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ON THE RANDOM NON-AUTONOMOUS LOGISTIC EQUATION WITH TIME-DEPENDENT COEFFICIENTS

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Abstract. In this paper, we deal with the non-autonomous logistic growth model with time-dependent intrinsic growth rate and carrying capacity. Accounting for errors in recorded data, randomness is incorporated into the equation by assuming that the input parameters are random variables. The uncertainty of the model output is quantified by approximations of the first probability density function via the random variable transformation method. A numerical example illustrates the results.

Keywords: logistic growth model; time-dependent carrying capacity; random parameters; probability density function.

AMS Classification 2010: 34F05; 92D25; 92D40.

1. Introduction

Growth models such as the logistic model are widely studied and applied in population and ecological modeling. Classically, the intrinsic growth rate and the carrying capacity of the logistic model have been considered constant. However, some works considered it as a function of time [6, 7], for instance, motivated by the principle that a changing environment may result in a significant change in the limiting capacity [17]; or in the case of periodic coefficients which is especially important for many biological problems due to a natural periodicity of the Earth rotations [14]. The model is then presented by the non-autonomous logistic equation

$$\begin{cases} N'(t) &= r(t)N(t)\left(1-\dfrac{N(t)}{K(t)}\right), \quad t\in\mathbb{R}, \\ N(t_0) &= N_0, \end{cases} \tag{1.1}$$

where $N_0 > 0$ is the initial condition, $r(t)$ is the intrinsic growth rate, and $K(t)$ is the carrying capacity.

Equation (1.1) is a Bernoulli-type ordinary differential equation. When subject to an initial condition $N(t_0) = N_0 > 0$, it has a unique solution $N(t)$ given by [6, 14]

$$N(t) = \frac{N_0 \exp\left(\int_{t_0}^{t} r(s)\,\mathrm{d}s\right)}{1 + N_0 \int_{t_0}^{t} \frac{r(s)}{K(s)} \exp\left(\int_{t_0}^{s} r(\nu)\,\mathrm{d}\nu\right)\,\mathrm{d}s}. \tag{1.2}$$

In [6], it is showed that if r and K are (measurable) functions on $\mathbb{R}$ for which the numbers

$$r_{\text{inf}} = \inf_{t\in\mathbb{R}} r(t),\;\; r_{\text{sup}} = \sup_{t\in\mathbb{R}} r(t),\;\; K_{\text{inf}} = \inf_{t\in\mathbb{R}} K(t),\;\; K_{\text{sup}} = \sup_{t\in\mathbb{R}} K(t)$$

obey the relations $0 < r_{\text{inf}}, r_{\text{sup}} < \infty$, $0 < K_{\text{inf}}, K_{\text{sup}} < \infty$, then the non-autonomous logistic equation (1.1) possesses a canonical solution on $\mathbb{R}$ which is approached, in the limit of large t, by each solution satisfying $N(t_0) = N_0 > 0$.

In the mathematical modeling of population and ecological processes, the parameters are either measured directly or determined by curve fitting. These parameters may have large variability that depends on the experimental method and its inherent error, on differences in the actual population sample size used, as well as other factors that are difficult to account for. In view of this fact, randomness is incorporated into equation (1.1) by assuming that the parameters N_0, $r(t) = r(t;\xi_1,\xi_2,...,\xi_m)$ and $K(t) = K(t;\xi_1,\xi_2,...,\xi_m)$ depend on the random vector $(N_0,\xi_1,\xi_2,...,\xi_m)$, being m a non-negative integer, with known joint probability distribution. Therefore, the general solution $N(t)$ to (1.1), given by (1.2), becomes a random variable that evolves with time, that is, a stochastic process [2, 15]. In this paper, we will assume that these random variables and stochastic process are defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the sample space consisting of outcomes $\omega \in \Omega$, $\mathcal{F}$ is the σ-algebra of events, and $\mathbb{P}$ is the probability measure.

