4 B_4 = X_4 C_4$. Moreover,

$$BB^{-\dagger} = U \begin{pmatrix} I_a & 0 \\ 0 & B_4 X_4 \end{pmatrix} U^* \quad \text{and} \quad CB^{-\dagger} = U \begin{pmatrix} I_a & 0 \\ C_3 D_a^{-1} + C_4 X_3 & C_4 X_4 \end{pmatrix} U^*.$$

Since $BB^{-\dagger} = CB^{-\dagger}$, we get $C_3 = -C_4 X_3 D_a$ and $B_4 X_4 = C_4 X_4$. Therefore,

$$C = U \begin{pmatrix} D_a & 0 \\ -C_4 X_3 D_a & C_4 \end{pmatrix} V^*. \tag{4.4}$$

That is, there exist matrices $X_3 \in \mathbb{C}^{(n-a)\times a}$ and $C_4 \in \mathbb{C}^{(m-a)\times(n-a)}$ such that C has the form given in (4.4). So, From Theorem 4.2 (a), we conclude $A \leq^{-\dagger} C$. Therefore, $\leq^{-\dagger}$ is transitive. □

Remark 4.4. Let $A, B \in \mathbb{C}^{m\times n}$. Note that, in general, $A \leq^{-\dagger} B$ does not imply $\mathcal{B}\{-\dagger\} \subseteq \mathcal{A}\{-\dagger\}$. Indeed, assume that $A \leq^{-\dagger} B$ holds and suppose that A is written as in (2.1) and B has the form found in Theorem 4.2. If $X \in \mathcal{B}\{-\dagger\}$, then X can be written as in Theorem 4.2 (b) (ii). By (2.3), it is clear that $X \in \mathcal{A}\{-\dagger\}$ if and only if $X_4 = 0$. Clearly, if we consider a matrix X such that $X_4 \neq 0$, the statement $A \leq^{-\dagger} B \Rightarrow \mathcal{B}\{-\dagger\} \subseteq \mathcal{A}\{-\dagger\}$ is false. Moreover, $X_4 = 0$ if and only if $B_4 = 0$ because $X_4 \in \mathcal{B}_4\{-\dagger\} = \mathcal{B}_4\{1,2,3\}$. Therefore, $\mathcal{B}\{-\dagger\} \subseteq \mathcal{A}\{-\dagger\}$ if and only if $A = B$.

5. MP1-inverse and the associated partial order. This section is devoted to present dual inverses of the 1MP-inverses introduced and characterized in the previous section and the associated partial order.

Proceeding as in Section 2, if $A \in \mathbb{C}^{m\times n}$ is written as in (2.1) and the general form for $\{1\}$-inverses of A is given by (2.2), we obtain

$$AA^- = U \begin{pmatrix} I_a & D_a A_{12} \\ 0 & 0 \end{pmatrix} U^*.$$

Now, by defining the equivalence relation: for $A^-, A^= \in \mathcal{A}\{1\}$,

$$A^- \sim_r A^= \quad \text{if and only if} \quad AA^- = AA^=,$$

we get

$$[A^-]_{\sim_r} = \left\{ V \begin{pmatrix} D_a^{-1} & A_{12} \\ B_{21} & B_{22} \end{pmatrix} U^* \in \mathcal{A}\{1\} : B_{21} \in \mathbb{C}^{(n-a)\times a}, B_{22} \in \mathbb{C}^{(n-a)\times(m-a)} \right\}.$$

A complete set of representatives of the partition on $\mathcal{A}\{1\}$ induced by $\sim_r$ is given by

$$\mathcal{R}_{\sim_r} := \left\{ V \begin{pmatrix} D_a^{-1} & A_{12} \\ 0 & 0 \end{pmatrix} U^* : A_{12} \in \mathbb{C}^{a\times(m-a)} \text{ is arbitrary} \right\}.$$

Now, we observe that any element in $\mathcal{R}_{\sim_r}$ can be written as $A^\dagger AA^-$. So, we consider a new type of generalized inverses which is the dual of the 1MP-inverses.

Definition 5.1. *Let $A \in \mathbb{C}^{m\times n}$. For each $A^- \in \mathcal{A}\{1\}$, the MP1-inverse of A, denoted by $A^{\dagger-}$, is the $n \times m$ matrix*

$$A^{\dagger-} := A^\dagger AA^-.$$

The symbol $\mathcal{A}\{\dagger-\}$ stands for the set of all MP1-inverses of A; clearly $A^\dagger$ is an element of this set, thus $\mathcal{A}\{\dagger-\} \neq \emptyset$. Hence, $\mathcal{A}\{\dagger-\} = \{A^\dagger AA^- : A^- \in \mathcal{A}\{1\}\}$. Therefore, MP1-inverses of A always exist; in general, they are not unique.

Since the development of MP1-inverses is analogous to 1MP-inverses, we only provide the results without proofs.

First of all, we observe that if $A \in \mathbb{C}^{m\times n}$, for each $A^- \in \mathcal{A}\{1\}$, the matrix $A^{\dagger-} \in \mathcal{A}\{1,2,4\}$.

