Kumar, A.; Gupta, DK.; Martínez Molada, E.; Hueso, JL. (2021). Convergence and dynamics of improved Chebyshev-Secant-type methods for non differentiable operators. Numerical Algorithms. 86(3):1051-1070. https://doi.org/10.1007/s11075-020-00922-9

Noname manuscript No.
(will be inserted by the editor)

Convergence and dynamics of improved Chebyshev-Secant-type methods for non differentiable operators

Abhimanyu Kumar · D.K. Gupta · Eulalia Martínez · José L. Hueso

Abstract In this paper, the convergence and dynamics of improved Chebyshev-Secant-type iterative methods are studied for solving nonlinear equations in Banach space settings. their semilocal convergence is established using recurrence relations under weaker continuity conditions on first order divided differences. Convergence theorems are established for the existence-uniqueness of the solutions. Next, center-Lipschitz condition is defined on the first order divided differences and its influence on the domain of starting iterates is compared with those corresponding to the domain of Lipschitz conditions. Several numerical examples including Automotive Steering problems and nonlinear mixed Hammerstein type integral equations are analyzed and the output results are compared with those obtained by some of similar existing iterative methods. It is found that improved results are obtained for all the numerical examples. Further, the dynamical analysis of the iterative method is carried out. It confirms that the proposed iterative method has better stability properties than its competitors.

keywords Nonlinear equations; Divided differences; Semilocal convergence; Domain of parameters; Dynamical analysis

Mathematical Subject Classification 2010: 47H99, 65H10, 49M15.

1 Introduction

Consider approximating a locally unique solution ρ^* of

$$\mathrm{F}(x) = 0, \tag{1.1}$$

where F is a continuous nonlinear operator defined on a non-empty open convex subset D of a Banach space X with values in another Banach space Y. This is one of the most important problems in applied mathematics and engineering. The second order Fréchet derivative free family

This research was partially supported by Ministerio de Economía y Competitividad under grant PGC2018-095896-B-C22.

Abhimanyu Kumar,
Department of Mathematics, Lalit Narayan Mithila University-846004,India, sukhjitmath@gmail.com

D. K. Gupta
Department of Mathematics, Indian Institute of Technology Kharagpur - 721302, India, dkg@maths.iitkgp.ernet.in

Eulalia Martínez , José L. Hueso
Instituto Universitario de Matemática Multidisciplinar, Universitat Politècnica de València, Spain, eumarti@mat.upv.es, jlhueso@mat.upv.es

of Chebyshev-type methods described in [1] is given for $k \geq 0$ by

$$
\begin{aligned}
y_k &= x_k - \mathrm{F}'(x_k)^{-1}\mathrm{F}(x_k),\\
z_k &= x_k + \alpha(y_k - x_k),\\
x_{k+1} &= x_k - \frac{1}{\alpha^2}\mathrm{F}'(x_k)^{-1}((\alpha^2+\alpha-1)\mathrm{F}(x_k) + \mathrm{F}(z_k)),
\end{aligned} \tag{1.2}
$$

where $x_0 \in \mathrm{D}$ is the starting iterate, the parameter $\alpha \in \mathbb{R} - \{0\}$ and $\mathrm{F}'(x_k)^{-1} \in \mathrm{L}(\mathrm{Y}, \mathrm{X})$, where $\mathrm{L}(\mathrm{Y}, \mathrm{X})$ denotes the set of bounded linear operators from Y to X. These methods can not be applied for problems involving non-differentiable operators.

Replacing the first derivative $\mathrm{F}'(x_k)$ of (1.2) by $[x_{k-1}, x_k; \mathrm{F}]$, we get

$$
\begin{aligned}
y_k &= x_k - [x_{k-1}, x_k; \mathrm{F}]^{-1}\mathrm{F}(x_k),\\
z_k &= x_k + \alpha(y_k - x_k),\\
x_{k+1} &= x_k - [x_{k-1}, x_k; \mathrm{F}]^{-1}(\beta\mathrm{F}(x_k) + \gamma\mathrm{F}(z_k)),
\end{aligned} \tag{1.3}
$$

where $x_{-1},\ x_0 \in \mathrm{D}$ are two starting iterates and $[x, y; \mathrm{F}] \in \mathrm{L}(\mathrm{X}, \mathrm{Y})$ satisfies $[x, y; \mathrm{F}](x - y) = \mathrm{F}(x) - \mathrm{F}(y)$ for $x, y \in \mathrm{D}$ and $x \neq y$, for $x = y$, $[x, y; \mathrm{F}] = \mathrm{F}'(x)$. Here, $\alpha,\ \beta$ and γ are nonnegative real parameters carefully chosen so that the sequence $\{x_k\}$ converges to ρ^*. This family of iterative methods used for the solution of (1.1) is known as the Chebyshev-Secant-type methods (CSTM). For $\alpha = 0,\ \ \beta = \gamma = 1/2$ and $y_k = x_{k+1}$, it becomes Secant method. Another quadratically convergent one point iterative method for $\alpha = \beta = \gamma = 1$ is described in [2]. The local and semilocal convergence of (1.3) is established in [3] and [4]. Another variant of (1.3) is given for $k \geq 0$ by

$$
\begin{aligned}
y_k &= x_k - [2x_k - x_{k-1}, x_k; \mathrm{F}]^{-1}\mathrm{F}(x_k),\\
z_k &= x_k + \alpha(y_k - x_k),\\
x_{k+1} &= x_k - [2x_k - x_{k-1}, x_k; \mathrm{F}]^{-1}(\beta\mathrm{F}(x_k) + \gamma\mathrm{F}(z_k)),
\end{aligned} \tag{1.4}
$$

where $x_{-1}, x_0 \in \mathrm{D}$ are two starting iterates and $\alpha,\ \beta$ and γ are nonnegative real parameters. It can be known as the Chebyshev-Kurchatov-type method (CKTM). The semilocal convergence of (1.4) is discussed in [5]. The improved Chebyshev-Secant-type method (ICSTM) proposed by us is given for $k \geq 0$ by

$$
\begin{aligned}
x_{k+1} &= x_k - \mathrm{B}_k^{-1}\mathrm{F}(x_k), \quad \mathrm{B}_k = [x_k, y_k; \mathrm{F}],\\
z_k &= x_k + \alpha(x_{k+1} - x_k),\\
y_{k+1} &= x_k - \mathrm{B}_k^{-1}(\beta\mathrm{F}(x_k) + \gamma\mathrm{F}(z_k)),
\end{aligned} \tag{1.5}
$$

where $x_0, y_0 \in \mathrm{D}$ are two starting iterates and $\alpha,\ \beta$ and γ are nonnegative real parameters. Considering $\alpha = \beta = \gamma = 1$ we obtain the double step Secant method [6], and [7], with order of convergence $1 + \sqrt{2}$. It can be easily seen that the number of functions evaluations and the corresponding divided differences used in CSTM and ICSTM are the same. The importance of the ICSTM lies in the fact that for $\alpha = \beta = \gamma = 1$, its convergence order is $1 + \sqrt{2}$, while the convergence order of the CSTM is 2.

Remark 11 *The convergence order of CKTM in [8] given as $\frac{1+\sqrt{17}}{2}$ is close to the convergence order of ICSTM for $\alpha = \beta = \gamma = 1$. However, it requires very specific starting iterates.*

The following example demonstrates that ICSTM performs better than CKTM and CSTM.

Example 11 *Consider the nonlinear non-differentiable system*

$$u^{3/2} - v - \frac{3}{4} + \frac{1}{9}|u-1| = 0,$$
$$v^{3/2} + \frac{2}{9}u - \frac{3}{8} + \frac{1}{9}|v| = 0$$

for $x = (u, v) \in \mathbb{R}^2$. The unique solution of this system is given by $\rho^* = (1, 0.25)$. Starting with $(x_0, y_0) = (x_{-1}, x_0) = ((2, 2), (1, 0))$, the Table 1 compares the error approximations $\|x_{k+1} - x_k\|_\infty$ of ICSTM with $\|y_k - x_k\|_\infty$ of CSTM and CKTM for $\alpha = \beta = 1/2,\ \gamma = 1$, where tolerance $\|\mathrm{F}(x_k)\|_\infty$ is less than 10^{-15}.

