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THE W-WEIGHTED DRAZIN-STAR MATRIX AND ITS DUAL*

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Abstract. After decades studying extensively two generalized inverses, namely Moore–Penrose inverse and Drazin inverse, currently, we found immersed in a new generation of generalized inverses (core inverse, DMP inverse, etc.). The main aim of this paper is to introduce and investigate a matrix related to these new generalized inverses defined for rectangular matrices. We apply our results to the solution of linear systems.

Key words. Moore–Penrose inverse, Drazin inverse, Weighted Drazin inverse.

AMS subject classification. 15A09.

1. Introduction. Let $\mathbb{C}^{m\times n}$ denote the set of all complex $m \times n$ matrices, and $\mathbb{C}_r^{m\times n}$ is set of all complex $m \times n$ matrices of rank r. For $A \in \mathbb{C}^{m\times n}$, the symbols A^*, $\operatorname{rank}(A)$, $R(A)$, and $N(A)$ denote conjugate transpose, rank, range space, and null space of A, respectively. The symbol I stands for the identity matrix of an appropriate order, and the symbol $P_{M,N}$ denotes the projector onto M along N, where M and N are two complementary subspaces of $\mathbb{C}^{m\times 1}$. The unique matrix $X \in \mathbb{C}^{n\times m}$, which is denoted by $A^\dagger$, satisfying the following equations:

$$AXA = A, \tag{1.1}$$

$$XAX = X, \tag{1.2}$$

$$(AX)^* = AX, \tag{1.3}$$

$$(XA)^* = XA, \tag{1.4}$$

is called the Moore–Penrose inverse of $A \in \mathbb{C}^{m\times n}$ [19]. A matrix $X \in \mathbb{C}^{n\times m}$ is called an outer inverse of $A \in \mathbb{C}^{m\times n}$ if X satisfies the equation (1.2). If $A \in \mathbb{C}_r^{m\times n}$, T is a subspace of $\mathbb{C}^n$ of dimension $t \leq r$ and S is a subspace of $\mathbb{C}^m$ of dimension $m - t$, then A has an outer inverse $X \in \mathbb{C}^{n\times m}$ with prescribed range $R(X) = T$ and null space $N(X) = S$ if and only if $AT \oplus S = \mathbb{C}^m$. In this case, X is unique and it is denoted by $A^{(2)}_{T,S}$ [2]. If $A \in \mathbb{C}^{n\times n}$, the index of A, denoted by $\operatorname{ind}(A)$, is the smallest nonnegative integer k such that $\operatorname{rank}(A^k) = \operatorname{rank}(A^{k+1})$. The unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying the following three equations:

$$A^{k+1}X = A^k, \quad XAX = X, \quad AX = XA,$$

is called the Drazin inverse of A and denoted by A^D, where $k = \operatorname{ind}(A)$. When $k = 1$, the Drazin inverse is called the group inverse, and denoted by $A^\#$ [7].

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The core inverse of a square complex matrix was introduced by Baksalary and Trenkler [1]. Later, Xu et al. [24] characterized the core inverse by three equations. Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) \leq 1$. The unique matrix X, denoted by $A^{\#\!\!\!\bigcirc}$, satisfying the following equations:

$$(AX)^* = AX, \quad AX^2 = X, \quad XA^2 = A,$$

is called the core inverse of A. We know that $A^{\#\!\!\!\bigcirc} = A^{\#}AA^{\dagger}$. Malik et al. [16] extended the core inverse from index at most one to an arbitrary index. These generalizations were called DMP inverse and dual DMP inverse. Zhu et al. defined and characterized the DMP inverse in rings [25]. Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. The DMP inverse of A, denoted by $A^{D,\dagger}$, is defined to be the matrix $A^{D,\dagger} = A^D AA^{\dagger}$. The dual DMP inverse of A, denoted by $A^{\dagger,D}$, is defined as $A^{\dagger,D} = A^{\dagger}AA^D$.

Let $A \in \mathbb{C}^{m\times n}$ and $0 \neq W \in \mathbb{C}^{n\times m}$. Cline et al. [4] extended the Drazin inverse from a square matrix to a rectangular matrix. A matrix $X \in \mathbb{C}^{m\times n}$ is the weighted Drazin inverse of A, and denoted by $A^{D,W}$, if X is the unique matrix satisfying $AWX = XWA$, $XWAWX = X$ and $XW(AW)^{k+1} = (AW)^k$, where $k = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$. Moreover, $A^{D,W} = A((WA)^D)^2 = ((AW)^D)^2A$ hold. In particular, if $k = 1$, the weighted Drazin inverse is the weighted group inverse and is denoted by $A^{\#,W}$. The DMP inverse was generalized from square matrices to rectangular matrices by Meng [17]. It is called the W-weighted DMP inverse and given by $A^{D,\dagger,W} = WA^{D,W}WAA^{\dagger}$, which is unique solution of the following equations:

$$XAX = X, \quad XA = WA^{D,W}WA, \quad (WA)^kX = (WA)^kA^{\dagger},$$

where $k = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$. Dually, it is easy to see that the W-weighted dual DMP inverse is given by $A^{\dagger,D,W} = A^{\dagger}AWA^{D,W}W$. In literature, a variety of algorithms for computing generalized inverses were designed (e.g., [6, 14, 20, 13]). Another papers related to weighted inverses that motivated our research are [11] by Kyrchei.

It is well known that one of the most important applications of the inverse of a matrix (square and nonsingular) is its involvement to solve linear systems. This application remains being important for rectangular and singular matrices by considering outer inverses [2]. In this sense, the W-weighted Drazin-star matrix introduced in this paper allows us to solve certain class of linear systems. Further potential works starting from our approach may be developed by studying Cramer's rules and determinantal representations for solving the class that we treat in this paper as well as the least-squares associated problem.

Recently, the Drazin-star matrix and the star-Drazin matrix were introduced by Mosić [18]. For $A \in \mathbb{C}^{n\times n}$ of index k, the Drazin-star matrix of A is defined as $A^{D,*} := A^DAA^*$, that is, it appears A^* instead of the Moore–Penrose inverse in the definition of the DMP inverse. The matrix $A^{D,*}$ is the unique matrix satisfying the following system of equations:

$$X(A^{\dagger})^*X = X, \quad A^kX = A^kA^*, \quad X(A^{\dagger})^* = A^DA.$$

The Drazin-star matrix introduced by Mosić in [18] is a (complex) square matrix. However, in practice, a wide range of real problems require generalized inverses of rectangular matrices and techniques involving Drazin-star matrix cannot be applied. The advantage of studying W-weighted Drazin-star matrix is that they allow us to tackle problems in areas such as (rectangular) linear system [2], cryptography [10], chemical equations [21], optimal problem [9], and rectangular descriptor control systems [12]. Motivated by above discussion, we investigate the Drazin-star matrix for rectangular matrices and present its properties and application (Section 5).

