The weak core inverse

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Abstract

In this paper, we introduce a new generalized inverse, called weak core inverse (or, in short, WC inverse) of a complex square matrix. This new inverse extends the notion of the core inverse defined by O.M. Baksalary and G. Trenkler in 2010. We investigate characterizations, representations, and properties for this generalized inverse. In addition, we introduce weak core matrices (or, in short, WC matrices) and we show that these matrices form a more general class than that given by the known weak group matrices, recently investigated by H. Wang and X. Liu.

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1 Introduction

The classical Moore-Penrose inverse [22] and Drazin inverse [10] were defined in the fifties and have been thoroughly studied since then. On the other hand, generalized inverses such as core inverses [2], BT inverses [3], core EP inverses [18], DMP inverses [16], CMP inverses [19], WG inverses [27], etc., were introduced in the last decade and, nowadays, they attract the attention of many researchers. In contrast to the classical ones, these recent generalized inverses (from 2010 onwards) allow us to tackle new problems and are opening up new horizons in this field both theoretical and applied.

Generalized inverses of matrices are applied in areas as varied as Markov chains [4], coding theory [29], chemical equations [23], robotics [9], geology [5], etc. Matrix partial orders is another important area in which generalized inverses are an essential tool towards which attention is directed [7, 8, 20, 25, 32].

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Because the core inverse was defined only for the class of index-one matrices and since the aforementioned extensions have enhanced their understanding, there is an obvious desire to extend it to new arbitrary-index classes. Motivated by these facts, our main aim is to introduce and investigate a new generalized inverse, namely the weak core inverse.

Let us now recall notions of several generalized inverses and notations.

We denote the set of all $m \times n$ complex matrices by $\mathbb{C}^{m\times n}$. For $A \in \mathbb{C}^{m\times n}$, the symbols A^*, A^{-1}, $\mathrm{rk}(A)$, $\mathcal{N}(A)$, and $\mathcal{R}(A)$ will stand for the conjugate transpose, the inverse ($m = n$), the rank, the kernel, and the range space of A, respectively. Moreover, I_n will refer to the $n \times n$ identity matrix. A matrix $X \in \mathbb{C}^{n\times m}$ that satisfies the equality $AXA = A$ is called an *inner inverse* or *$\{1\}$-inverse* of A, and a matrix $X \in \mathbb{C}^{n\times m}$ that satisfies the equality $XAX = X$ is called an *outer inverse* or *$\{2\}$-inverse* of A. An $n \times m$ matrix X satisfying $AXA = A$ and $XAX = X$ is called a *reflexive inverse* or *$\{1, 2\}$-inverse* of A.

For $A \in \mathbb{C}^{m\times n}$, the *Moore-Penrose inverse* of A is the unique matrix $A^\dagger \in \mathbb{C}^{n\times m}$ satisfying the following four equations [1]

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A.$$

The Moore-Penrose inverse can be used to represent orthogonal projectors $P_A := AA^\dagger$ and $Q_A := A^\dagger A$ onto $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$, respectively.

Let $A \in \mathbb{C}^{n\times n}$. The smallest nonnegative integer k for which $\mathcal{R}(A^k) = \mathcal{R}(A^{k+1})$ is called the *index* of A and is denoted by $\mathrm{Ind}(A)$.

We recall that the *Drazin inverse* of $A \in \mathbb{C}^{n\times n}$ is the unique matrix $A^d \in \mathbb{C}^{n\times n}$ satisfying the following three equations [1]

$$A^d AA^d = A^d, \quad AA^d = A^d A, \quad A^d A^{k+1} = A^k,$$

where $k = \mathrm{Ind}(A)$.

The $k = 0$ case corresponds to nonsingular matrices. If $k = 1$, the Drazin inverse of A is called the *group inverse* of A and is denoted by $A^\#$.

The well-known class of EP matrices is defined by the square complex matrix A that commutes with its Moore-Penrose inverse $A^\dagger$, that is,

$$\mathbb{C}_n^{\mathrm{EP}} = \{A \in \mathbb{C}^{n\times n} : AA^\dagger = A^\dagger A\}.$$

In 2010, the core inverse was introduced in the paper [O. Baksalary and G. Trenkler, *Core inverse of matrices*, Linear and Multilinear Algebra, 58 (6) (2010) 681-697] (see [2]), which considerably revitalized this research area. For a given matrix $A \in \mathbb{C}^{n\times n}$, the core inverse of A is defined to be the unique matrix $A^{\textcircled{\#}} \in \mathbb{C}^{n\times n}$ satisfying the conditions

$$AA^{\textcircled{\#}} = P_A, \quad \mathcal{R}(A^{\textcircled{\#}}) \subseteq \mathcal{R}(A). \tag{1}$$

It was proved that a matrix A is core invertible if and only if $\mathrm{Ind}(A) \leq 1$.