Table 1: Comparison of errors

k	ICSTM	CSTM	CKTM
0	1.83512	1.83512	1.81228
1	7.73059×10^{-02}	0.49249	0.50231
2	7.82988×10^{-03}	6.31812×10^{-02}	6.92736×10^{-02}
3	1.60152×10^{-05}	5.09870×10^{-03}	2.13478×10^{-03}
4	2.20401×10^{-10}	1.23059×10^{-04}	1.08211×10^{-04}
5		4.53173×10^{-06}	2.78909×10^{-06}
6		4.96587×10^{-11}	1.29562×10^{-07}
7		3.60821×10^{-13}	6.01912×10^{-09}
8		2.26963×10^{-14}	3.16729×10^{-10}
9			1.14641×10^{-11}

We run these algorithms with Matlab 2018 in a PC with windows 10 and processor `intel(R) Core(TM)` $i7-4790$ `CPU 3.60GHz`, the average CPU time in seconds taken by ICSTM, CSTM and CKTM are 0.482991, 0.991029 and 1.423315, respectively, which shows the competitiveness of the ICSTM algorithm.

The rest of the paper is structured as follows. In Section (2), the improved Chebyshev-Secant-type method (ICSTM) and it's semilocal convergence analysis is given. In Section (3), the domain of parameters for the guaranteed convergence of ICSTM is established. In section (4), the dynamic behaviour of the methods is analyzed. In Section (5), some numerical examples are given to show the efficiency of ICSTM and justify the theoretical results obtained in this study. Finally, conclusions are given in Section 6.

2 Semilocal convergence of ICSTM

In this section, the semilocal convergence of ICSTM for solving (1.1) is established. Let $\mathcal{B}(x, r)$ and $\overline{\mathcal{B}}(x, r)$ denote open and closed balls with center at x and radius r, respectively. For suitably chosen initial approximations x_0 and y_0, we define a class $\mathrm{S}(\Theta, \delta, \eta, \sigma)$, where $\Theta > 0,\ \delta > 0,\ \eta > 0$ are some positive real numbers and σ is to be defined. The triplet $(\mathrm{F}, x_0, y_0) \in \mathrm{S}(\Theta, \delta, \eta, \sigma)$ if

[C$_1$] $\|x_0 - y_0\| \le \Theta$ for $x_0, y_0 \in \mathrm{D}$.
[C$_2$] $\mathrm{B}_0^{-1} \in \mathrm{L}(\mathrm{Y}, \mathrm{X})$ such that $\|\mathrm{B}_0^{-1}\| \le \delta$.
[C$_3$] $\|\mathrm{B}_0^{-1}\mathrm{F}(x_0)\| \le \eta$.
[C$_4$] $\|([x, y; \mathrm{F}] - [u, v; \mathrm{F}])\| \le \sigma(\|x - u\|, \|y - v\|)$, where $\sigma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and non decreasing function in its both arguments for $x, y, u, v \in \mathrm{D}$.
[C$_5$] $(1 - \beta) = (1 - \alpha)\gamma$ and $\alpha \in (0, 1]$.

[C$_6$] The equation

$$(1 - g(t))t - \eta = 0$$

where, $g(t) = \dfrac{M}{1 - \delta\sigma(t, t + \Theta)}$,
and $M = \max(\alpha\gamma\delta\sigma(\eta, \Theta), \alpha\delta\sigma(\eta, \Theta), \delta\sigma(\eta, \Theta), \alpha\gamma\delta\sigma(\eta, (1 + p)\eta))$, where $p = \alpha\gamma\delta\sigma(\eta, \Theta)$, has at least one positive root. The smallest positive root is denoted by R and verifies:

[C$_7$] $\mathcal{B}(x_0, R) \subseteq \mathrm{D}$.

[C$_8$] $\delta\sigma(R, R + \Theta) < 1$.

Lemma 21 *For the improved Chebyshev-Secant-type method (ICSTM) proposed in (1.5) it is verified for all $k \geq 0$:*

(i) $\mathrm{F}(z_k) = \alpha([z_k, x_k; \mathrm{F}] - \mathrm{B}_k)(x_{k+1} - x_k) + (1 - \alpha)\mathrm{F}(x_k)$.

(ii) $\mathrm{F}(x_{k+1}) = ([x_{k+1}, x_k; \mathrm{F}] - [x_k, y_k; \mathrm{F}])\,(x_{k+1} - x_k)$.

Proof The proof follows obviously by (1.5) and the application of the usual property of the divided difference operator, $[x, y, \mathrm{F}](x - y) = \mathrm{F}(x) - \mathrm{F}(y)$, hence omitted here.

Lemma 22

For method ICSTM proposed in (1.5) under conditions $[\mathrm{C}_1] - [\mathrm{C}_8]$ *and for* $(\mathrm{F}, x_0, y_0) \in \mathrm{S}(\Theta, \delta, \eta, \sigma)$, *we obtain the following bounds for all $k \geq 0$:*

(i) *There exists* B_k^{-1} *satisfying* $\|\mathrm{B}_k^{-1}\| \leq \frac{\delta}{1 - \delta\sigma(R, R+\Theta)}$,

(ii) $\|x_{k+1} - x_k\| \leq g(R)\|x_k - x_{k-1}\|$,

(iii) $\|y_{k+1} - x_{k+1}\| \leq g(R)\|x_{k+1} - x_k\|$.

Proof The above inequalities can be proved by using mathematical induction. Using Lemma 21 and the definition of class $\mathrm{S}(\Theta, \delta, \eta, \sigma)$, we get $\|x_1 - x_0\| \leq \eta < R$, $\|z_0 - x_0\| \leq \eta < R$, for being R a root of (2.1). Now, by definition of the method (1.5) it holds:

$$\|y_1 - x_1\| = \|y_1 - x_0 + x_0 - x_1\| = \| - \alpha\gamma \mathrm{B}_0^{-1}([x_0, z_0, F] - \mathrm{B}_0)(x_1 - x_0)\| \leq \alpha\gamma\delta\sigma(\eta, \Theta))\|x_1 - x_0\| < g(R)\eta < R,$$

where in the last inequality we have used that $g(R) < 1$ what follows from definition of R and from the fact of $\eta > 0$. Moreover,

$$\|y_1 - x_0\| \leq \|y_1 - x_1\| + \|x_1 - x_0\| = (1 + g(R))\eta < \frac{\eta}{1 - g(R)} = R,$$

where we have bounded two adds of a geometrical progression with ratio less than one. Thus, lemma holds for $n = 0$. Suppose that it holds for every $n \leq k - 1$. Notice that by using the induction hypothesis it can be obviously deduced that:

$$\|x_k - x_{k-1}\| \leq g(R)\|x_{k-1} - x_{k-2}\|,$$
$$\|y_k - x_k\| \leq g(R)\|x_k - x_{k-1}\|,$$
$$\|y_k - x_{k-1}\| \leq (1 + g(R))\|x_k - x_{k-1}\|,$$

and so it follows

$$\|y_k - x_0\| \leq \sum_{j=0}^{k} g(R)^j \eta < R.$$

Now, in order to obtain the assertion for $n = k$, we use previous bounds and $[\mathrm{C}_8]$, so it holds:

$$\|I - \mathrm{B}_0^{-1}\mathrm{B}_k\| \leq \delta\sigma(\|x_k - x_0\|, \|y_k - x_0\| + \|y_0 - x_0\|) \leq \delta\sigma(R, R+\Theta) < 1.$$

So, by Banach's lemma on invertible operators [9] it is verified

$$\|\mathrm{B}_k^{-1}\| \leq \frac{\delta}{1 - \delta\sigma(R, R+\Theta)}.$$

Using these staments and Lemma 21 once more, we get

$$\begin{aligned}\|x_{k+1} - x_k\| &\leq \|\mathrm{B}_k^{-1}\|\|\mathrm{F}(x_k)\|\\ &\leq \frac{\delta\sigma(\|x_k - x_{k-1}\|, \|x_{k-1} - y_{k-1}\|)}{1 - \delta\sigma(R, R+\Theta)}\|x_k - x_{k-1}\|\\ &\leq g(R)\|x_k - x_{k-1}\|.\end{aligned}$$

In consequence we have:

$$\|x_{k+1} - x_0\| \leq \sum_{j=0}^{k} g(R)^j \eta < R,$$

$$\|z_k - x_0\| \leq \sum_{j=0}^{k} g(R)^j \eta < R.$$

Now,

$$\|y_{k+1} - x_{k+1}\| \leq \|\alpha\gamma \mathrm{B}_k^{-1}\sigma(\|z_k - x_k\|, \|x_k - y_k\|)(x_{k+1} - x_k)\| \leq g(R)\|x_{k+1} - x_k\|,$$

which proves the lemma.