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This paper is organized as follows. In Section 2, we introduce the W-weighted Drazin-star matrix as the unique solution of a suitable systems of equations. This matrix extends the Drazin-star matrix from a square matrix to a rectangular matrix. In Section 3, we investigate several equivalent characterizations of the W-weighted Drazin-star matrix. In Section 4, we compute the W-weighted Drazin-star matrix by weighted core-EP decomposition and singular value decomposition. Two canonical forms of the W-weighted Drazin-star matrix are obtained. In Section 5, we design two algorithms to solve a system of linear equations by the W-weighted Drazin-star matrix and we give examples to illustrate them. In Section 6, the corresponding results of the W-weighted star-Drazin matrix are presented. Simultaneously, we study the relationships between the W-weighted Drazin-star matrix, the W-weighted star-Drazin matrix, and other generalized inverses.

2. Definition of the W-weighted Drazin-star matrix. In this section, we introduce the W-weighted Drazin-star matrix. Several properties of the W-weighted Drazin-star matrix are studied. Throughout this paper, we will use a nonzero matrix $W \in \mathbb{C}^{n\times m}$ that will play the role of the weighted matrix.

Firstly, we have the following system of equations.

THEOREM 2.1. *Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $k = \max\{\operatorname{ind}(AW), \operatorname{ind}(WA)\}$. Then the system*

$$X(A^\dagger)^* X = X, \quad (WA)^k X = (WA)^k A^*, \quad X(A^\dagger)^* = WA^{D,W}WA, \tag{2.5}$$

is consistent and it has a unique solution given by $X = WA^{D,W}WAA^*$.

Proof. Assume that $X := WA^{D,W}WAA^*$. Then we have

$$X(A^\dagger)^* = WA^{D,W}WAA^*(A^\dagger)^* = WA^{D,W}WA,$$

$$(WA)^k X = (WA)^k WA^{D,W}WAA^* = (WA)^k(WA)^D WAA^* = (WA)^k A^*,$$

and

$$X(A^\dagger)^* X = WA^{D,W}WAWA^{D,W}WAA^* = WA^{D,W}WAA^* = X.$$

So, $X = WA^{D,W}WAA^*$ satisfies the three equations in (2.5). If there exist two $n \times m$ matrices X_1 and X_2 such that the equations in (2.5) hold, then

$$\begin{aligned} X_1 &= X_1(A^\dagger)^* X_1 = WA^{D,W}WAX_1 = (WA)^D WAX_1 = ((WA)^D)^k (WA)^k X_1 \\ &= ((WA)^D)^k (WA)^k A^* = ((WA)^D)^k (WA)^k X_2 = (WA)^D WAX_2 \\ &= WA^{D,W}WAX_2 = X_2(A^\dagger)^* X_2 = X_2. \end{aligned}$$

Therefore, the solution of the system (2.5) is unique. ∎

Theorem 2.1 justifies the following definition.

DEFINITION 2.2. *Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $k = \max\{\operatorname{ind}(AW), \operatorname{ind}(WA)\}$. The W-weighted Drazin-star matrix of A is defined as $A^{W-D,*} = WA^{D,W}WAA^*$.*

REMARK 2.3. For $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $k = \max\{\operatorname{ind}(AW), \operatorname{ind}(WA)\}$, we have $A^{W-D,*} = (WA)^D WAA^*$. When $m = n$ and $W = I$, it is easy to check that $A^{W-D,*} = A^D AA^*$ is a Drazin-star matrix. In particular, if $k = 1$, then the W-weighted Drazin-star matrix of A is called the W-weighted group-star matrix of A and denoted by $A^{W-\#,*}$.

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In the following lemma, we consider the weighted Drazin-star matrix as an outer inverse with prescribed range and null space.

LEMMA 2.4. *Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $k = \max\{\operatorname{ind}(AW), \operatorname{ind}(WA)\}$. Then*

(i) *$(A^\dagger)^*A^{W-D,*}$ is a projector onto $R((A^\dagger)^*WA^{D,W})$ along $N(A^{D,W}A^*)$;*
(ii) *$A^{W-D,*}(A^\dagger)^*$ is a projector onto $R((WA)^k)$ along $N((WA)^k)$;*
(iii) *$A^{W-D,*} = ((A^\dagger)^*)^{(2)}_{R((WA)^k),\ N(A^{D,W}A^*)}$.*

Proof. (i): Since $A^{W-D,*}(A^\dagger)^*A^{W-D,*} = A^{W-D,*}$, $(A^\dagger)^*A^{W-D,*}$ is a projector. From

$$(A^\dagger)^*A^{W-D,*} = (A^\dagger)^*WA^{D,W}WAA^*,$$

and

$$(A^\dagger)^*WA^{D,W} = (A^\dagger)^*WA^{D,W}WAA^*(A^\dagger)^*WA^{D,W},$$

we have $R((A^\dagger)^*A^{W-D,*}) = R((A^\dagger)^*WA^{D,W})$.

On the other hand, since

$$N((A^\dagger)^*A^{W-D,*}) = N((A^\dagger)^*WAWA^{D,W}A^*) \supseteq N(A^{D,W}A^*),$$

and

$$\begin{aligned} N(A^{D,W}A^*) &= N(A^{D,W}WA^{D,W}WAA^*) = N(A^{D,W}A^{W-D,*}) \\ &= N(A^{D,W}WA^{D,W}WAA^*(A^\dagger)^*A^{W-D,*}) \supseteq N((A^\dagger)^*A^{W-D,*}), \end{aligned}$$

we get $N((A^\dagger)^*A^{W-D,*}) = N(A^{D,W}A^*)$.

(ii): Since $A^{W-D,*}(A^\dagger)^* = (WA)^DWA$, we get

$$R(A^{W-D,*}(A^\dagger)^*) = R((WA)^k) \quad\text{and}\quad N(A^{W-D,*}(A^\dagger)^*) = N((WA)^k).$$

(iii): From $R(A^{W-D,*}) = R(A^{W-D,*}(A^\dagger)^*) = R((WA)^k)$ and

$$N(A^{W-D,*}) = N((A^\dagger)^*A^{W-D,*}) = N(A^{D,W}A^*),$$

we have $A^{W-D,*} = ((A^\dagger)^*)^{(2)}_{R((WA)^k),\ N(A^{D,W}A^*)}$. ∎

3. Characterizations of W-weighted Drazin-star matrices. This section provides both algebraic and geometrical characterizations of the W-weighted Drazin-star matrix. Two canonical forms of the W-weighted Drazin-star matrix are also developed.