The aim of this work is to provide and illustrate approximations of the first probability density function (pdf), $f_N(q;t)$, of the solution stochastic process $N(t)$ from (1.2). By definition, the pdf is a non-negative Borel measurable function characterized by $\mathbb{P}[N(t) \in \mathcal{B}] = \int_{\mathcal{B}} f_N(q;t)\mathrm{d}q$ for any Borel set $\mathcal{B}$ in $\mathbb{R}$. A random variable or vector is said to be absolutely continuous when it has a pdf.

The paper is organized as follows. In Section 2, an approximation of the first pdf of the solution stochastic process to (1.1) is constructed. This approximation is based on the random variable transformation method. Section 3 is addressed to illustrate numerical approximations of the first pdf of $N(t)$ for a particular example. Finally, in Section 4 our main conclusions are presented.

2. Approximation of the PDF of the solution process

In this section, we assume that the random quantities N_0, $r(t)$, and $K(t)$ of (1.1) depend on the random vector $(N_0,\xi_1,\xi_2,...,\xi_m)$, which has a known probability distribution. We compute approximations of the pdf of $N(t)$ from (1.2), for each fixed instant t.

Let $(N_0, \xi_1, \xi_2, ..., \xi_m)$ be an absolutely continuous real random vector in $(\Omega, \mathcal{F}, \mathbb{P})$. Obviously, all of its components depend on the sample parameter, for example $N_0 = N_0(\omega)$, $\omega \in \Omega$, but as usual this notation will be hidden hereinafter.

To apply the random variable transformation method [3], [5, Th. 2.1.5], let us consider the mapping

$$(N_0, \xi_1, ..., \xi_m) \quad \mapsto \quad (N, X_1, ..., X_m) = \left(\frac{N_0\, \Theta}{1 + N_0\, \eta}, \xi_1, ..., \xi_m \right), \tag{2.1}$$

where the auxiliary random variables $X_1 = \xi_1$, ..., $X_m = \xi_m$ have been conveniently chosen, $N = N(t)$, for a fixed t, and

$$\Theta = \Theta(t;\, \xi_1, ..., \xi_m) = \exp\left(\int_{t_0}^{t} r(s)\, \mathrm{d}s \right) \tag{2.2}$$

and

$$\eta = \eta(t;\, \xi_1, ..., \xi_m) = \int_{t_0}^{t} \frac{r(s)}{K(s)} \exp\left(\int_{t_0}^{s} r(\nu)\, \mathrm{d}\nu \right) \mathrm{d}s. \tag{2.3}$$

It is not difficult to verify that the function defined by (2.1) is invertible and its inverse is given by

$$(N, X_1, ..., X_m) \quad \mapsto \quad (N_0, \xi_1, ..., \xi_m) = \left(\frac{N}{\Theta - N\, \eta}, \xi_1, ..., \xi_m \right). \tag{2.4}$$

From the random variable transformation method, the density function of $N(t)$, for a fixed t, can be presented as

$$\begin{aligned} f_N(N; t) &= \int_{\mathcal{D}(X_1,...,X_m)} f_{(N,X_1,...,X_m)}(N, X_1, ..., X_m)\, \mathrm{d}X_1 \dots \mathrm{d}X_m = \\ &= \int_{\mathcal{D}(\xi_1,...,\xi_m)} f_{(N_0,\xi_1,...,\xi_m)}(N_0, \xi_1, ..., \xi_m)\, |J(N, X_1, ..., X_m)|\, \mathrm{d}\xi_1 \dots \mathrm{d}\xi_m, \end{aligned}$$

where $f_{(N,X_1,...,X_m)}$ is the joint density of the random vector $(N, X_1, ..., X_m)$; $f_{(N_0,\xi_1,...,\xi_m)}$ is the joint density of $(N_0, \xi_1, ..., \xi_m)$; $\mathcal{D}$ denotes the support of the corresponding random vector; and $J(N, X_1, ..., X_m)$ is the determinant Jacobian of the function given by (2.4), that is,

$$\begin{aligned} J(N, X_1, ..., X_m) &= \det\left(\frac{\partial(N_0, \xi_1, ..., \xi_m)}{\partial(N, X_1, ..., X_m)} \right) = \frac{\partial N_0}{\partial N} = \\ &= \frac{\partial}{\partial N}\left(\frac{N}{\Theta - N\, \eta} \right) = \frac{\Theta}{(\Theta - N\, \eta)^2} > 0, \end{aligned}$$

almost surely, since Θ in (2.2) is positive.