The symbol $\mathbb{C}_n^{\mathrm{CM}}$ denotes the set of so called core matrices (also referred to as group matrices), which is given by all the $n \times n$ complex matrices for which A is core invertible, that is,

$$\mathbb{C}_n^{\mathrm{CM}} = \{A \in \mathbb{C}^{n\times n} : \mathrm{rk}(A) = \mathrm{rk}(A^2)\}.$$

In 2014, O. Baksalary and G. Trenkler also introduced the *BT inverse* of $A \in \mathbb{C}^{n\times n}$ (originally referred to as generalized core inverse) as the matrix

$$A^{\diamond} := (AP_A)^{\dagger}. \tag{2}$$

Two other new generalizations of the core inverse for $n \times n$ complex matrices of arbitrary index k were also introduced in 2014. K. Manjunatha Prasad and K.S. Mohana [18] defined the *core EP inverse* of $A \in \mathbb{C}^{n\times n}$ as the unique matrix $A^{\textcircled{\dagger}} \in \mathbb{C}^{n\times n}$ satisfying

$$A^{\textcircled{\dagger}} A A^{\textcircled{\dagger}} = A^{\textcircled{\dagger}} \qquad \text{and} \qquad \mathcal{R}(A^{\textcircled{\dagger}}) = \mathcal{R}((A^{\textcircled{\dagger}})^*) = \mathcal{R}(A^k).$$

In addition, it was proved that $A^{\textcircled{\dagger}} = A^k \left((A^*)^k A^{k+1}\right)^{\dagger} (A^*)^k$. And, S. Malik and N. Thome [16] introduced the *DMP inverse* of $A \in \mathbb{C}^{n\times n}$ as the unique matrix $A^{d,\dagger} \in \mathbb{C}^{n\times n}$ satisfying

$$A^{d,\dagger} A A^{d,\dagger} = A^{d,\dagger}, \qquad A^{d,\dagger} A = A^d A, \quad \text{and} \quad A^k A^{d,\dagger} = A^k A^{\dagger}. \tag{3}$$

Moreover, it was proved that $A^{d,\dagger} = A^d A A^{\dagger}$. The authors also introduced another outer inverse associated with a square matrix, namely $A^{\dagger,d} = A^{\dagger} A A^d$ called the *dual DMP inverse* of A. For computational aspects of core and core EP inverses, we refer the reader to [30] and for those of DMP inverses to [15].

From 2018 onwards, three new generalized inverses were introduced for complex square matrices, namely CMP inverses, WG inverses, and MPCEP inverses. Firstly, M. Mehdipour and A. Salemi [19] introduced the *CMP inverse* of $A \in \mathbb{C}^{n\times n}$ as the matrix

$$A^{c,\dagger} := Q_A A^d P_A. \tag{4}$$

Secondly, H. Wang and J. Chen [27] by using the core EP inverse of a matrix, introduced the *weak group inverse* (or, in short, *WG inverse*) of a matrix $A \in \mathbb{C}^{n\times n}$ as the unique matrix $A^{\textcircled{w}} \in \mathbb{C}^{n\times n}$ satisfying

$$A(A^{\textcircled{w}})^2 = A^{\textcircled{w}} \quad \text{and} \quad AA^{\textcircled{w}} = A^{\textcircled{\dagger}} A. \tag{5}$$

If $k = 1$, the WG inverse and the group inverse coincide.

Recently, H. Wang and X. Liu [28] introduced a new class of matrices defined by the square complex matrix A that commutes with its WG inverse $A^{ⓦ}$, that is,

$$\mathbb{C}_n^{\mathrm{WG}} = \{A \in \mathbb{C}^{n\times n} : AA^{ⓦ} = A^{ⓦ}A\}.$$

A matrix $A \in \mathbb{C}_n^{\mathrm{WG}}$ is called a *WG matrix*.