PROPOSITION 3.1. *Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $k = \max\{\operatorname{ind}(AW), \operatorname{ind}(WA)\}$. The W-weighted Drazin-star matrix $X \in \mathbb{C}^{n\times m}$ of A satisfies the following matrix equations:*

(A) $(WA)^kX = (WA)^kA^*$;
(B) $X(A^\dagger)^* = WA^{D,W}WA$;
(C) $AX = AWA^{D,W}WAA^*$;
(D) $XA = WA^{D,W}WAA^*A$;
(E) $(A^\dagger)^*X = (A^\dagger)^*WA^{D,W}WAA^*$;
(F) $WA^{D,W}WAX = X$;

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(G) $WA^{D,W}WAXAA^\dagger = X$;
(H) $XAA^\dagger = X$;
(I) $(WA)^k X(A^\dagger)^* = (WA)^k$;
(J) $X(A^\dagger)^*(WA)^k = (WA)^k$;
(K) $X(A^\dagger)^* A^\dagger = A^{D,\dagger,W}$;
(L) $X(A^\dagger)^* WAWA^{D,W}A^* = X$;
(M) $(A^\dagger)^* WAWA^{D,W}X = (A^\dagger)^* WA^{D,W}WAA^*$;
(N) $X(A^\dagger)^* WA^{D,W}WA = WA^{D,W}WA$.

Proof. A standard computation allows us to verify that items (A) − (N) hold by applying Theorem 2.1 and Definition 2.2. ∎

The equivalent conditions for a rectangular matrix to be a weighted Drazin-star matrix are studied in the following theorem.

THEOREM 3.2. *Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $k = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$ and consider the notation of items in Proposition 3.1. Then $X \in \mathbb{C}^{n\times m}$ is the W-weighted Drazin-star matrix of A if and only if any of the following statements is satisfied:*

(i) (A) *and* (F);
(ii) (G) *and* (I);
(iii) (C) *and* (F);
(iv) (E) *and* (F);
(v) (B) *and* (H);
(vi) (J) *and* (L);
(vii) (H) *and* (K);
(viii) (D) *and* (H);
(ix) (F) *and* (M);
(x) (L) *and* (N).

Proof. Let $X := WA^{D,W}WAA^*$. It is clear that conditions (i) − (x) hold by Proposition 3.1. Conversely, it is sufficient to verify that every condition (i) − (x) implies $X = WA^{D,W}WAA^*$. In fact,

(i): Assume that $WA^{D,W}WAX = X$ and $(WA)^k X = (WA)^k A^*$. Then

$$X = WA^{D,W}WAX = ((WA)^D)^k (WA)^k X = ((WA)^D)^k (WA)^k A^* = WA^{D,W}WAA^*.$$

(ii): Suppose that $WA^{D,W}WAXAA^\dagger = X$ and $(WA)^k X(A^\dagger)^* = (WA)^k$. Then

$$X = WA^{D,W}WAXAA^\dagger = ((WA)^D)^k (WA)^k X(A^\dagger)^* A^* = WA^{D,W}WAA^*.$$

(iii): Let $WA^{D,W}WAX = X$ and $AX = AWA^{D,W}WAA^*$. Then

$$X = WA^{D,W}WAX = WA^{D,W}WAWA^{D,W}WAA^* = WA^{D,W}WAA^*.$$

(iv): Set $WA^{D,W}WAX = X$ and $(A^\dagger)^* X = (A^\dagger)^* WA^{D,W}WAA^*$. Then

$$X = WA^{D,W}WAX = WA^{D,W}WAA^*(A^\dagger)^* X = WA^{D,W}WAA^*.$$

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(v): Assume that $XAA^\dagger = X$ and $X(A^\dagger)^* = WA^{D,W}WA$. Then

$$X = XAA^\dagger = X(A^\dagger)^*A^* = WA^{D,W}WAA^*.$$

In a similar way, we can prove the rest of items. $\square$

Remark 3.3. We would like to highlight that while the system (2.5) that defines the W-weighted Drazin-star matrix requires three equations, every item in Theorem 3.2 needs only two matrix equations.

Now, we characterize the W-weighted Drazin-star matrix from a geometrical point of view.

Theorem 3.4. *Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $k = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$. Then $A^{W-D,*}$ is the unique matrix X that satisfies*

$$(A^\dagger)^*X = P_{R((A^\dagger)^*WA^{D,W}),\ N(A^{D,W}A^*)}, \quad R(X) \subseteq R((WA)^k). \tag{3.6}$$

Proof. Assume that $X = A^{W-D,*}$. By Lemma 2.4 (i), it is easy to check that $A^{W-D,*}$ is a solution of both conditions in system (3.6). It is sufficient to prove that the solution of system (3.6) is unique. Suppose that X_1 and X_2 satisfy both conditions in (3.6). From $(A^\dagger)^*(X_1 - X_2) = 0$, we have

$$R(X_1 - X_2) \subseteq N((A^\dagger)^*) = N(A) \subseteq N(WA^{D,W}WA) = N((WA)^k).$$

Since $R(X_1) \subseteq R((WA)^k)$ and $R(X_2) \subseteq R((WA)^k)$, we get

$$R(X_1 - X_2) \subseteq R((WA)^k) \cap N((WA)^k) = \{0\}.$$

Hence, $X_1 = X_2$. $\square$

4. Computing the W-weighted Drazin-star matrix. In order to compute numerically the W-weighted Drazin-star matrix, we present two methods by using the weighted core-EP decomposition and the singular value decomposition in Theorems 4.2 and 4.4, respectively.

Lemma 4.1 ([8]). *Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $k = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$. Then there exist two unitary matrices $U \in \mathbb{C}^{m\times m}$, $V \in \mathbb{C}^{n\times n}$, two nonsingular matrices $A_1,\ W_1 \in \mathbb{C}^{t\times t}$, and two matrices $A_2 \in \mathbb{C}^{(m-t)\times(n-t)}$ and $W_2 \in \mathbb{C}^{(n-t)\times(m-t)}$ such that A_2W_2 and W_2A_2 are nilpotent of indices $\mathrm{ind}(AW)$ and $\mathrm{ind}(WA)$, respectively, with*

$$A = U\begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix}V^*, \quad W = V\begin{pmatrix} W_1 & W_{12} \\ 0 & W_2 \end{pmatrix}U^*.$$

The decomposition given in Lemma 4.1 is called weighted core-EP decomposition of the pair $\{A, W\}$.

Theorem 4.2. *Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $k = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$. If A and W are given as in Lemma 4.1, then*

$$A^{W-D,*} = V\begin{pmatrix} A_1^* + R_{WA}A_{12}^* & R_{WA}A_2^* \\ 0 & 0 \end{pmatrix}U^*,$$

where $R_{WA} = \sum_{i=0}^{k}(W_1A_1)^{i-k-1}(W_1A_{12} + W_{12}A_2)(W_2A_2)^{k-i}$.