Theorem 21 *Let $F : D \subseteq X \to Y$ be a continuous nonlinear operator, and consider the triplet $(\mathrm{F}, x_0, y_0) \in \mathrm{S}(\Theta, \delta, \eta, \sigma)$ defined in section 2, with $x_0, y_0 \in \mathrm{D}$ verifiying conditions $[\mathrm{C}_1] - [\mathrm{C}_8]$. Then, by taking x_0, y_0 as starting points, the sequences $\{x_k\}$, $\{y_k\}$ and $\{z_k\}$ generated by (1.5) for $k \geq 0$ are well defined and belong to $\mathcal{B}(x_0, R) \subseteq D$. Also, the iterate x_k, y_k and z_k converge to $\rho^* \in \overline{\mathcal{B}}(x_0, R)$, where ρ^* is the unique solution of (1.1) in $\overline{\mathcal{B}}(x_0, R) \cap \mathrm{D}$.*

Proof Using Lemma 21 and Lemma 22, we see that the iterates x_k and y_k are well defined and belong to $\mathcal{B}(x_0, R) \subseteq D$. It is sufficient to show that $\{x_k\}$ is a Cauchy sequence. For fixed k and $m \geq 1$, we get

$$\begin{aligned}\|x_{k+m} - x_k\| &\leq \|x_{k+m} - x_{k+m-1}\| + \ldots + \|x_{k+1} - x_k\|\\ &\leq \left(g(R)^{m-1} + g(R)^{m-2} + \ldots + g(R) + 1\right)\|x_{k+1} - x_k\|\\ &\leq \left(\frac{1 - g(R)^m}{1 - g(R)}\right) g(R)^k \|x_1 - x_0\|.\end{aligned}$$

Therefore the sequence $\{x_k\}$ is a Cauchy sequence in a Banach space and so it is convergent, Let ρ^* the limit. Now, we show that ρ^* is a solution of (1.1). From Lemma 21, we get

$$\|\mathrm{F}(x_{k+1})\| \leq \|[x_{k+1}, x_k; \mathrm{F}] - [x_k, y_k; \mathrm{F}]\|\|x_{k+1} - x_k\| \to 0 \text{ as } k \to \infty.$$

From the continuity of F, it is assured that $\mathrm{F}(\rho^*) = 0$. Analogously we can obtain the thesis for sequences $\{y_k\}$ and $\{z_k\}$. To show the uniqueness of ρ^*, let $\widehat{\rho}$ be another solution of (1.1) in $\overline{\mathcal{B}}(x_0, R)$ such that $\mathrm{F}(\widehat{\rho}) = 0$. For $\mathrm{B}^* = [\rho^*, \widehat{\rho}; \mathrm{F}]$, we get

$$\|I - \mathrm{B}_0^{-1}\mathrm{B}^*\| \leq \delta\sigma(\|\rho^* - x_0\|, \|\widehat{\rho} - x_0\| + \|y_0 - x_0\|) \leq \delta\sigma(R, R+\Theta) < 1.$$

This shows that B^* is invertible and from the identity $[\rho^*, \widehat{\rho}; \mathrm{F}](\rho^* - \widehat{\rho}) = \mathrm{F}(\rho^*) - \mathrm{F}(\widehat{\rho})$, taking norms on both sides, we get $\rho^* = \widehat{\rho}$. This implies the uniqueness of ρ^*.

3 Domain of parameters of ICSTM

In this section, after the semilocal convergence analysis has been established, the domain of parameters associated to Theorem 21 for ICSTM is discussed. It is defined as the region of a plane whose points allow us to guarantee the convergence of ICSTM from the conditions imposed on Theorem 21. For this purpose, we will use a special case of $[\mathrm{C}_4]$ for non differentiable operators. We define F between Banach spaces $\mathrm{X} = \mathrm{Y} = \mathbb{R}^n$ by

$$\mathrm{F}(t) = t - g - W(\xi_1 v_t + \xi_2 w_t) = 0$$

where g is a nonlinear vector function of size $n \times 1$, W is a matrix of size $n \times n$, $t = (t_1, t_2, \ldots, t_n)^T$, $v_t = (t_1^2, t_2^2, \ldots, t_n^2)^T$, $w_t = (|t_1|, |t_2|, \ldots, |t_n|)^T$ and $\xi_1, \xi_2 \in \mathbb{R} - \{0\}$. We have used the following definition of first order divided difference.
For $x, y \in \mathbb{R}^n$ and $\mathrm{F} \in \mathrm{L}(\mathbb{R}^n, \mathbb{R}^n)$, we define

$$[x, y, \mathrm{F}]_{j,k} = \frac{\mathrm{F}_j(x_1, \ldots, x_k, y_{k+1}, \ldots, y_n) - \mathrm{F}_j(x_1, \ldots, x_{k-1}, y_k, \ldots, y_n)}{x_k - y_k}, \quad j, k = 1, \ldots, n.$$

If $x_k = y_k$ then $[x, y, \mathrm{F}]_{jk} = 0$. Then, for the nonlinear operator (3.1) we have $[x, y; F] = I - (\xi_1 w_{ij}(x_j + y_j) + \xi_2 w_{ij} \frac{|x_j| - |y_j|}{xj - y_j})$, as consequence

$$\|[x, y; \mathrm{F}] - [u, v; \mathrm{F}]\| \leq K_1 + K_2(\|x - u\| + \|y - v\|), \ x, y, u, v \in \mathbb{R}^n \tag{3.1}$$

where, $K_1 = 2|\xi_2|\|W\|$ and $K_2 = |\xi_1|\|W\|$. Notice that if $K_1 = 0$, we obtain the condition for differential operators. Since (1.3) involves the parameters α, β and γ, we must express it in terms of one of them, α. For this, we take $\sigma(\|x - u\|, \|y - v\|) = K_1 + K_2(\|x - u\| + \|y - v\|)$ as a special case of $[\mathrm{C}_4]$ of class $\mathrm{S}(\Theta, \delta, \eta, \sigma)$. Taking $\alpha\gamma = 1$ and using (1.5), class $\mathrm{S}(\Theta, \delta, \eta, \sigma)$ and $[\mathrm{C}_5]$, ICSTM becomes

$$\begin{aligned} x_{k+1} &= x_k - \mathrm{B}_k^{-1}\mathrm{F}(x_k) \\ z_k &= x_k + \alpha(x_{k+1} - x_k) \\ y_{k+1} &= x_k - \mathrm{B}_k^{-1}\left((2 - \frac{1}{\alpha})\mathrm{F}(x_k) + \frac{1}{\alpha}\mathrm{F}(z_k)\right). \end{aligned} \tag{3.2}$$

where, $\alpha \in (0, 1]$. For each value of α, we get different methods. Now, we discuss the semilocal convergence of (3.2) satisfying (3.1).

Theorem 31 *Let $\mathrm{F} : \mathrm{D} \subseteq \mathrm{X} \to \mathrm{Y}$ and R be the smallest positive real number satisfying*

$$(1 + \mathrm{L}_2 - \mathrm{L}_3 - \mathrm{L}_4(R))\eta - R(1 - \mathrm{L}_1 - \mathrm{L}_4(R)) = 0, \tag{3.3}$$

where $\mathrm{L}_1 = \delta(K_1 + K_2(\alpha+1)\eta)$, $\mathrm{L}_2 = \delta(K_1 + K_2(\eta + \Theta))$, $\mathrm{L}_3 = \delta(K_1 + 2K_2\eta)$ and $\mathrm{L}_4(R) = \delta(K_1 + K_2(2R + \Theta))$. If $\mathrm{L}_1 + \mathrm{L}_4(R) < 1$, $\mathrm{L}_2 + \mathrm{L}_4(R) < 1$ and $\mathrm{L}_3 + \mathrm{L}_4(R) < 1$. Then starting with $x_0, y_0 \in \mathrm{D}$, the sequences $\{x_k\}$, $\{y_k\}$ and $\{z_k\}$ given in (3.2) are well defined, remain in $\mathcal{B}(x_0, R)$ and converge to the unique solution $\rho^* \in \overline{\mathcal{B}}(x_0, R)$ of (1.1).