Finally, J. Chen, D Mosić, and S. Xu [6] defined the *MPCEP inverse* of $A \in \mathbb{C}^{n\times n}$ as the matrix

$$A^{\dagger,\textcircled{\dagger}} := Q_A A^{\textcircled{\dagger}}. \tag{6}$$

By using the core EP decomposition, in this paper we introduce one more generalization of the core inverse of a square matrix. Note that, while the core inverse is restricted to index-one matrices, this new generalized inverse exists for any square matrix. We also give some of its characterizations, representations, properties and applications. Furthermore, by using this new notion of generalized inverse we define and investigate a new class of matrices which extend the class of WG matrices.

This paper is organized as follows. In Section 2, some preliminaries are given. Section 3 introduces the generalized core inverse. In Section 4, we derive some properties of WC inverses. Section 5 is devoted to the study of weak core matrices. Section 6 offers some more characterizations of WC inverses. Finally, Section 7 analyzes the WC binary relation.

2 Preliminaries

In this section, we present some preliminary results.

Lemma 2.1. *[26, Core EP decomposition] Let $A \in \mathbb{C}^{n\times n}$ with Ind$(A) = k$. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that*

$$A = A_1 + A_2, \quad A_1 := U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^*, \quad A_2 := U\begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix}U^*, \tag{7}$$

where T is nonsingular with $t := rk(T) = rk(A^k)$ and N is nilpotent of index k. The representation of A given in (7) satisfies Ind$(A_1) \leq 1$, $A_2^k = 0$, and $A_1^*A_2 = A_2A_1 = 0$. Moreover, it is unique in this last sense and is called the core EP decomposition of A.

In addition, the core EP inverse of A is

$$A^{\textcircled{\dagger}} = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*. \tag{8}$$

Notice that if A is nonsingular (that is, $k = 0$) in Lemma 2.1 then $t = n$. So, N and S are absent in (7), and $A = T$ (with $U = I_n$). Thus, (7) provides a powerful tool for analyzing singular matrices, which are the object of study in this paper. Henceforth, we can assume $\mathrm{Ind}(A) = k \geq 1$ when the core EP decomposition is applied.

From (7) and (8) it is easy to check that

$$A_1 = AA^{\textcircled{\dagger}}A \quad \text{and} \quad A_2 = A - AA^{\textcircled{\dagger}}A. \tag{9}$$

Theorem 2.2. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix written as in (7) and let $\Delta := (TT^* + S(I_{n-t} - Q_N)S^*)^{-1}$ and $\widetilde{T} := \sum_{j=0}^{k-1} T^j S N^{k-1-j}$. Then*

(a) [12, Theorem 3.7] the Moore-Penrose inverse of A is

$$A^\dagger = U \begin{bmatrix} T^*\Delta & -T^*\Delta S N^\dagger \\ (I_{n-t} - Q_N)S^*\Delta & N^\dagger - (I_{n-t} - Q_N)S^*\Delta S N^\dagger \end{bmatrix} U^*,$$

(b) [12, Theorem 3.9] the Drazin inverse of A is

$$A^d = U \begin{bmatrix} T^{-1} & T^{-(k+1)}\widetilde{T} \\ 0 & 0 \end{bmatrix} U^*,$$

(c) [12, Theorem 3.11] the DMP inverse of A is

$$A^{d,\dagger} = U \begin{bmatrix} T^{-1} & T^{-(k+1)}\widetilde{T}P_N \\ 0 & 0 \end{bmatrix} U^*,$$

(d) [12, Corollary 3.12] the CMP of A is

$$A^{c,\dagger} = U \begin{bmatrix} T^*\Delta & T^*\Delta T^{-k}\widetilde{T}P_N \\ (I_{n-t} - Q_N)S^*\Delta & (I_{n-t} - Q_N)S^*\Delta T^{-k}\widetilde{T}P_N \end{bmatrix} U^*,$$

(e) [27, Theorem 3.1] the weak group inverse of A is

$$A^{\textcircled{w}} = U \begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^*,$$

(f) [6, Lemma 2.3] the MPCEP inverse of A is

$$A^{\dagger,\textcircled{\dagger}} = U \begin{bmatrix} T^*\Delta & 0 \\ (I_{n-t} - Q_N)S^*\Delta & 0 \end{bmatrix} U^*.$$