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Proof. According to [5, 8], we have $(WA)^D = V\begin{pmatrix}(W_1A_1)^{-1} & \widetilde{T}_{WA}\\ 0 & 0\end{pmatrix}V^*$, where

$$\widetilde{T}_{WA} = \sum_{i=0}^{k-1}(W_1A_1)^{i-k-1}(W_1A_{12}+W_{12}A_2)(W_2A_2)^{k-1-i}.$$

Thus,

$$\begin{aligned}A^{W-D,*} &= WA^{D,W}WAA^* = (WA)^D WAA^*\\ &= V\begin{pmatrix}(W_1A_1)^{-1} & \widetilde{T}_{WA}\\ 0 & 0\end{pmatrix}\begin{pmatrix}W_1A_1 & W_1A_{12}+W_{12}A_2\\ 0 & W_2A_2\end{pmatrix}\begin{pmatrix}A_1^* & 0\\ A_{12}^* & A_2^*\end{pmatrix}U^*\\ &= V\begin{pmatrix}A_1^*+R_{WA}A_{12}^* & R_{WA}A_2^*\\ 0 & 0\end{pmatrix}U^*,\end{aligned}$$

where $R_{WA} = \sum_{i=0}^{k}(W_1A_1)^{i-k-1}(W_1A_{12}+W_{12}A_2)(W_2A_2)^{k-i}$. ∎

Lemma 4.3 ([2] (Singular value decomposition)). *The matrices $A \in \mathbb{C}_r^{m\times n}$ and $W \in \mathbb{C}_s^{n\times m}$ have the following singular value decompositions, respectively,*

$$A = U\begin{pmatrix}\Sigma_A & 0\\ 0 & 0\end{pmatrix}V^*, \quad W = \widetilde{V}\begin{pmatrix}\Sigma_W & 0\\ 0 & 0\end{pmatrix}\widetilde{U}^*,$$

where $U = (U_1,\ U_2) \in \mathbb{C}^{m\times m}$, $\widetilde{U} = (\widetilde{U}_1,\ \widetilde{U}_2) \in \mathbb{C}^{m\times m}$, $V = (V_1,\ V_2) \in \mathbb{C}^{n\times n}$ and $\widetilde{V} = (\widetilde{V}_1,\ \widetilde{V}_2) \in \mathbb{C}^{n\times n}$ are unitary matrices, $U_1 \in \mathbb{C}^{m\times r}$, $\widetilde{U}_1 \in \mathbb{C}^{m\times s}$, $V_1 \in \mathbb{C}^{n\times r}$, $\widetilde{V}_1 \in \mathbb{C}^{n\times s}$, and $\Sigma_A = diag(\sigma_1,\ldots,\sigma_r)$, $\Sigma_W = diag(\beta_1,\ldots,\beta_s)$, $\sigma_1 \geq \cdots \geq \sigma_r > 0$, and $\beta_1 \geq \cdots \geq \beta_s > 0$.

Theorem 4.4. *Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $k = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$. If A and W are represented as in Lemma 4.3, then*

$$A^{W-D,*} = \widetilde{V}\begin{pmatrix}\Sigma_W S_{11}\Lambda\Sigma_W S_{11}\Sigma_A^2 & 0\\ 0 & 0\end{pmatrix}U^*,$$

where $S_{11} = \widetilde{U}_1^*U_1$, $S_{12} = \widetilde{U}_1^*U_2$, $T_{11} = V_1^*\widetilde{V}_1$, $T_{12} = V_1^*\widetilde{V}_2$ and $\Lambda = (\Sigma_A T_{11})^{D,\Sigma_W S_{11}}$.

Proof. Denote $S = \widetilde{U}^*U = \begin{pmatrix}S_{11} & S_{12}\\ S_{21} & S_{22}\end{pmatrix}$ and $T = V^*\widetilde{V} = \begin{pmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{pmatrix}$, where $U = (U_1, U_2)$, $\widetilde{U} = (\widetilde{U}_1, \widetilde{U}_2)$, $V = (V_1, V_2)$ and $\widetilde{V} = (\widetilde{V}_1, \widetilde{V}_2)$. Then $S_{11} = \widetilde{U}_1^*U_1$ and $T_{11} = V_1^*\widetilde{V}_1$. From [22], we have

$$W = \widetilde{V}\begin{pmatrix}\Sigma_W S_{11} & \Sigma_W S_{12}\\ 0 & 0\end{pmatrix}U^*, \quad A = U\begin{pmatrix}\Sigma_A T_{11} & \Sigma_A T_{12}\\ 0 & 0\end{pmatrix}\widetilde{V}^*,$$

and $A^{D,W} = U\begin{pmatrix}\Lambda & \Lambda\Sigma_W S_{11}\Lambda\Sigma_W S_{11}\Sigma_A T_{12}\\ 0 & 0\end{pmatrix}\widetilde{V}^*$, where $\Lambda = (\Sigma_A T_{11})^{D,\Sigma_W S_{11}}$. By direct computation, we can get the results. ∎

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5. Applications.

This section presents some applications.

5.1. Algorithms to solve linear systems.

In this section, we design two algorithms to solve a linear system of equations by using the W-weighted Drazin-star matrix. In addition, an illustrating example is presented.

THEOREM 5.1. *Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $\max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\} = k \le s$, let $b \in \mathbb{C}^m$. Then $A^{W-D,*}b$ is a solution of equation*

$$(WA)^s x = (WA)^s A^* b, \tag{5.7}$$

and its general solution is given by

$$x = A^{W-D,*}b + (I - WA^{D,W}WA)y, \tag{5.8}$$

for arbitrary $y \in \mathbb{C}^n$.

Proof. By Definition 2.5, it is easy to see that $A^{W-D,*}b$ is a solution of the equation (5.7). Suppose that $x = A^{W-D,*}b + (I - WA^{D,W}WA)y$, for arbitrary $y \in \mathbb{C}^n$. Then

$$\begin{aligned}(WA)^s x &= (WA)^s(A^{W-D,*}b + (I - (WA)^D WA)y)\\ &= (WA)^s A^{W-D,*}b = (WA)^s A^* b,\end{aligned}$$

that is, x is a solution of the equation (5.7). Assume that x is a solution of the equation (5.7). Then, we have

$$\begin{aligned}WA^{D,W}WAx = (WA)^D WAx = ((WA)^D)^s(WA)^s x &= ((WA)^D)^s(WA)^s A^* b\\ = WA^{D,W}WAA^*b &= A^{W-D,*}b.\end{aligned}$$

So, $x = A^{W-D,*}b + x - WA^{D,W}WAx = A^{W-D,*}b + (I - WA^{D,W}WA)x$, which is of the form (5.8). ◻

Now, let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $k = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$ and assume that $b \in R((A^\dagger)^*(WA)^k)$. We consider the linear system

$$(A^\dagger)^* x = b, \tag{5.9}$$

and we are interested in obtaining solutions in $R((WA)^k)$. We note that the expression (5.9) is not a restriction because every linear system $Bx = b$ can be written as $(A^\dagger)^* x = b$ by setting $A := (B^*)^\dagger$. Next, algorithms allow us to solve this problem.