In this case, the set $\mathcal{A}\{\dagger-\}$ can be characterized as follows

$$\mathcal{A}\{\dagger-\} = \mathcal{A}\{1,2,4\} = \{X \in \mathbb{C}^{n\times m} : XAX = X, XA = A^\dagger A\}.$$

Moreover, if A is written as in (2.1) then there exists $A^- \in \mathcal{A}\{1\}$ such that $Z := A^\dagger AA^- \in \mathcal{A}\{\dagger-\}$ if and only if

$$Z = V \begin{pmatrix} D_a^{-1} & Z_{12} \\ 0 & 0 \end{pmatrix} U^*,$$

for arbitrary $Z_{12} \in \mathbb{C}^{a\times(m-a)}$.

A 1-parametrized formula for MP1-inverses can be also established. Let $A \in \mathbb{C}^{m\times n}$. The following conditions are equivalent: $Z \in \mathcal{A}\{\dagger-\}$ if and only if there exists $A^{\dagger-} \in \mathcal{A}\{\dagger-\}$ such that $Z = A^{\dagger-} + A^{\dagger-}AW(I - AA^{\dagger-})$, for arbitrary W of suitable size.

Remark 5.2. Let $A \in \mathbb{C}^{m\times n}$. Note that, for each $A^{-} \in \mathcal{A}\{1\}$, any MP1-inverse $A^{\dagger-}$ of A can be seen as an outer inverse $A^{(1,2)}_{T,S}$ of A, by setting $T := \mathcal{R}(A^*)$ and $S := \mathcal{N}(AA^{-})$.

We can state an interesting relationship between 1MP- and MP1-inverses.

Proposition 5.3. *Let* $A \in \mathbb{C}^{m\times n}$. *Then*

$$\mathcal{A}\{-\dagger\} \cap \mathcal{A}\{\dagger-\} = \{A^{\dagger}\}.$$

Proof. It directly follows from $\mathcal{A}\{-\dagger\} = \mathcal{A}\{1,2,3\}$ and $\mathcal{A}\{\dagger-\} = \mathcal{A}\{1,2,4\}$. □

Remark 5.4. Let $A \in \mathbb{C}^{n\times n}$. If $ind(A) \le 1$, it is immediate that $A^{\oplus} \in \mathcal{A}\{-\dagger\}$. In addition, it is easy to see that if $A^{D,\dagger} \in \mathcal{A}\{-\dagger\}$ then $AA^{D}A = A$. Thus, $ind(A) \le 1$, from where $A^{D,\dagger} = A^{\oplus}$. That is, $A^{D,\dagger} \in \mathcal{A}\{-\dagger\} \Longleftrightarrow A^{D,\dagger} = A^{\oplus}$. Similarly, for dual core inverses: $A^{\dagger,D} \in \mathcal{A}\{\dagger-\}$ if and only if $A_{\oplus} = A^{\dagger,D}$. In addition, $A^{c,\dagger} \in \mathcal{A}\{-\dagger\}$ if and only if $A^{c,\dagger} \in \mathcal{A}\{\dagger-\}$ if and only if $A^{c,\dagger} = A^{\dagger}$.

Definition 5.5. *Let* $A, B \in \mathbb{C}^{m\times n}$. *We will say that* A *is below* B *under the binary relation* $\le^{\dagger-}$, *and it is denoted by* $A \le^{\dagger-} B$, *if there exists* $A^{\dagger-} \in \mathcal{A}\{\dagger-\}$ *such that* $A^{\dagger-}A = A^{\dagger-}B$ *and* $AA^{\dagger-} = BA^{\dagger-}$.

For a fixed matrix $A \in \mathbb{C}^{m\times n}$, the next result provides all matrices $B \in \mathbb{C}^{m\times n}$ such that $A \le^{\dagger-} B$ and, in this case, the general form for $B^{\dagger-}$.

Theorem 5.6. *Let* $A \in \mathbb{C}^{m\times n}$ *written as in* (2.1).

(a) *The following conditions are equivalent:*
   (i) *There exists* $B \in \mathbb{C}^{m\times n}$ *such that* $A \le^{\dagger-} B$.
   (ii) *There exist matrices* $A_{12} \in \mathbb{C}^{a\times(m-a)}$ *and* $B_4 \in \mathbb{C}^{(m-a)\times(n-a)}$ *such that*

$$B = U \begin{pmatrix} D_a & -D_aA_{12}B_4 \\ 0 & B_4 \end{pmatrix} V^*.$$

(b) *Let* $B \in \mathbb{C}^{m\times n}$ *such that* $A \le^{-\dagger} B$. *The following conditions are equivalent:*
   (i) $X \in \mathcal{B}\{\dagger-\}$.

(ii) *There are matrices X_2 and X_4 of suitable sizes such that*

$$X = V \begin{pmatrix} D_a^{-1} & X_2 \\ 0 & X_4 \end{pmatrix} U^*,$$

where $X_2 B_4 = A_{12} B_4$, and $X_4 \in \mathcal{B}_4\{\dagger-\}$.

We close this section by stating that $\leq^{\dagger-}$ is a partial order on $\mathbb{C}^{m\times n}$.

THEOREM 5.7. *The binary relation $\leq^{\dagger-}$ defined on $\mathbb{C}^{m\times n}$ is a partial order.*

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