Summarizing, the following result has been established.

Theorem 2.1. *For a fixed t, the pdf of $N(t)$ in (1.2), f_N, is given by*

$$f_N(q; t) \;=\; \int_{\mathcal{D}(\xi_1,...,\xi_m)} f_{(N_0,\xi_1,...,\xi_m)}\left(\frac{q}{\Theta - q\, \eta}, \xi_1, ..., \xi_m \right) \frac{\Theta}{(\Theta - q\, \eta)^2}\, \mathrm{d}\xi_1 \dots \mathrm{d}\xi_m, \tag{2.5}$$

where $\Theta = \Theta(t;\, \xi_1, ..., \xi_m)$ and $\eta = \eta(t;\, \xi_1, ..., \xi_m)$ are given in (2.2) and (2.3), respectively.

It is important to note that if some input random parameter is independent of the rest, then the joint pdf in the integrand in (2.5) can be factorized as a product. For instance, in the particular case that N_0, ξ_1, ..., ξ_m are independent random variables, the integrand of (2.5) reads

$$f_{(N_0,\xi_1,\dots,\xi_m)}\left(\frac{q}{\Theta - q\,\eta}, \xi_1, \dots, \xi_m\right) = f_{N_0}\left(\frac{q}{\Theta - q\,\eta}\right) f_{\xi_1}(\xi_1) \dots f_{\xi_m}(\xi_m).$$

In that case, (2.5) can be presented parametrically [4] as

$$f_N(q;t) = \mathbb{E}\left[f_{N_0}\left(\frac{q}{\Theta - q\,\eta}\right)\frac{\Theta}{(\Theta - N\,\eta)^2}\right]. \tag{2.6}$$

3. Numerical example

Let us consider the logistic model (1.1) driven by the time-varying intrinsic growth rate, $r(t)$, and the time-varying carrying capacity, $K(t)$, that take the following forms:

Case 1. $K(t) = a + b\sin(2\pi t/p)$ and $r(t) = r_0 + \beta\sin(2\pi t/p)$;

Case 2. $K(t) = a + b\sin(2\pi t/p)$ and $r(t) = r_0$;

Case 3. $K(t) = K_1K_2/\left[K_1 + (K_2 - K_1)\mathrm{e}^{-ct}\right]$ and $r(t) = r_0$;

Case 4. $K(t) = K_1K_2/\left[K_1 + (K_2 - K_1)\mathrm{e}^{-ct}\right]$ and $r(t) = r_0 + \beta\sin(2\pi t/p)$,

where $a \sim \text{Uniform}[4.85, 5.15]$ (it is a random variable that follows a uniform distribution); $b \sim \text{Uniform}[1.93, 2.07]$; $p \sim \text{Uniform}[1.97, 2.03]$; $r_0 \sim \text{Uniform}[0.94, 1.06]$; $\beta \sim \text{Uniform}[0.77, 0.83]$; $K_1 \sim \text{Uniform}[4.86, 5.14]$; $K_2 \sim \text{Uniform}[6.74, 7.26]$; $c \sim$ Exponential(1/0.8) (it has an exponential distribution with rate parameter 1/0.8); $N_0 \sim$ Exponential(1/2) in Cases 1–2; and $N_0 \sim \text{Uniform}[1.84, 2.16]$ in Cases 3–4. Moreover, all the involved random variables are assumed to be independent. Those several functional forms for $K(t)$ and $r(t)$ have been used in the literature [6, 7, 13, 16, 9, 17]. Essentially, they are periodic or steady for the intrinsic growth rate, and periodic or sigmoid for the carrying capacity. Fluctuating and periodic functions are quite common in scientific fields, not only in Ecology which this paper belongs to, but also in Physics [11, 12] or Epidemiology [8, 1], for instance.