Algorithm 1: Input: $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$ and $b \in \mathbb{C}^m$.

Output: The solution of system (5.9) (if possible).

Step 1: Compute $(A^\dagger)^*$ and $R((WA)^k)$.

Step 2: Check whether $b \in R((A^\dagger)^*(WA)^k)$. If not, go to Step 6.

Step 3: Perform the weighted core-EP decomposition of the pair $\{A,\ W\}$ (see Lemma 4.1).

Step 4: Compute $A^{W-D,*}$ with Theorem 4.2.

Step 5: The solution of system (5.9) is $x = A^{W-D,*}b$. Go to end.

Step 6: There exists no solution of system (5.9).

End.

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Now, we have to justify the Algorithm 1. Suppose that $b \in R((A^\dagger)^*(WA)^D)$, then there exists $z \in \mathbb{C}^n$ such that $b = (A^\dagger)^*(WA)^D z$. Set $x = A^{W-D,*}b$. So, we have

$$\begin{aligned}(A^\dagger)^*x &= (A^\dagger)^*A^{W-D,*}b = (A^\dagger)^*WA^{D,W}WAA^*b = (A^\dagger)^*(WA)^D WAA^*b\\ &= (A^\dagger)^*(WA)^D WAA^*(A^\dagger)^*(WA)^D z = (A^\dagger)^*(WA)^D z = b.\end{aligned}$$

That is, x is a solution of equation (5.9). Now, we justify the uniqueness of solution in equation (5.9). Assume that the equation (5.9) has two solutions x_1 and x_2 in $R((WA)^k)$. By Lemma 2.4 (ii), we have $N(A^{W-D,*}(A^\dagger)^*) = N((WA)^k)$. Therefore,

$$\begin{aligned}x_1 - x_2 &\in R((WA)^k) \cap N((A^\dagger)^*) \subseteq R((WA)^k) \cap N(A^{W-D,*}(A^\dagger)^*)\\ &= R((WA)^k) \cap N((WA)^k) = \{0\},\end{aligned}$$

that is, $x_1 = x_2$.

Algorithm 2: Input: $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, and $b \in \mathbb{C}^m$.

Output: The solution of system (5.9) (if possible).

Step 1: Compute $(A^\dagger)^*$ and $R((WA)^k)$.

Step 2: Check whether $b \in R((A^\dagger)^*(WA)^k)$. If not, go to Step 6.

Step 3: Perform the singular value decomposition of the pair $\{A,\ W\}$ (See Lemma 4.3).

Step 4: Compute $A^{W-D,*}$ with Theorem 4.4.

Step 5: The solution of system (5.9) is $x = A^{W-D,*}b$. Go to end.

Step 6: There exists no solution of system (5.9).

End.

The following example illustrates the performance of the above algorithms.

Example 5.2. Let $A = \begin{pmatrix} \frac{1}{2} & \frac{5}{2} & \frac{1}{2} & 1 & 1\\ -\frac{1}{2} & -\frac{5}{2} & -\frac{1}{2} & 0 & 0\\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & -1\\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -1 & 0\end{pmatrix}$, $W = \begin{pmatrix} 2 & 0 & 0 & 0\\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\\ 0 & 0 & 0 & 0\end{pmatrix}$, and $b = \begin{pmatrix}\frac{5}{12}\\ \frac{1}{4}\\ \frac{5}{12}\\ \frac{5}{12}\end{pmatrix}$. It is easy to check that $\max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\} = 3$. From Algorithm 1, by weighted core-EP decomposition, we get $A = U\begin{pmatrix} A_1 & A_{12}\\ 0 & A_2\end{pmatrix}V^*$, $W = V\begin{pmatrix} W_1 & W_{12}\\ 0 & W_2\end{pmatrix}U^*$, where $U = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1\\ -1 & -1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 2\\ 0 & 3\end{pmatrix}$, $A_{12} = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 1\end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$, $W_1 = \begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}$, $W_{12} = \begin{pmatrix} 1 & 1\\ 0 & 0\end{pmatrix}$, $W_2 = \begin{pmatrix} 1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}$ and V is identity matrix of size 5. Then, by the canonical form of W-weighted Drazin-star matrix in Theorem 4.2, we have

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$$(WA)^D = \begin{pmatrix} 1 & -\frac{5}{3} & 1 & \frac{7}{9} & \frac{10}{27} \\ 0 & \frac{1}{3} & 0 & \frac{1}{9} & \frac{4}{27} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A^{W-D,*} = \begin{pmatrix} \frac{31}{9} & -1 & -\frac{1}{9} & -\frac{1}{3} \\ \frac{59}{18} & -\frac{5}{2} & -\frac{17}{18} & -\frac{5}{6} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since

$$(A^\dagger)^* = \begin{pmatrix} \frac{5}{12} & -\frac{1}{6} & \frac{5}{12} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ \frac{5}{12} & -\frac{1}{6} & \frac{5}{12} & \frac{1}{2} & -\frac{1}{2} \\ \frac{5}{12} & -\frac{1}{6} & \frac{5}{12} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

and we check that $b = (\frac{5}{12}, \frac{1}{4}, \frac{5}{12}, \frac{5}{12})^* \in R((A^\dagger)^*(WA)^k) = R((A^\dagger)^*(WA)^D)$, by direct computation, we obtain $A^{W-D,*}b = (1,0,0,0,0)^*$ and $(A^\dagger)^* A^{W-D,*} b = b$.

Now, we use Algorithm 2. By singular value decomposition (see Lemma 4.3), we have

$$\Sigma_A = diag\left(\frac{3637}{936}, \frac{2367}{1526}, 1, \frac{951}{1351}\right), \quad \Sigma_W = diag\left(\frac{2051}{937}, 1, 1, \frac{937}{2051}\right),$$

$$U = \begin{pmatrix} \frac{254}{339} & -\frac{329}{1914} & 0 & -\frac{1212}{1895} \\ -\frac{2311}{3661} & -\frac{1251}{2620} & 0 & -\frac{514}{841} \\ -\frac{101}{713} & \frac{1653}{2713} & -\frac{985}{1393} & -\frac{514}{1559} \\ -\frac{101}{713} & \frac{1653}{2713} & \frac{985}{1393} & -\frac{514}{1559} \end{pmatrix} \quad and \quad \widetilde{V} = \begin{pmatrix} -\frac{171}{188} & 0 & 0 & \frac{1444}{3475} & 0 \\ -\frac{327}{1363} & -\frac{1488}{1921} & \frac{496}{1921} & -\frac{825}{1571} & 0 \\ -\frac{327}{1363} & \frac{486}{2969} & -\frac{1995}{2494} & -\frac{825}{1571} & 0 \\ -\frac{327}{1363} & \frac{493}{807} & \frac{831}{1534} & -\frac{825}{1571} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $\mathrm{rank}(A) = 4 = \mathrm{rank}(W)$, by the proof of Theorem 4.4, we have $S = S_{11}$ in this example. Then

$$S = \widetilde{U}^* U = \begin{pmatrix} -\frac{572}{787} & \frac{169}{640} & 0 & \frac{473}{746} \\ -\frac{1004}{1677} & \frac{73}{715} & \frac{228}{721} & -\frac{868}{1191} \\ \frac{365}{1829} & -\frac{73}{2145} & \frac{684}{721} & \frac{868}{3573} \\ \frac{151}{557} & \frac{1339}{1397} & 0 & -\frac{430}{4863} \end{pmatrix},$$