From (7) and Theorem 2.2 (e) we derive the following expressions for the projectors $AA^{ⓦ}$ and $A^{ⓦ}A$, which will be often used

$$AA^{ⓦ} = U \begin{bmatrix} I_t & T^{-1}S \\ 0 & 0 \end{bmatrix} U^* \quad \text{and} \quad A^{ⓦ}A = U \begin{bmatrix} I_t & T^{-1}S + T^{-2}SN \\ 0 & 0 \end{bmatrix} U^*. \tag{10}$$

Lemma 2.3. *[14, Hartwig-Spindelböck decomposition] Let $A \in \mathbb{C}^{n\times n}$ of rank $r > 0$. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that*

$$A = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*, \tag{11}$$

where $\Sigma = diag(\sigma_1 I_{r_1}, \sigma_2 I_{r_2}, \ldots, \sigma_t I_{r_t})$ is the diagonal matrix of singular values of A, $\sigma_1 > \sigma_2 > \cdots > \sigma_t > 0$, $r_1 + r_2 + \cdots + r_t = r$, and $K \in \mathbb{C}^{r\times r}$, $L \in \mathbb{C}^{r\times(n-r)}$ satisfy $KK^* + LL^* = I_r$.

Theorem 2.4. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix written as in (11). Then*

(a) [2, Formula (1.13)] the Moore-Penrose inverse of A is

$$A^\dagger = U \begin{bmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{bmatrix} U^*,$$

(b) [11, Formula (10)] the core EP inverse of A is

$$A^{\textcircled{\dagger}} = U \begin{bmatrix} (\Sigma K)^{\textcircled{\dagger}} & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

3 Definitions and characterizations of generalized core inverses

In this section we introduce a new generalized inverse of $A \in \mathbb{C}^{n\times n}$ by using the Drazin, the WG, and the Moore-Penrose inverses of A. We begin with a definition and some properties.

Definition 3.1. *Let $A \in \mathbb{C}^{n\times n}$. The weak core part of A, denoted by C, is defined to be the product $C := AA^{ⓦ}A$.*

We note that from (10) it is easy to obtain

$$C = AA^{ⓦ}A = U \begin{bmatrix} T & S + T^{-1}SN \\ 0 & 0 \end{bmatrix} U^*. \tag{12}$$

Remark 3.2. *Notice that if $A \in \mathbb{C}_n^{\mathrm{CM}}$ is written as in (7) then $N = 0$. Thus, the weak core part of A is given by*

$$C = U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} U^* = A_1 = A.$$

Now, we establish some interesting properties of the weak core part of A.

Proposition 3.3. *Let $A \in \mathbb{C}^{n\times n}$ with $Ind(A) = k$. Then the weak core part C of A satisfies the following properties:*

(a) $A^\dagger$ is an inner inverse of C,

(b) A^d is an inner inverse of C,

(c) $CA^k = A^{k+1}$,

(d) $C^k = U \begin{bmatrix} T^k & T^{k-1}S + T^{k-2}SN \\ 0 & 0 \end{bmatrix} U^*$,

(e) $C = A^{\textcircled{\dagger}} A^2$,

(f) $(I - AA^d)C = 0$,

(g) $(I - A^{\textcircled{\dagger}} A)C = (I - AA^{\textcircled{w}})C = 0$,

(h) $C(I - Q_A) = 0$.

As a consequence, C can be represented in terms of A^d, $A^{\textcircled{\dagger}}$, $A^{\textcircled{w}}$, and $A^\dagger$ as $C = AA^dY$, $C = A^{\textcircled{\dagger}} AZ = AA^{\textcircled{w}} Z$, and $C = WQ_A$, for arbitrary Y, Z, W of adequate sizes.

Proof. From [27, Remark 3.4 and Remark 3.5], we know that $A^{\textcircled{w}}$ is an outer inverse of A and

$$\mathcal{R}(A^{\textcircled{w}}) = \mathcal{R}(A^k) \quad \text{and} \quad A^{\textcircled{w}} A^{k+1} = A^k. \tag{13}$$