We point out that the uniform distribution corresponds to the maximum entropy distribution when only prior information about the bounded support is known, while the exponential distribution is the maximum entropy distribution for a positive random quantity with known mean value [10, 18]. In modeling, the support and the mean value of an input random parameter may be inferred from its physical interpretation, experimental measurements or curve fittings.

To illustrate our results, in Figure 1 we present approximations of $f_N(q;t)$ given by (2.5) for $(q;t) \in [0,8] \times [0,6]$. They were computed by using the crude Monte Carlo (MC) method with 200 000 realizations of the involved random variables to estimate the expectation in (2.6). As the integrand in (2.5), in Cases 3–4, has jump discontinuities in f_{N_0}, the MC method has been utilized in all cases instead of computing the integral via quadrature techniques. The graphical results obtained are easily interpreted physically. In the top two panels for Cases 1–2, the stochastic solution has an oscillating behavior as time passes. Indeed, it is observed that the probability density scrolls in the q domain from left to right and vice versa with time. This is due to the periodic definition of the carrying capacity. In the bottom two panels for Cases 3–4, by contrast, the carrying capacity is increasing towards K_2 with time in a sigmoid form. In Case 3, $r(t)$ is steady and positive, so $N(t)$ increases with time in a sigmoid manner. This is the reason of the density function traveling towards higher q values. In Case 4, $N(t)$ tends to the carrying capacity as time goes on, but oscillations are appreciated, which tend to disappear with time. Those fluctuations are a consequence of the periodicity of $r(t)$.

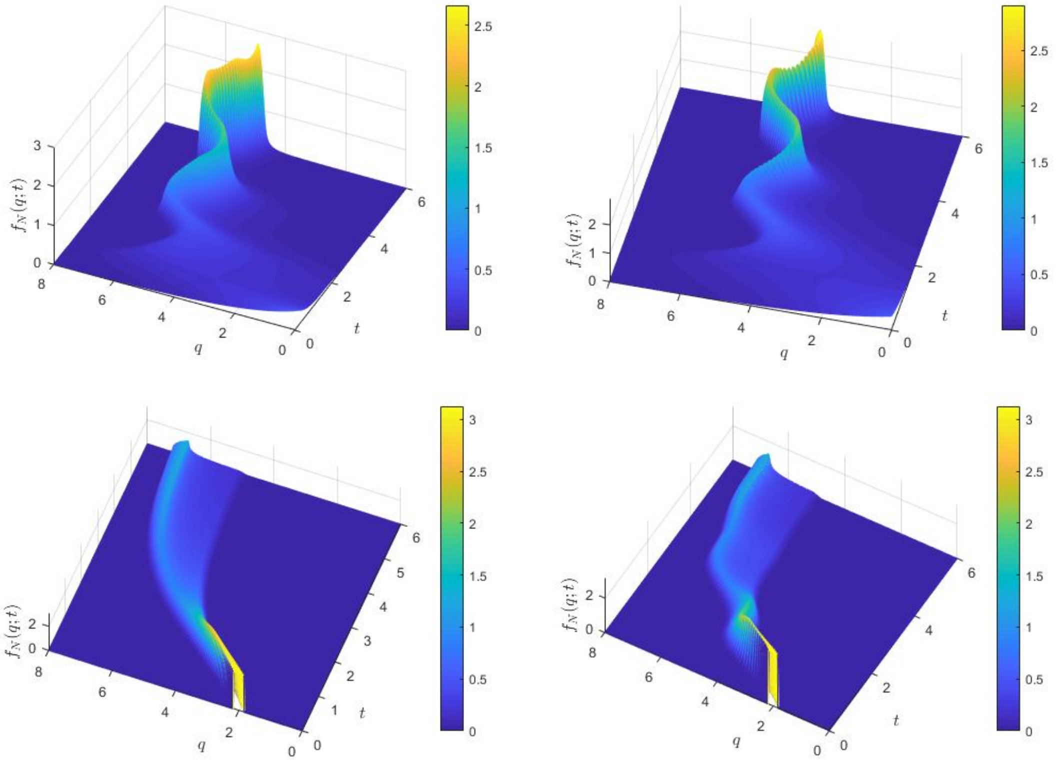

FIGURE 1. Approximations of $f_N(q;t)$, $(q;t) \in [0,8] \times [0,6]$. Cases 1–2 (top left–right). Cases 3–4 (bottom left–right).