Proof The proof is similar to Theorem 21 and hence omitted here.

The condition (3.1) can further be weakened by introducing the center-Lipschitz condition

$$\|[x, y; \mathrm{F}] - [x_0, y_0; \mathrm{F}]\| \leq K_1 + K_0(\|x - x_0\| + \|y - y_0\|), \ x, y \in \mathrm{X}. \tag{3.4}$$

If we take $K_1 = 0$, then it holds for the differentiable operators and many researchers have established the semilocal and local convergence analysis for different methods using this condition.

Theorem 32 *Let* $\mathrm{F}:\mathrm{D}\subseteq\mathrm{X}\to\mathrm{Y}$ *and* R *be the smallest positive real number satisfying*

$$(1+\mathrm{T}_1-\mathrm{T}_2-\mathrm{T}_3(R))\eta-R(1-\mathrm{T}_2-\mathrm{T}_3(R))=0, \tag{3.5}$$

where $\mathrm{T}_1=\delta(K_1+K_0(\eta+\Theta))$, $\mathrm{T}_2=\delta(K_1+2K_2\eta)$ *and* $\mathrm{T}_3(R)=\delta(K_1+K_0(2R+\Theta))$. *If* $\mathrm{T}_1+\mathrm{T}_3(R)<1$, $\mathrm{T}_2+\mathrm{T}_3(R)<1$, *then, sequences* $\{x_k\}$, $\{y_k\}$ *and* $\{z_k\}$ *given by (3.2) are well defined, remain in* $\mathcal{B}(x_0,R)$ *and converge to the unique solution* $\rho^*\in\overline{\mathcal{B}}(x_0,R)\subseteq D$ *of (1.1).*

Proof The proof is similar to Theorem 21 and hence omitted here.

Now, we study the domain of parameters corresponding to Theorem 31 and Theorem 32. This establishes a relationship among the different constants involved in class $\mathrm{S}(\Theta,\delta,\eta,\sigma)$ that ensure the guaranteed convergence of (3.2). Using Theorem 32, (3.5) can be transformed to a quadratic equation

$$\begin{aligned} b_0t^2+b_1t+b_2&=0, \\ b_0&=2K_0\delta, \\ b_1&=\delta(2K_1+2K_2\eta+K_0\Theta-2K_0\eta)-1 \quad \text{and} \\ b_2&=\eta(\delta(K_0\eta-K_1-2K_2\eta)+1) \end{aligned} \tag{3.6}$$

Now, we study when (3.6) has two positive real roots, in the following lemma.

Lemma 31 *If* $b_2>0$ *and* $b_1+\sqrt{4b_0b_2}<0$ *then (3.6) are transformed to the conditions*

$$\eta(\delta(K_0\eta-K_1-2K_2\eta)+1)>0 \tag{3.7}$$

and

$$\delta(2K_1+2K_2\eta+K_0\Theta-2K_0\eta)+\sqrt{8K_0\eta\delta(\delta(K_0\eta-K_1-2K_2\eta)+1)}<1. \tag{3.8}$$

Also, the smallest root denoted by R_1 *is given by*

$$R_1=\frac{1}{4K_0\delta}\left(1-\delta(2K_1+2K_2\eta+K_0\Theta-2K_0\eta)-\sqrt{\Delta}\right),$$

where $\Delta=\sqrt{(\delta(2K_1+2K_2\eta+K_0\Theta-2K_0\eta))^2-8K_0\eta\delta(1+\delta(K_0\eta-K_1-2K_2\eta))}$.

Proof The proof is simple and hence omitted here.

Now, we compare Theorem 31 and Theorem 32 and observe the enlargement of the domain of the parameters. Since, $K_0\le K_2$, we take $\lambda=\frac{K_0}{K_2}\in(0,1]$. For different values of λ, the domain of the parameters vary. If $\lambda=1$, the domain of the parameters match with the domain obtained with Theorem 31. Otherwise, it gives these corresponding to the Theorem 32. Now, we take $x=K_2\delta$, $y=\eta$ and $N_1=\frac{K_1}{K_2}$. Using conditions imposed in Theorem 31, Theorem 32 and Lemma 31, we define the domain of parameters for (3.2) as the region of x-y plane whose points satisfy $b_2>0$, $b_1+\sqrt{4b_0b_2}<0$, $T_1+T_3(R_1)<1$ and $T_2+T_3(R_1)<1$. It is to be noted that we can choose different combinations of $K_2\delta$ and η for a given value of Θ.The number of suitable starting iterates increases for a larger size of the domain of parameters. We observe that the domain of parameters of (3.2) depends on the following three cases.

1 $K_1=0$ implying differentiability of F,
2 $K_1\neq0$ implying non-differentiability of F.
3 Θ representing the distance between x_0 and y_0.

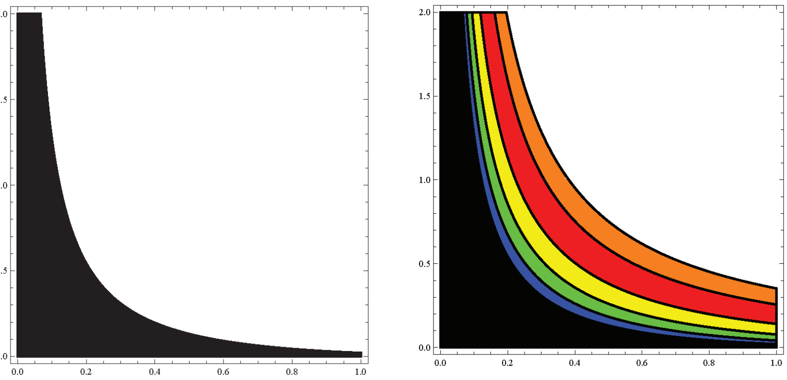


Fig. 1: Domain of parameters corresponding to Theorem 31 (Left), a comparison between Theorem 31 and Theorem 32 for domain of parameters represented by orange, red, yellow, green, blue and black colour, respectively for $\Theta = 0.6$ and $\lambda = 0.1, 0.2, 0.4, 0.6, 0.8, 1$ (Right).

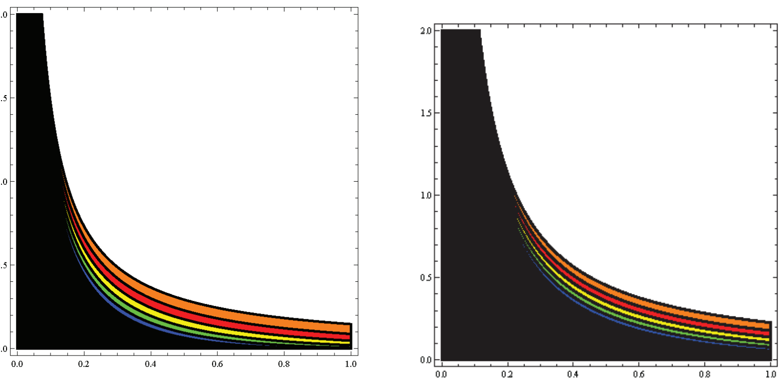


Fig. 2: The domain of parameters for fixed $\lambda = 1$ and $\Theta = 0, 0.2, 0.4, 0.6, 0.8, 1$ represented by orange, red, yellow, green, blue and black colour, respectively (Left), The domain of parameters for fixed $\lambda = 1/2$ and $\Theta = 0, 0.2, 0.4, 0.6, 0.8, 1$ represented by orange, red, yellow, green, blue and black colour, respectively (Right).