(a) Now it is clear that $CA^\dagger C = AA^{\textcircled{w}} AA^\dagger AA^{\textcircled{w}} A = AA^{\textcircled{w}} AA^{\textcircled{w}} A = AA^{\textcircled{w}} A = C$.
(b) We observe that $\mathcal{R}(AA^{\textcircled{w}}) = A\mathcal{R}(A^{\textcircled{w}}) = A\mathcal{R}(A^k) = \mathcal{R}(A^{k+1}) = \mathcal{R}(A^k)$, from where $AA^{\textcircled{w}} = A^kB$ for some matrix B. Then

$$CA^dC = AA^{\textcircled{w}} AA^d(AA^{\textcircled{w}})A = AA^{\textcircled{w}} A^{k+1}A^dBA = AA^{\textcircled{w}} A^kBA = AA^{\textcircled{w}} AA^{\textcircled{w}} A = AA^{\textcircled{w}} A = C.$$

(c) It directly follows that $CA^k = (AA^{\textcircled{w}} A)A^k = AA^{\textcircled{w}} A^{k+1} = A^{k+1}$.
(d) It follows from (12) after a simple computation.
(e) From (5) we have that $AA^{\textcircled{w}} = A^{\textcircled{\dagger}} A$, hence $C = AA^{\textcircled{w}} A = A^{\textcircled{\dagger}} A^2$.
(f) As in the proof of point (b) we obtain $AA^{\textcircled{w}} = A^kB$, for some matrix B. Pre- and postmultiplying this last equality by A^dA and A, respectively, we have $A^dAAA^{\textcircled{w}} A = A^dAA^kBA = A^kBA = AA^{\textcircled{w}} A$, thus the result follows.
(g) and (h) are obvious.

The consequences follow directly by solving (in C) the matrix equations from (f), (g), and (h) by means of [1, Theorem 1, p. 52] and using that the projectors $I - A^{\textcircled{\dagger}}A$, $I - AA^{\textcircled{w}}$, and $I - Q_A$ are $\{1\}$-inverses of themselves. $\square$

Let $A \in \mathbb{C}^{n\times n}$ and let C be its weak core part. We consider the following system of equations:

$$XAX = X, \quad AX = CA^{\dagger}, \quad \text{and} \quad XA = A^dC. \tag{14}$$

Theorem 3.4. *Let $A \in \mathbb{C}^{n\times n}$ and C as in Definition 3.1. The system (14) is consistent and its unique solution is the matrix $X = A^dCA^{\dagger}$.*

Proof. It is easy to see that the matrix $X := A^dCA^{\dagger}$ satisfies the three equations in system (14). In fact, Proposition 3.3 (f) implies $AX = AA^dCA^{\dagger} = CA^{\dagger}$. On the other hand, Proposition 3.3 (h) yields $XA = A^dCA^{\dagger}A = A^dC$. Finally, $XAX = A^dCX = A^dAA^{\textcircled{w}}AX = A^dAA^{\textcircled{w}}CA^{\dagger} = A^dCA^{\dagger} = X$, where the last equality follows from Proposition 3.3 (g).

For the uniqueness, we assume that X_1 and X_2 are two solutions of the system (14). From $AX_1 = CA^{\dagger} = AX_2$ and $X_1A = A^dC = X_2A$, we have $X_2 = X_2(AX_2) = X_2(AX_1) = (X_2A)X_1 = X_1AX_1 = X_1$. $\square$

Theorem 3.4 allows us to give the following definition.

Definition 3.5. *Let $A \in \mathbb{C}^{n\times n}$ with Ind$(A) = k$ and C as in Definition 3.1. The weak core inverse (or, in short, WC inverse) of A, denoted as $A^{\textcircled{w},\dagger}$, is defined to be the solution to the system (14).*

Theorem 3.6. *Let $A \in \mathbb{C}^{n\times n}$. Then*

$$A^{\textcircled{w},\dagger} = A^{\textcircled{w}}AA^{\dagger}. \tag{15}$$

Proof. From [27, Remark 3.4] we have $\mathcal{R}(A^{\textcircled{w}}) = \mathcal{R}(A^k)$, and so $A^{\textcircled{w}} = A^kZ$ for some matrix Z. Then,

$$A^{\textcircled{w},\dagger} = A^dCA^{\dagger} = A^dAA^{\textcircled{w}}AA^{\dagger} = A^dAA^kZAA^{\dagger} = A^kZAA^{\dagger} = A^{\textcircled{w}}AA^{\dagger}.$$

$\square$

In this way, the equality (15) justifies the notation $A^{\textcircled{w},\dagger}$ for the WC inverse in Definition 3.5.