We also compare the densities $f_N(q;t)$ for several values of t with those ones obtained by employing a kernel density estimation method (with normal kernel and Silverman's selection of the bandwidth), a non-parametric way to estimate the pdf of a random variable, and the classical Runge–Kutta scheme with 200 000 realizations of the involved random variables. It is observed full agreement, see Figure 2.

Figure 3 illustrates $f_N(q;6)$, for Cases 3–4, calculated by using the MC method in (2.6) using 1 500 000 realizations. Notice that, in contrast to kernel density estimation (non-parametric nature), our parametric method is able to capture the density features (in this case non-differentiability points).

To emphasize the relevance of the variability of the parameters, we compare the expectation of $N(t)$, $\mathbb{E}[N(t)]$, with the solution of the simplified version of (1.1) and (1.2), where the random parameters are replaced by their respective means.

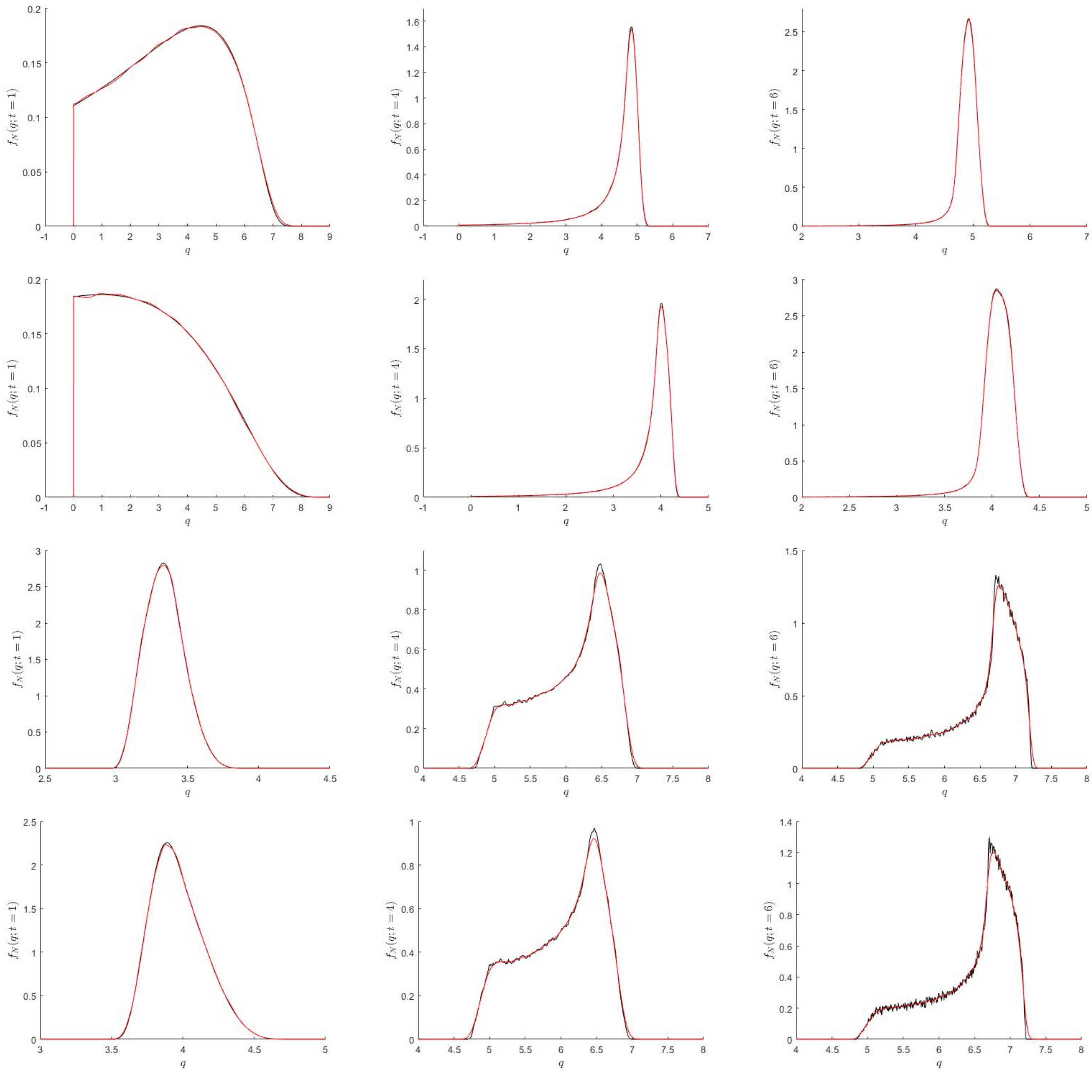

FIGURE 2. Estimations of $f_N(q;t)$, $t = 1, 4, 6$: by computing the expectation in (2.6) using the MC method with 200 000 realizations (black line); by a kernel density estimation method (red line). Case number $i = 1, 2, 3, 4$ in row i.