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and

$$T_{11} = V_1^*\widetilde{V}_1 = \begin{pmatrix} -\dfrac{487}{1109} & -\dfrac{787}{1423} & \dfrac{3930}{15719} & -\dfrac{1031}{1659} \\ -\dfrac{1361}{2909} & -\dfrac{929}{3051} & -\dfrac{2063}{3223} & \dfrac{254}{1605} \\ \dfrac{3182}{18757} & -\dfrac{562}{1301} & -\dfrac{321}{838} & \dfrac{443}{1193} \\ \dfrac{817}{1411} & -\dfrac{619}{977} & \dfrac{510}{2063} & \dfrac{254}{2769} \end{pmatrix},$$

Therefore, by using MATLAB, it is easy to obtain

$$(\Sigma_A T_{11}\Sigma_W S)^D = \begin{pmatrix} \dfrac{888}{4453} & -\dfrac{293}{2075} & -\dfrac{318}{4973} & -\dfrac{989}{622} \\ -\dfrac{797}{4413} & \dfrac{509}{4743} & -\dfrac{345}{1601} & -\dfrac{2885}{1273} \\ 0 & 0 & 0 & 0 \\ \dfrac{740}{6573} & -\dfrac{671}{9719} & \dfrac{306}{2891} & \dfrac{1042}{1015} \end{pmatrix}.$$

and

$$\Lambda = (\Sigma_A T_{11})^{D,\Sigma_W S} = \begin{pmatrix} -\dfrac{1123}{2065} & \dfrac{1469}{1462} & -\dfrac{931}{2314} & \dfrac{215}{1656} \\ -\dfrac{1107}{1897} & \dfrac{521}{306} & -\dfrac{2512}{3161} & \dfrac{1799}{3232} \\ 0 & 0 & 0 & 0 \\ \dfrac{455}{1808} & -\dfrac{539}{683} & \dfrac{877}{2341} & -\dfrac{370}{1337} \end{pmatrix}.$$

Hence, $A^{W-D,*} = \widetilde{V}\begin{pmatrix}\Sigma_W \\ 0\end{pmatrix} S\Lambda\Sigma_W S\Sigma_A^2 U^* = \begin{pmatrix} \frac{31}{9} & -1 & -\frac{1}{9} & -\frac{1}{3} \\ \frac{59}{18} & -\frac{5}{2} & -\frac{17}{18} & -\frac{5}{6} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. It illustrates that Algorithm 2 also works.

Remark 5.3. Theorem 5.1 has been presented as an application to solve certain class of linear equation by means of the W-weighted Drazin-star matrix. In Lemma 2.4 (iii), we have proved that the W-weighted Drazin-star matrix is an outer inverse matrix with prescribed range and null space. Since outer inverses with prescribed range and null space have a remarkable significance in Matrix Theory, the W-weighted Drazin-star matrix can provide theoretical value for future practice [11, 17].

5.2. Solving restricted linear systems. Let $A \in \mathbb{C}^{m\times n}$, $T \in \mathbb{C}^n$, and $S \in \mathbb{C}^m$. In [3, 23], it was shown that $A^{(2)}_{T,S}b$ is the solution of the restricted linear equation $Ax = b$, with $x \in T$, $b \in AT$. In [3, Theorem 2.1], the author proved that the system $Ax = b$ restricted to $x \in T$ has a unique solution if and only if $b \in AT$ and $T \cap N(A) = \{0\}$. In the following lemma, we give an auxiliary lemma.

Lemma 5.4. *Let $B \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $\mathrm{ind}(W(B^\dagger)^*) = k$. Then $N(B)\cap R((W(B^\dagger)^*)^k) = \{0\}$.*

Proof. Let $x \in N(B)\cap R((W(B^\dagger)^*)^k)$. Then there exists $y \in \mathbb{C}^n$ such that $x = (W(B^\dagger)^*)^k y$ and $Bx = 0$. Since $(B^\dagger)^*B^\dagger B = (B^\dagger)^*$, by Drazin inverse definition we have

$$x = (W(B^\dagger)^*)^D(W(B^\dagger)^*)B^\dagger B(W(B^\dagger)^*)^k y = (W(B^\dagger)^*)^D(W(B^\dagger)^*)B^\dagger Bx = 0.$$

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Next, we show that the W-weighted Drazin-star matrix can solve the restricted linear equation.

THEOREM 5.5. *Let $B \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $\max\{\mathrm{ind}((B^\dagger)^*W), \mathrm{ind}(W(B^\dagger)^*)\} = k$ and let $S := R((W(B^\dagger)^*)^k)$. If $b \in BS$, then the unique solution of*

$$Bx = b, \quad \textit{restricted to} \quad x \in S,$$

is given by

$$x = B^{(2)}_{R((W(B^\dagger)^*)^k), N((W(B^\dagger)^*)^D B^\dagger)} b = ((B^\dagger)^*)^{W-D,*} b.$$

Proof. From Lemma 2.4 (iii), it is easy to obtain that

$$((B^\dagger)^*)^{W-D,*} = B^{(2)}_{R((W(B^\dagger)^*)^k), N((W(B^\dagger)^*)^D B^\dagger)}.$$

By Lemma 5.4, we know that $S \cap N(B) = \{0\}$. Assuming that $b \in BS$, the rest proof is similar to the proof of [3, Theorem 2.2]. ∎

LEMMA 5.6. *Let $B \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $\mathrm{ind}(W(B^\dagger)^*) = k$, let $S := R((W(B^\dagger)^*)^k)$. Then for any $b \in \mathbb{C}^m$, Xb is the minimum-norm least-squares solution of*

$$Bx = b, \quad \textit{restricted to} \quad x \in S,$$

if and only if $X = P_S(BP_S)^\dagger$ is the S-restricted Moore–Penrose inverse of B.

Proof. It is clear by [2, Page 113, Ex20, Ex21, Ex22]. ∎

6. The W-weighted star-Drazin matrix. Dually, we give the related results of W-weighted star-Drazin matrices.

In a similar way, the system is considered in the following result.