The domain of parameters associated to Theorem 31 for case 1 for fixed $\Theta = 0.6$ and $N_1 = 0$ is given in Figure 1(Left). Now, we use Theorem 32 and see the difference in the domain of parameters for the different values of λ in Figure 1 (Right). We take a fixed value of Θ and denote the domain of starting points for a particular value of λ by D_λ. In particular, the following relation can be established between λ and D_λ,

$$\mathrm{D}_{\lambda_0} \subset \mathrm{D}_{\lambda_1} \subset \ldots \subset \mathrm{D}_{\lambda_k} \quad \text{for } \lambda_k \geq \lambda_{k-1} \geq \ldots \geq \lambda_0 = 1. \tag{3.9}$$

It can be seen that the domain of starting points decreases by increasing the value of λ. It remains to see the effect of the distance between the starting points (Θ) on the domain of parameters. For this, we fix the value of $\lambda = 1$ and then observe the variation in the domain of parameters for different values of Θ in Figure 2 (Left). It is found that the domain of parameter decreases with the enlargement of Θ. However, for a smaller value of λ, the domain for starting points increases, as shown in Figure 2 (Right). We must note that the influence on the domain of the parameters of the values of Θ is less remarkable compared with the solutions obtained with the different values of λ. In particular, the following relation can be established between Θ and D_Θ for a fixed value

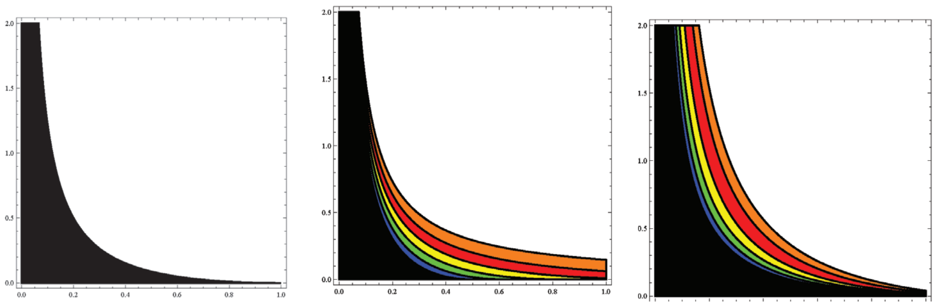


Fig. 3: The domain of parameters from Theorem 31 for $\lambda = 1$ and $N_1 = 0.5$(Left), The domain of parameters from Theorem 31 for $\lambda = 1$ $\Theta = 0$ and $N_1 = 0, 0.2, 0.4, 0.6, 0.8, 1$ represented by orange, red, yellow, green,blue and black colour, respectively (Middle), The domain of parameters for non differentiable case, $N_1 = 0.4$, $\Theta = 0$ and $\lambda = 0.1, 0.2, 0.4, 0.6, 0.8, 1$ represented by orange, red, yellow, green, blue and black colour, respectively (Right).

of λ, where D_Θ stands for the domain for a particular value of Θ.

$$\mathrm{D}_{\Theta_0} \subset \mathrm{D}_{\Theta_1} \subset \ldots \subset \mathrm{D}_{\Theta_k} \quad \text{for } 0 = \Theta_k \leq \Theta_{k-1} \leq \ldots \leq \Theta_0.$$

Now, we consider $K_1 \neq 0$. In this case, the domain of the parameters depends on these three factors, λ, Θ and K_1. The domain of the parameter associated to Theorem 31 for fixed $\lambda = 1$ and $N_1 = 0.5$ is given in Figure 3(Left). We can observe that the domain of the parameter is enlarged for decreasing values of N_1. When it maximizes, it leads to the differentiable case. If λ and Θ are fixed and D_{N_1} denotes the domain of parameters for some N_1, then the following relation can be established between N_1 and the domain D_{N_1} as illustrated in Figure 3 (Middle).

$$\mathrm{D}_{N_{1_k}} \subset \mathrm{D}_{N_{1_{k-1}}} \subset \ldots \subset \mathrm{D}_{N_{1_0}} \quad \text{for } 0 = N_{1_0} \leq \ldots N_{1_{k-1}} \leq N_{1_k}.$$

The domain of the parameters is quite similar for the different values of Θ taken. However, for $\Theta = 0$, it gives the largest domain of parameters. To see the effect of λ on the most suitable domain of parameters, we fix $\Theta = 0$ and $N_1 = 0.4$ and take different values of λ. The relation (3.9) also hold in this case. This situation can be seen in Figure 3(Right). From the above discussion, we conclude that the domain of the parameter is mainly affected by the value of λ and not much influenced by K_1 and Θ. The most optimal situation arises when all of the factors are taken as small as possible. This is true for both differential and non differentiable cases.

4 Dynamics of a method with memory

The behaviour of iterative methods has been examined from a global point of view by using ideas of dynamical systems. Complex dynamics is the most common tool used for the study of iterative methods without memory, not only because the analytic functions have better properties in the complex domain, but also because they provide good pictorial representations, (see [10], [11], [12], [13]).

Let us recall some basic concepts of discrete dynamics, in order to fix the notation. Consider a function $\mathrm{G} : \mathbb{C} \longrightarrow \mathbb{C}$. The set of successive images of a point p by G: $p, \mathrm{G}(p), \mathrm{G}(\mathrm{G}(p)), \ldots,$ is the orbit of p. A point $q \in \mathbb{C}$ is called a fixed point of G, if $\mathrm{G}(q) = q$. A fixed point q is attracting if the orbits

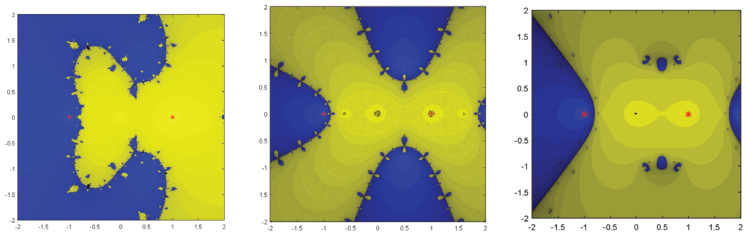


Fig. 4: Dynamics of CSTM(Left), CKTM(Middle) and ICSTM(Right) for $x^2 - 1 = 0$.

of all the points in a neighborhood of q tend to q. The set of all the points whose orbit converges to a fixed point q is called the attraction basin of q.

The attraction basins of the fixed points form the so called Fatou set. Its complement, the Julia set, establishes the borders between the basins of attraction.

The basins of attraction of the different fixed points of G are graphically represented by colouring each basin in a different colour, forming the so called dynamical plane. If G is the iteration function of a numerical method for solving equations, the attraction basins of G give an idea of the behaviour of the method and its sensitivity to the initial guess.

We are going to compare the dynamic properties of methods (1.3), (1.4) and (1.5) by applying them to scalar functions of a complex variable in order to analyze the sensitivity of the iterative method to the initial estimates of the root. The parameters of the methods are set to $\alpha = \beta = \gamma = 1$, which simplifies their application.

For the comparison of the dynamics of these methods, instead of taking two initial estimates, x_{-1} and x_0 or x_0 and y_0, it is more useful to consider only one starting point x_0, and an initial value of the divided difference B_0.

Accordingly, (1.3) can be expressed, for $k > 0$, as

$$\begin{aligned} y_k &= x_k - B_k^{-1}\mathrm{F}(x_k) \\ x_{k+1} &= y_k - B_k^{-1}\mathrm{F}(y_k) \\ B_{k+1} &= [x_k, x_{k+1}; \mathrm{F}]. \end{aligned} \tag{4.1}$$

In the same way, (1.4) becomes

$$\begin{aligned} y_k &= x_k - B_k^{-1}\mathrm{F}(x_k) \\ x_{k+1} &= y_k - B_k^{-1}\mathrm{F}(y_k)) \\ B_{k+1} &= [2x_{k+1} - x_k, x_{k+1}; \mathrm{F}]. \end{aligned} \tag{4.2}$$

And, finally, (1.5) becomes

$$\begin{aligned} x_{k+1} &= x_k - B_k^{-1}\mathrm{F}(x_k) \\ y_{k+1} &= x_{k+1} - B_k^{-1}\mathrm{F}(x_{k+1})) \\ B_{k+1} &= [x_{k+1}, y_{k+1}; \mathrm{F}]. \end{aligned} \tag{4.3}$$

We apply these iterative methods to polynomial equations $F_n(x) = x^n - 1 = 0$, for $n = 2, 3, 4$, and compare the behaviour of these methods. Starting from an arbitrary point $x_0 \in \mathbb{C}$ and an initial estimation of the divided difference $B_0 \in \mathbb{C}$, we check whether the sequence x_k converges to a root of the polynomial, diverges to infinity, or has periodic or a more complex behaviour.