Figure 4 illustrates the two approaches: the fat line refers to the simplified version of $K(t)$; the red one refers to $\mathbb{E}[K(t)]$ computed using the crude MC method; the dots correspond to the numerical solution of the simplified version of $N(t)$ employing the classical Runge-Kutta scheme; the blue line refers to $\mathbb{E}[N(t)]$ computed using the crude MC method. In the top panels, the carrying capacity is periodic, and $N(t)$ tends to move within its range oscillating. This interesting behavior coincides with the deterministic situation, where convergence towards a periodic canonical solution holds [7]. In the bottom panels, by contrast, the carrying capacity increases with time in a sigmoid fashion. This is also the case for $N(t)$ in Case 3, because $r(t)$ is time-independent. In Case 4, instead, $N(t)$ presents small oscillations, which die out

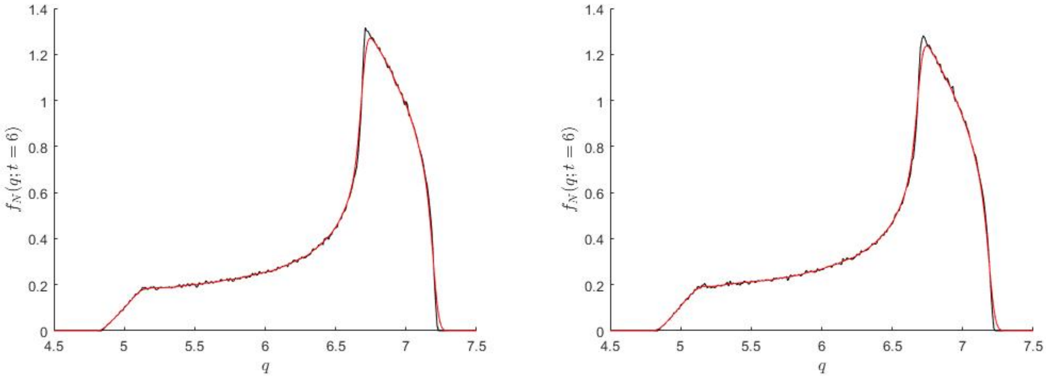

FIGURE 3. Illustration of $f_N(q;6)$: by computing the expectation in (2.6) using the MC method with 1 500 000 realizations (black line); by a kernel density estimation method (red line). Case 3 (left) and Case 4 (right).

in the end, due to the periodicity of $r(t)$. Observe that $\mathbb{E}[N(t)]$ and the simplified $N(t)$ differ, with $\mathbb{E}[N(t)]$ being smaller for all t.