THEOREM 6.1. *Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $k = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$. Then the system:*

$$X(A^\dagger)^*X = X, \quad X(AW)^k = A^*(AW)^k, \quad (A^\dagger)^*X = AWA^{D,W}W,$$

is consistent and it has a unique solution given by $X = A^*AWA^{D,W}W$.

DEFINITION 6.2. *Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $k = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$. Then the W-weighted star-Drazin matrix of A is defined as $A^{*,W-D} = A^*AWA^{D,W}W$.*

REMARK 6.3. Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $k = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$. If $k = 1$, then the W-weighted star-Drazin matrix of A is reduced to the weighted star-group matrix of A and denoted by $A^{*,W-\#}$.

In the following lemma, we consider the W-weighted star-Drazin matrix as an outer inverse with prescribed range and null space.

LEMMA 6.4. *Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $k = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$. Then*

(i) $(A^\dagger)^*A^{*,W-D}$ *is a projector onto $R((AW)^k)$ along $N((AW)^k)$;*
(ii) $A^{*,W-D}(A^\dagger)^*$ *is a projector onto $R(A^*A^{D,W})$ along $N(A^{D,W}W(A^\dagger)^*)$;*
(iii) $A^{*,W-D} = ((A^\dagger)^*)^{(2)}_{R(A^*A^{D,W}),\ N((AW)^k)}$.

We give the equivalent conditions of the W-weighted star-Drain matrix.

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Proposition 6.5. *Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $k = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$. The W-weighted star-Drazin matrix $X \in \mathbb{C}^{n\times m}$ of A satisfies the following equations:*

(A) $X(AW)^k = A^*(AW)^k$;
(B) $(A^\dagger)^*X = AWA^{D,W}W$;
(C) $AX = AA^*AWA^{D,W}W$;
(D) $XA = A^*AWA^{D,W}WA$;
(E) $X(A^\dagger)^* = A^*AWA^{D,W}W(A^\dagger)^*$;
(F) $XAWA^{D,W}W = X$;
(G) $A^\dagger AXAWA^{D,W}W = X$;
(H) $A^\dagger AX = X$;
(I) $(A^\dagger)^*X(AW)^k = (AW)^k$;
(J) $(AW)^k(A^\dagger)^*X = (AW)^k$;
(K) $A^\dagger(A^\dagger)^*X = A^{\dagger,D,W}$;
(L) $A^*AWA^{D,W}W(A^\dagger)^*X = X$;
(M) $XA^{D,W}WAW(A^\dagger)^* = A^*AWA^{D,W}W(A^\dagger)^*$;
(N) $AWA^{D,W}W(A^\dagger)^*X = AWA^{D,W}W$.

Theorem 6.6. *Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $k = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$ and consider the notation of items in Proposition 6.5. Then $X \in \mathbb{C}^{n\times m}$ is the W-weighted star-Drazin matrix of A if and only if any of the following statements is satisfied:*

(i) (A) *and* (F);
(ii) (G) *and* (I);
(iii) (D) *and* (F);
(iv) (E) *and* (F);
(v) (B) *and* (H);
(vi) (J) *and* (L);
(vii) (H) *and* (K);
(viii) (C) *and* (H);
(ix) (F) *and* (M);
(x) (L) *and* (N).

We characterize the W-weighted star-Drazin matrix by applying a geometrical method in the following theorem.

Theorem 6.7. *Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $k = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$. Then $A^{*,W-D}$ is the unique matrix X satisfies*

$$(A^\dagger)^*X = P_{R((AW)^k),\ N((AW)^k)}, \quad R(X) \subseteq R(A^*A^{D,W}).$$

Theorem 6.8. *Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $k = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$. If A and W are given as in Lemma 4.1, then*

$$A^{*,W-D} = V \begin{pmatrix} A_1^* & A_1^*R_{AW} \\ A_{12}^* & A_{12}^*R_{AW} \end{pmatrix} U^*,$$

where $R_{AW} = \sum_{i=0}^{k}(A_1W_1)^{i-k-1}(A_1W_{12} + A_{12}W_2)(A_2W_2)^{k-i}$.

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THEOREM 6.9. *Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $k = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$. If A and W are represented as in Lemma 4.3, then*

$$A^{*,W-D} = \widetilde{V}\begin{pmatrix} T_{11}^*\Sigma_A^2 T_{11}\Sigma_W S_{11}\Lambda\Sigma_W S_{11} & T_{11}^*\Sigma_A^2 T_{11}\Sigma_W S_{11}\Lambda\Sigma_W S_{12} \\ T_{12}^*\Sigma_A^2 T_{11}\Sigma_W S_{11}\Lambda\Sigma_W S_{11} & T_{12}^*\Sigma_A^2 T_{11}\Sigma_W S_{11}\Lambda\Sigma_W S_{12}\end{pmatrix} U^*,$$

where $S_{11} = \widetilde{U}_1^* U_1$, $S_{12} = \widetilde{U}_1^* U_2$, $T_{11} = V_1^*\widetilde{V}_1$, $T_{12} = V_1^*\widetilde{V}_2$ and $\Lambda = (\Sigma_A T_{11})^{D,\Sigma_W S_{11}}$.

Finally, the relations between the W-weighted Drazin-star matrix, the W-weighted star-Drazin matrix, and some generalized inverses are given.

THEOREM 6.10. *Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $k = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$. Then the following conditions hold:*

- (i) *$A^{W-D,*} = A^{*,W-D}$ if and only if $R((WA)^k) = R(A^*A^{D,W})$ and $N(A^{D,W}A^*) = N((AW)^k)$;*
- (ii) *$A^{W-D,*}(A^\dagger)^* = (A^\dagger)^*A^{*,W-D}$ if and only if $(WA)^D WA = AW(AW)^D$;*
- (iii) *$A^{W-D,*} = A^* = A^{*,W-D}$ if and only if $A^{D,\dagger,W} = A^\dagger = A^{\dagger,D,W}$;*
- (iv) *$AA^{W-D,*}(A^\dagger)^* = (A^\dagger)^*A^{*,W-D}A$ if and only if $AA^{D,\dagger,W}A = AA^{\dagger,D,W}A$;*
- (v) *$A^{W-D,*}A = (WA)^D WA$ if and only if $A^{W-D,*} = A^{D,\dagger,W}$;*
- (vi) *$AA^{*,W-D} = AW(AW)^D$ if and only if $A^{*,W-D} = A^{\dagger,D,W}$.*

Proof. (i): It is evident by Lemmas 2.4 (iii) and 6.4 (iii).

(ii): By definition, the condition $A^{W-D,*}(A^\dagger)^* = (A^\dagger)^*A^{*,W-D}$ can be rewritten as $WA^{D,W}WA = AWA^{D,W}W$ which is equivalent to $(WA)^D WA = AW(AW)^D$.