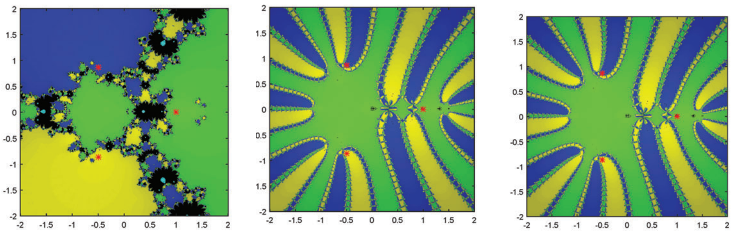


Fig. 5: Dynamics of CSTM(Left), CKTM(Middle) and ICSTM(Right) for $x^3 - 1 = 0$.

The dynamical plane is obtained by colouring the point x_0 of in the complex plane accordingly to the root to which the sequence converges. This colour is darkened according to the number of iterations required to fulfill a given tolerance. Let us set $B_0 = 1$ and take as starting points $x_0 \in \mathbb{C}$ the nodes of a regular mesh on a rectangle of side 4 centered at the origin. We run the iterations until a tolerance $|x_{k+1} - x_k| < 10^{-12}$ is fulfilled or a maximum of 100 iterations is reached.

Figures 4 to 6 show that the complexity of the basins increases with the degree of the polynomial. The roots are marked in red for the second and third degree polynomials and in white in the case of the fourth degree one. The basins of the roots of the polynomial are successively coloured in yellow, blue, green and red, according to its degree, and they are darkened to represent the number of iterations needed for fulfilling the convergence conditions. Methods CKTM and ICSTM have better behaviour in terms of convergence to the roots than CSTM, whose dynamical planes present more zones depicted in cyan or black, corresponding to starting points for which the iterations diverge to infinity or do not converge after the allowed number of iterations, respectively.

For the second degree equation $x^2 - 1 = 0$, the three methods show a similar behaviour. Independently from the starting point, the iterations always converge to a root for methods CSTM and ICSTM, and almost always for CSTM. See Figure 4, where the roots are marked with an asterisk. The attraction basins of the roots are relatively simple. The basins are wider about the root 1 but this is due to the election of B_0. Taking the opposite value of B_0, the figure changes by a symmetry about the imaginary axis.

The third degree polynomial $x^3 - 1 = 0$ allows us to observe more differences among the methods. CSTM presents small cyan zones where the method does not converge and even bigger black zones of divergence to infinity. The divergence zones are inappreciable for CKTM and do not exist in the case of ICSTM. Moreover, the basins are more fractionated for CKTM than for ICSTM. (See Figure 5)

The differences among methods are more evident for the fourth degree equation, $x^4 - 1 = 0$. CSTM presents more zones of divergence to infinity and not convergence after 100 iterations, as shown in Figure 6. The divergence zones are more evident for ICSTM than for CKTM, but the basins of CKTM have more branches (Figure 6).

5 Numerical Examples

In this section, the numerical experiments are carried out to show the efficacy of ICSTM. We have used the first order divided difference defined in (3.1). Now, We have tested the performance of ICSTM with CSTM and CKTM by working out a number of numerical examples. The performance

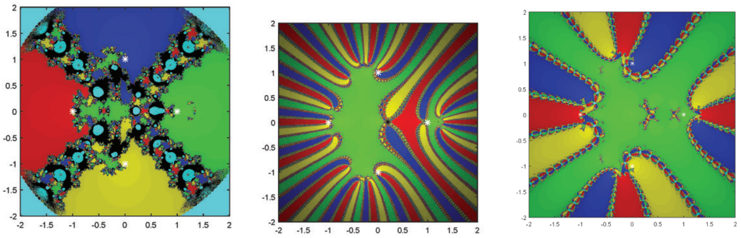


Fig. 6: Dynamics of CSTM(Left), CKTM(Middle) and ICSTM(Right) for $x^4 - 1 = 0$.

measures taken are the number of iterations and CPU time. All calculations are carried out in Matlab 2018 in a PC with windows 10 and processor intel(R) Core(TM) $i7 - 4790$ CPU 3.60**GHz.**

Example 51 *[5]* Consider $\mathrm{X} = \mathrm{Y} = \mathbb{R}^2$ and $\mathrm{F} : \mathrm{X} \to \mathrm{Y}$ given by

$$\mathrm{F}(x) = (x_1^3 - 0.75, x_2^3 - 0.75). \tag{5.1}$$

The solution of $\mathrm{F}(x) = 0$ **is given by** $\rho^* = (0.908560296, 0.908560296)$**. Starting with** $(x_{-1}, x_0) = ((0.999, 0.999), (1, 1))$ **for (1.3) and (1.4) and** $(x_0, y_0) = ((0.999, 0.999), (1, 1))$ **for (1.5), the error approximation is given in Table 2.**

Table 2: Absolute error between two last iterates, $\|x_{k+1} - x_k\|_\infty$ obtained by (1.3), (1.4) and (1.5) for $\alpha = \beta = 1/2$ and $\gamma = 1$

k	ICSTM	CSTM	CKTM
0	8.24167×10^{-02}	8.24167×10^{-02}	8.23333×10^{-02}
1	7.96787×10^{-03}	9.08442×10^{-02}	9.13769×10^{-02}
2	5.51107×10^{-05}	6.34526×10^{-03}	6.39438×10^{-03}
3		3.35609×10^{-04}	4.28183×10^{-05}
4		1.20551×10^{-06}	

Example 52 *Consider*

$$\begin{aligned}
\mathrm{F}_j(x) &= x_j \left(\sum_{k=1}^{n} \mathrm{L}_{jk} x_k - 1 \right), \text{ where} \\
\mathrm{L}_{j,j} &= 4(j-1) + 1, \ j = 1, \ldots, n \\
\mathrm{L}_{j,k} &= \mathrm{L}_{j,j} + 1, \ j = 1, \ldots, n-1 \ k = j+1, \ldots, n \\
\mathrm{L}_{j,k} &= \mathrm{L}_{j,j} + 1, \ k = 1, \ldots, n-1 \ j = k+1, \ldots, n.
\end{aligned}$$

Starting with $(x_{-1}, x_0) = \big((0.1, \underbrace{\ldots}_{n-2}, 0.1)^T, (0.2, \underbrace{\ldots}_{n-2}, 0.2)^T\big)$ **for (1.3) and (1.4) and** $(x_0, y_0) = \big((0.1, \underbrace{\ldots}_{n-2}, 0.1)^T, (0.2, \underbrace{\ldots}_{n-2}, 0.2)^T\big)$ **for (1.5), the solution of** $\mathrm{F}(x) = 0$ **is given by** $\rho^* = (0, \ldots, 0)^T$**. In Table 3, the total number of iterations are used by (1.3), (1.4) and (1.5) for** $\alpha = \beta = 1/2$ **and** $\gamma = 1$ **to converge with tolerance** $\|\mathrm{F}(x_k)\|_\infty < 10^{-15}$**. '-' denotes if these methods do not converge within 100 iterations.**

Example 53 *[14] Consider the nonlinear system*

$$F_k = \begin{cases} x_k^2 x_{k+1} - 1 = 0, & \text{if } 1 \le k \le n-1, \\ x_n^2 x_1 - 1 = 0, & \text{if } k = n, \end{cases}$$

.