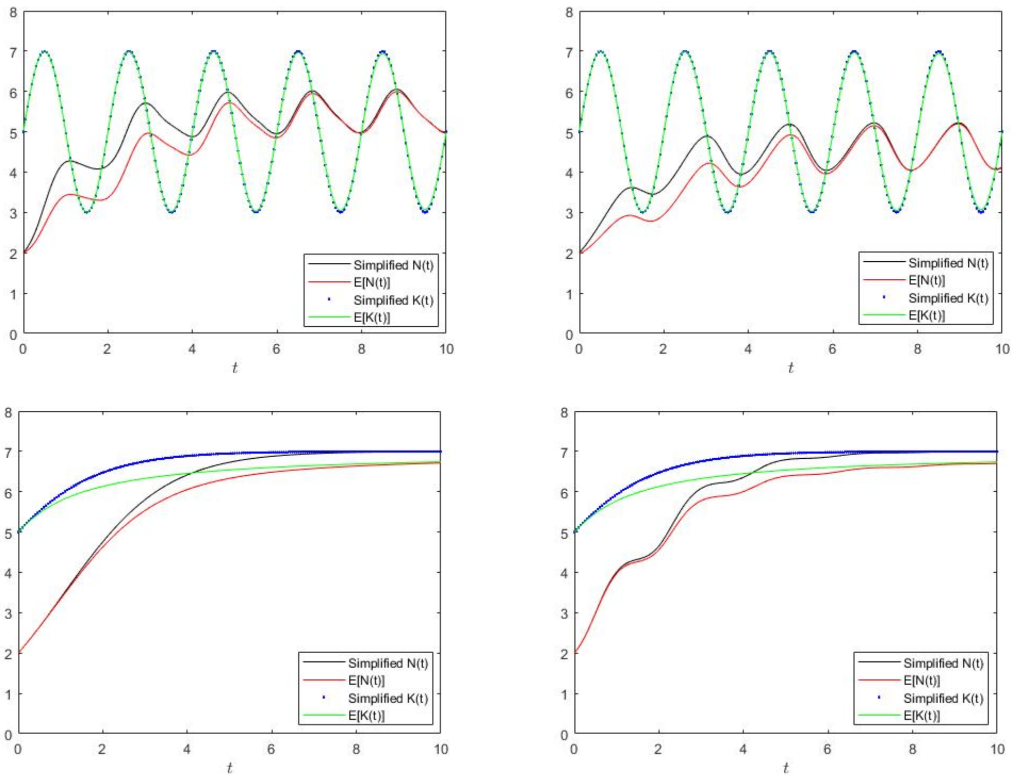

FIGURE 4. Illustration of $\mathbb{E}[N(t)]$ and the simplified version of $N(t)$, $t \in [0, 10]$. Cases 1–2 (top left–right). Cases 3–4 (bottom left–right).

4. Discussion

In this paper we have extended, to the random setting, the non-autonomous logistic model whose intrinsic growth rate and environmental carrying capacity are time-dependent. Time dependency is necessary to better describe changes in the environment. Randomness is needed to account for errors in recorded data and in model assumptions. Solving a random differential equation problem means to understand the statistical content of the solution. This can be done by computing its probability density function. The random variable transformation technique is an exact method to derive the probability density function, by employing the explicit input-output relation of the system and the change of variables formula for integration. Although the exact expression for the density function may involve multidimensional integration, independence of the input random parameters allows for writing it as a simple expectation. Such expectation is parametrically estimated by crude Monte Carlo simulation, therefore kernel density estimation, which is non-parametric, gets improved. These aspects have been illustrated numerically, by considering several functional forms of the intrinsic growth rate and the carrying capacity that have been used in the scientific literature. Essentially, those functional forms are periodic or steady for the intrinsic growth rate, and periodic or sigmoid for the carrying capacity.

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Conflict of Interest Statement

The authors declare that there is no conflict of interests regarding the publication of this article.

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