(iii): Assume that $A^{W-D,*} = A^* = A^{*,W-D}$. Then

$$WA^{D,W}WAA^* = A^\dagger AA^* \text{ and } A^*AWA^{D,W}W = A^*AA^\dagger.$$

By left and right $*$-cancellable property, we have $WA^{D,W}WA = A^\dagger A$ and $AWA^{D,W}W = AA^\dagger$. So,

$$\begin{aligned} A^{D,\dagger,W} &= (WA^{D,W}WA)A^\dagger = (A^\dagger A)A^\dagger = A^\dagger = A^\dagger(AA^\dagger) \\ &= A^\dagger AWA^{D,W}W = A^{\dagger,D,W}. \end{aligned}$$

Now, suppose that $A^{D,\dagger,W} = A^\dagger = A^{\dagger,D,W}$. Then

$$\begin{aligned} A^{W-D,*} &= WA^{D,W}WAA^* = WA^{D,W}WAA^\dagger AA^* = A^{D,\dagger,W}AA^* = A^* \\ &= A^*AA^\dagger = A^*AA^{\dagger,D,W} = A^*AA^\dagger AWA^{D,W}W = A^{*,W-D}. \end{aligned}$$

(iv): We have that

$$\begin{aligned} AA^{W-D,*}(A^\dagger)^* = (A^\dagger)^*A^{*,W-D}A &\Leftrightarrow AWA^{D,W}WAA^*(A^\dagger)^* = (A^\dagger)^*A^*AWA^{D,W}WA \\ &\Leftrightarrow AWA^{D,W}WAA^\dagger A = AA^\dagger AWA^{D,W}WA \\ &\Leftrightarrow AA^{D,\dagger,W}A = AA^{\dagger,D,W}A. \end{aligned}$$

(v): Suppose that $A^{W-D,*}A = (WA)^D WA$. Then

$$(WA)^D WAA^*A = (WA)^D W(A^\dagger)^*A^*A.$$

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By right $*$-cancellable property, we obtain

$$A^{W-D,*} = (WA)^D WAA^* = (WA)^D W(A^\dagger)^* A^* = (WA)^D WAA^\dagger = A^{D,\dagger,W}.$$

The converse is evident.

(vi): It is similar to the proof of (v). □

THEOREM 6.11. *Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $k = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$. If A and W are given as in Lemma 4.1, then $A^{W-D,*} = A^{*,W-D}$ if and only if $A_{12} = 0$ and $R_{WA}A_2^* = A_1^* R_{AW}$, where R_{WA} and R_{AW} as in Theorems 4.2 and 6.8, respectively.*

Proof. The result follows by applying Theorems 4.2 and 6.8. □

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REFERENCES

[1] O.M. Baksalary and G. Trenkler. Core inverse of matrices. *Linear Multilinear Algebra*, 58(6):681–697, 2010.
[2] A. Ben-Israel and T.N.E. Greville. *Generalized Inverses: Theory and Applications*. 2nd edition. Springer, New York, 2003.
[3] Y.L. Chen. A crammer rule for solution of the general restricted linear equation. *Linear Multilinear Algebra,* 34:177–186, 1993.
[4] R.E. Cline and T.N.E. Greville. A Drazin inverse for rectangular matrices. *Linear Algebra Appl.*, 29:53–62, 1980.
[5] D.S. Cvetković and V. Pavlović. Drazin invertibility of upper triangular operator matrices. *Linear Multilinear Algebra*, 66(2):260–267, 2018.
[6] D.S. Djordjević and P.S. Stanimirović. Iterative methods for computing generalized inverses related with optimization methods. *J. Aust. Math. Soc.*, 78:257–272, 2005.
[7] M.P. Drazin. Pseudo-inverses in associative rings and semigroups. *Amer. Math. Monthly*, 65:506–514, 1958.
[8] D.E. Ferreyra, V. Orquera, and N. Thome. A weak group inverse for rectangular matrices. *Revista de la Real Academia de Cliencias Exactas, Fisicas y Naturales. Ser. A. Matemáticas*, 113:3727–3740, 2019.
[9] S. Gigola, L. Lebtahi, and N. Thome. The inverse eigenvalue problem for a Hermitian reflexive matrix and optimization problem. *J. Comput. Appl. Math.*, 291:449–457, 2016.
[10] R.E. Hartwig. The weighted $*$-core-nilpotent decomposition. *Linear Algebra Appl.*, 211:101–111, 1994.
[11] I.I. Kyrchei, Determinantal representations of the weighted core-EP, DMP, MPD, and CMP inverses. *J. Math.*, Article ID 9816038, 2020.
[12] C. Lin, J. Chen, B. Chen, L. Guo, Q.G. Wang, and Z. Zhang. Stabilization for a class of rectangular descriptor systems via time delayed dynamic compensator. *J. Franklin Inst.*, 356(4):1944–1954, 2019.
[13] X.J. Liu and N.P. Cai. High-order iterative methods for the DMP inverse. *J. Math. Article*, ID: 8175935, 2018.
[14] X.J. Liu, C.M. Hu, and Y.M. Yu. Further results on iterative methods for computing generalized inverses. *J. Comput. Appl. Math.*, 234:684–694, 2010.
[15] H. Ma and T. Li. Characterizations and representations of the core inverse and its applications. *Linear Multilinear Algebra*, 2019, https://doi.org/10.1080/03081087.2019.1588847.
[16] S.B. Malik and N. Thome. On a new generalized inverse for matrices of an arbitrary index. *Appl. Math. Comput.*, 226:575–580, 2014.
[17] L.S. Meng. The DMP inverse for rectangular matrices. *Filomat*, 31(19):6015–6019, 2017.
[18] D. Mosić. Drazin-star and star-Drazin matrices. *Results Math.*, Doi: 10.1007/s00025-020-01191-7.
[19] R. Penrose. A generalized inverse for matrices. *Proc. Cambridge Philos. Soc.*, 51:406–413, 1955.

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Volume 37, pp. 72-87, January 2021.

[20] X.P. Sheng. Computation of weighted Moore-Penrose inverse through Gauss-Jordan elimination on bordered matrices. *Appl. Math. Comput.*, 323:64–74, 2018.

[21] F. Soleimani, P.S. Stanimirović, and F. Soleymani. Some matrix iterations for computing generalized inverses and balancing chemical equations. *Algorithms*, 8:982–998, 2015.

[22] N. Thome. A simultaneous canonical form of a pair of matrices and applications involving the weighted Moore-Penrose inverse. *Appl. Math. Letters*, 53:112–118, 2016.

[23] G.R. Wang, Y.M. Wei, and S.Z. Qiao. *Generalized Inverses: Theory and Computations*. Science Press, Beijing, 2004.

[24] S.Z. Xu, J.L. Chen, and X.X. Zhang. New characterizations for core inverses in rings with involution. *Front. Math. China*, 12(1):231–246, 2017.

[25] H.H. Zhu and P. Patrício. On DMP inverses and m-EP elements in rings. *Linear Multilinear Algebra*, 67:756–766, 2019.