Table 3: Number of iterations by (1.3), (1.4) and (1.5) for $\alpha = \beta = 1/2$ and $\gamma = 1$

n	5	10	15	20
CSTM	18	30	24	28
CKTM	62	32	25	–
ICSTM	13	11	11	17

Table 4: Different values of α, β and γ

Trials	α	β	γ
1	1/2	1/2	1
2	1/4	1/4	1/2
3	1	1	1
4	1/4	1	1/2
5	1/4	5/8	1/2

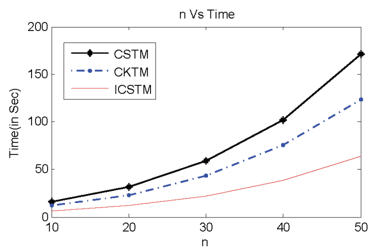


Fig. 7: Comparison of time for Example 53

Starting with $(x_{-1}, x_0) = \left((0.1, \underbrace{\ldots}_{n-2}, 0.1)^T, (0.2, \underbrace{\ldots}_{n-2}, 0.2)^T\right)$ **for (1.3) and (1.4) and** $(x_0, y_0) = \left((0.1, \underbrace{\ldots}_{n-2}, 0.1)^T, (0.2, \underbrace{\ldots}_{n-2}, 0.2)^T\right)$ **for (1.5), the solution of** $\mathrm{F}(x) = 0$ **is given by** $\rho^* = (1, \ldots, 1)^T$**. With different values of** α, β **and** γ**, we have compared the total number of iteration taken by these methods for n=10, 20, 30, 40, 50. The values taken by these parameters are given in Table 4. In Table 5, the total number of iterations are given by different methods to converge with tolerance** $\|\mathrm{F}(x_k)\|_\infty < 10^{-15}$**. '-' denotes that the method does not converge within 100 iterations. However, a comparison of CPU time taken by these methods is shown in Figure 7 for** $\alpha = \beta = 1/2$ **and** $\gamma = 1$**.**

Table 5: A comparison to number of iteration in Example 53

Method↓ No. of trials→	1	2	3	4	5
CSTM	34	91	–	48	30
CKTM	25	57	25	45	20
ICSTM	13	16	12	13	13

Example 54 Consider the Automotive Steering problem described in [15]. It is given in the form of a system of nonlinear equations for i=0,1,2,3 by

$$\begin{aligned}
\mathrm{F}_i(\psi_i,\phi_i) &= (\mathrm{E}_i(x_2\sin\psi_i - x_3) - \mathrm{G}_i(x_2\sin(\phi_i) - x_3))^2 + \big(\mathrm{G}_i(1 + x_2\cos\phi_i) \\
&\quad - \mathrm{E}_i(x_2\cos\psi_i - 1)\big)^2 - \big((1 + x_2\cos(\phi_i))(x_2\sin(\psi_i - x_3))x_1 \\
&\quad - (x_2\sin(\phi_i) - x_3)(x_2\cos(\psi_i) - x_3)x_1\big)^2, \quad \text{where} \\
\mathrm{E}_i &= x_2(\cos\phi_i - \cos\phi_0) - x_2x_3(\sin\phi_i - \sin\phi_0) - (x_2\sin\phi_i - x_3)x_1 \\
\mathrm{G}_i &= -x_2\cos\psi_i - x_2x_3\sin\psi_i + x_2\cos\psi_0 + x_1x_3 + (x_3 - x_1)x_2\sin\psi_0.
\end{aligned}$$

Table 6: The values of ψ_i and ϕ_i in Example 54

i	ψ_i	ϕ_i
0	1.3954170041	1.7461756494
1	1.7444828545	2.0364691127
2	2.0656234051	2.2390977868
3	2.4600678478	2.4600678409

Table 7: Different values of $\alpha,\ \beta$ and γ

Trials	α	β	γ	Initial Guess
1	1/2	1/2	1	$((0.7, 0.7, 0.7), (0.6, 0.6, 0.6))$
2	1/2	1/2	1	$((0.7, 0.6, 0.5), (0.6, 0.5, 0.4))$
3	1/4	1/2	1	$((0.7, 0.6, 0.5), (0.6, 0.5, 0.4))$
4	1/4	1/4	1/2	$((0.7, 0.6, 0.5), (0.6, 0.5, 0.4))$
5	1	1	1	$((0.7, 0.6, 0.5), (0.6, 0.5, 0.4))$

Table 8: A comparison to number of iteration in Example 54

Method↓ No. of trials→	1	2	3	4	5
CSTM	63	5	5	10	5
CKTM	–	4	4	6	4
ICSTM	21	3	3	6	3

ψ_i and ϕ_i for i=0,1,2,3 are given in Table 6. There are many solutions to this problem. A small change in the initial approximation leads to different solutions. However, with different values of the parameters $\alpha,\ \beta,\gamma$ and starting initial approximations given in Table 7, we checked the performance of these different methods using tolerance $\|\mathrm{F}(x_k)\|_2 < 10^{-3}$. We assume the solution is obtained once the tolerance is reached by the method. However, starting with $x_0 = (0.7, 0.6, 0.5),\ y_0 = (0.6, 0.5, 0.4)$ and for $\alpha = \beta = 1/2$ and $\gamma = 1$, we obtained the solution $(0.16359404..., 0.21418068..., 0.1419589...)$ for this problem. A comparison of the number of iterations accomplished by the different methods is given in Table 8.

In most of the cases, it is found that ICSTM performs better than CSTM and CKTM.

Example 55 [2] Consider the nonlinear integral equation of mixed Hammerstein type

$$x(s) = \mathrm{g}(s) + \int_a^b (\mathrm{G}(s,t)\big(\lambda_1(x(t) - \mathrm{f}(t))^2\big) + \mu_1|x(t) - \mathrm{f}(t)|)dt,\ s \in [a,b], \tag{5.2}$$

where $x, \mathrm{f} \in \mathcal{C}[a,b]$ and $\lambda_1, \mu_1 \in \mathbb{R}$. $\mathrm{G}(s,t)$ is the green function, given by

$$\mathrm{G}(s,t) = \begin{cases} (b-s)(t-a)/(b-a) & \text{if } t \leq s, \\ (s-a)(b-t)/(b-a) & \text{if } s \leq t, \end{cases}$$

We consider $a = 0,\ b = 1,\ \mathrm{g}(s) = 1,\ \mathrm{f}(t) = 0$ and $\lambda_1 = \mu_1 = 1/2$. In order to find a numerical solution of (5.2), we approximate the integral by the Gauss-Legendre formula with eight nodes

$$\int_0^1 h(t)dt = \sum_{k=1}^{8} w_k h(t_k)$$

where w_k and t_k are weight and nodes, respectively. Denote the approximation of $x(t_k)$ by $x_k,\ k = 1, \ldots, 8$. We obtain the following nonlinear system of equations:

$$x_j = 1 + \frac{1}{2}\sum_{k=1}^{8} a_{jk}(|x_k| + x_k^2),\ j = 1, \ldots, 8, \tag{5.3}$$

where

$$a_{jk} = \begin{cases} w_k t_k(1-t_j) & \text{if } k \leq j, \\ w_k t_j(1-t_k) & \text{if } j \leq k. \end{cases}$$

Now, (5.3) can be written as a system of nonlinear equations, by

$$\mathrm{F}(\mathbf{x}) \equiv \mathbf{x} - \mathbf{1} - \frac{1}{2}\mathbf{A}(\overline{\mathbf{x}} + \hat{\mathbf{x}}),$$

where $\mathrm{F} : \mathbb{R}^8 \to \mathbb{R}^8,\ \mathbf{x} = (x_1, \ldots, x_8)^T,\ \mathbf{1} = (1, \ldots, 1)^T,\ \mathbf{A} = (a_{jk})_{j,k=1}^8,\ \overline{\mathbf{x}} = (x_1^2, \ldots, x_8^2)^T$ and $\hat{\mathbf{x}} = (|x_1|, \ldots, |x_8|)^T$. Using (3.1), we obtain for $\mathbf{u},\ \mathbf{v} \in \mathbb{R}^8$, $[\mathbf{u}, \mathbf{v}; \mathrm{F}] = I - \frac{1}{2}(L + M)$, where $L = (L_{jk})_{j,k=1}^8$ with $L_{jk} = a_{jk}(u_k + v_k)$ and $M = (M_{jk})_{j,k=1}^8$ with $M_{jk} = a_{jk}\frac{|u_k| - |v_k|}{u_k - v_k}$.

Taking $\alpha = \beta = 1/2$ and $\gamma = 1$, we take starting points x_0 and y_0 as $x_0 = (1, \ldots, 1)^T$ and $y_0 = (0.9, \ldots, 0.9)^T$. Using sup-norm, we get $\eta = 0.145981421473560$, $\delta = 1.211673061136663$, $\Theta = 0.1$, $\sigma(u,v) = \frac{1}{2}(0.123558992073184)(u + v + 2)$. In addition, we get $M = 0.086202128806093$ and $R = 0.160243447853229$. We see that $g(R) = 0.089002243590837 \in (0, 0.618034)$. Hence, all conditions of Theorem 21 are satisfied and hence the solution exists in $\mathcal{B}(x_0, 0.16024...)$ and is unique in $\overline{\mathcal{B}}(x_0, 0.16024...)$. Now, we use the tolerance $\|\mathrm{F}(x_k)\|_\infty < 10^{-15}$ and the approximated solution is given in Table 9. It can be visualized in Figure 8.

Table 9: Approximate Numerical solution of (5.3)

k	ρ_k^*	k	ρ_k^*	k	ρ_k^*	k	ρ_k^*
1	1.01150108...	3	1.10989250...	5	1.14814397...	7	1.05467811...
2	1.05467811...	4	1.14814397...	6	1.10989251...	8	1.01150109...

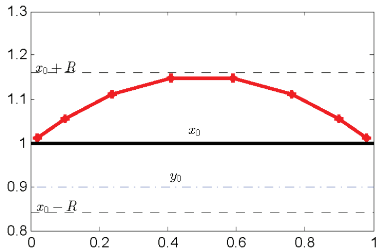


Fig. 8: Approximate solution of (5.3)

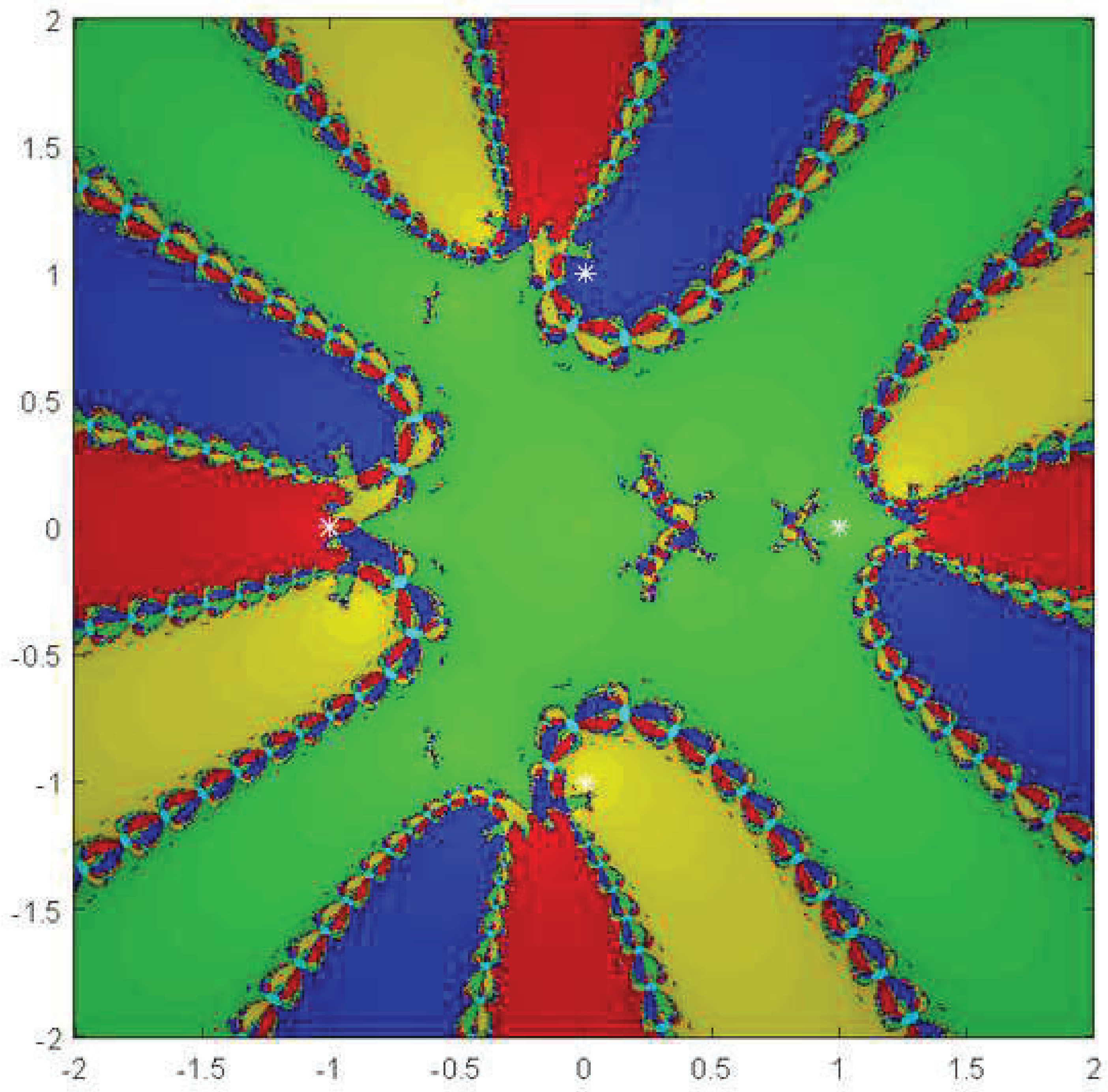
6 Conclusions

An improvement of the Chebyshev-Secant-type iterative methods requiring an equal number of functions evaluations used in Chebyshev-Secant-type iterative methods is described for solving nonlinear equations in a Banach space setting. Using recurrence relations, its semilocal convergence is discussed under weaker continuity conditions on first order divided differences. Convergence theorems are established for the existence-uniqueness of the solutions. Next, center-Lipschitz condition is defined on the first order divided differences and its influence on the domain of starting iterates is compared with the corresponding domain for Lipschitz conditions. A number of numerical examples including Automotive Steering problems and nonlinear mixed Hammerstein type integral equations are worked out and results obtained are compared with some of the existing similar iterative methods. It is observed that better results are obtained for all the numerical examples. This demonstrates the novelty and applicability of our study.

References

1. MA Hernández. Chebyshev's approximation algorithms and applications. *Computers & Mathematics with Applications*, 41(3-4):433–445, 2001.
2. JA Ezquerro, Miquel Grau-Sánchez, and MA Hernández. Solving non-differentiable equations by a new one-point iterative method with memory. *Journal of Complexity*, 28(1):48–58, 2012.
3. Ioannis K Argyros, JA Ezquerro, JM Gutiérrez, MA Hernández, and Saïd Hilout. On the semilocal convergence of efficient Chebyshev-Secant-type methods. *Journal of Computational and Applied Mathematics*, 235(10):3195–3206, 2011.
4. Hongmin Ren and Ioannis K Argyros. Local convergence of efficient Secant-type methods for solving nonlinear equations. *Applied Mathematics and computation*, 218(14):7655–7664, 2012.
5. Ioannis K Argyros and Hongmin Ren. On the semilocal convergence of derivative free methods for solving nonlinear equations. *Journal of Numerical Analysis and Approximation Theory*, 41(1):3–17, 2012.
6. Hongmin Ren and Ioannis K Argyros. On the convergence of King-Werner-type methods of order $1+\sqrt{2}$ free of derivatives. *Applied Mathematics and Computation*, 256:148–159, 2015.
7. Abhimanyu Kumar, DK Gupta, Eulalia Martínez, and Sukhjit Singh. Semilocal convergence of a Secant-type method under weak Lipschitz conditions in Banach spaces. *Journal of Computational and Applied Mathematics*, 330:732–741, 2018.
8. Miquel Grau-Sánchez, Miquel Noguera, and José M Gutiérrez. Frozen Iterative Methods Using Divided Differences "à la Schmidt–Schwetlick". *Journal of Optimization Theory and Applications*, 160(3):931–948, 2014.
9. Louis B Rall. *Computational solution of nonlinear operator equations*. Wiley New York, 1969.
10. Paul Blanchard. The dynamics of Newton's method. In *Proc. Symp. Appl. Math*, volume 49, pages 139–154, 1994.
11. Parisa Bakhtiari, Alicia Cordero, Taher Lotfi, Kathayoun Mahdiani, and Juan R Torregrosa. Widening basins of attraction of optimal iterative methods. *Nonlinear Dynamics*, 87(2):913–938, 2017.

12. Changbum Chun and Beny Neta. The basins of attraction of Murakami's fifth order family of methods. *Applied Numerical Mathematics*, 110:14–25, 2016.
13. Ángel Alberto Magreñán. A new tool to study real dynamics: the convergence plane. *Applied Mathematics and Computation*, 248:215–224, 2014.
14. Ramandeep Behl, Alicia Cordero, Sandile S Motsa, and Juan R Torregrosa. Stable high-order iterative methods for solving nonlinear models. *Applied Mathematics and Computation*, 303:70–88, 2017.
15. S Pramanik. Kinematic synthesis of a six-member mechanism for automotive steering. *Transactions American Society of Mechanical Engineers Journal of Mechanical Design*, 124(4):642–645, 2002.



2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2