Remark 3.7. *(a) Note that, if $A \in \mathbb{C}_n^{\mathrm{CM}}$ then the WC inverse coincides with the core inverse of A, that is, $A^{\textcircled{w},\dagger} = A^{\textcircled{\#}}$ (see [27, Remark 3.3]). This fact gives the name to the weak core inverse.*

(b) If $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$ then $\mathcal{R}(A^{\textcircled{w},\dagger}) = \mathcal{R}(A^k)$. In fact, by (14) we obtain

$$\mathcal{R}(A^{\textcircled{w},\dagger}) \subseteq \mathcal{R}(A^{\textcircled{w},\dagger}A) = \mathcal{R}(A^dC) \subseteq \mathcal{R}(A^d) = \mathcal{R}(A^k).$$

On the other hand, $\mathcal{R}(A^k) \subseteq \mathcal{R}(A^{\circledW,\dagger})$ *because applying* (14) *and using Proposition 3.3 (c) we have*

$$A^{\circledW,\dagger}A^{k+1} = (A^{\circledW,\dagger}A)A^k = A^dCA^k = A^dA^{k+1} = A^k. \tag{16}$$

Hence, $\mathcal{R}(A^k) = \mathcal{R}(A^{\circledW,\dagger})$.

(c) *Notice that the expression* $A^\dagger CA^d$ *provides a new representation for the dual DMP inverse of* A*, that is* $A^{\dagger,d} = A^\dagger CA^d$*. In fact, we know that* $A^d = A^kZ$ *for some matrix* Z *and also* $A^{\circledW}A^{k+1} = A^k$ *by (13). Thus,*

$$A^\dagger CA^d = A^\dagger AA^{\circledW}AA^d = A^\dagger AA^{\circledW}AA^kZ = A^\dagger AA^kZ = A^\dagger AA^d = A^{\dagger,d}.$$

We conclude that the expression $A^\dagger CA^d$ *can not be considered as a new generalized inverse of* A*. However, in view of Theorem 3.6, we can consider another (outer) inverse associated with a complex square matrix* A*, namely* $A^{\dagger,\circledW} := A^\dagger AA^{\circledW}$*. This new inverse will be called the dual weak core inverse (or, in short, dual WC inverse) of* A*. In particular, if* $A \in \mathbb{C}_n^{\mathrm{CM}}$*, this new generalized inverse coincides with the well-known dual core inverse* $A_{\#}$ *of* A *[18, Remark 3.4].*

Remark 3.8. *From* (16) *we have that* $A^{\circledW,\dagger}$ *is a weak Drazin inverse of* A *(see [4, Definition 9.7.1, p. 203]).*

Remark 3.9. *Notice that the weak core inverse is a particular case of the OMP inverse recently introduced by Mosić and Stanimirović in [21].*

The rest of the paper is devoted to studying the WC inverse.

In the following example, we check that the WC inverse provides a different class than those of Moore-Penrose, Drazin, BT, core EP, DMP, CMP, WG, and MPCEP inverses.

Example 3.10. *Let*

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 5 & 2 & 6 \\ 0 & -2 & -1 & -3 \end{bmatrix}.$$

It is easy to check that $Ind(A) = 3$*. The Moore-Penrose inverse* $A^\dagger$ *and the Drazin inverse* A^d *are*

$$A^\dagger = \begin{bmatrix} \frac{10}{19} & \frac{53}{209} & -\frac{35}{209} & -\frac{6}{209} \\ 0 & -\frac{7}{11} & \frac{4}{11} & \frac{1}{11} \\ \frac{9}{19} & \frac{80}{209} & -\frac{41}{209} & -\frac{13}{209} \\ -\frac{3}{19} & \frac{81}{209} & -\frac{18}{209} & -\frac{21}{209} \end{bmatrix} \quad \text{and} \quad A^d = \begin{bmatrix} 1 & 10 & 7 & 18 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Moreover, the BT inverse $A^{\diamond}$, the core EP inverse $A^{\textcircled{\dagger}}$, and the DMP inverse $A^{d,\dagger}$ are

$$A^{\diamond} = \begin{bmatrix} \frac{10}{19} & 0 & -\frac{27}{190} & \frac{9}{190} \\ 0 & 0 & \frac{3}{10} & -\frac{1}{10} \\ \frac{9}{19} & 0 & -\frac{3}{19} & \frac{1}{19} \\ -\frac{3}{19} & 0 & -\frac{9}{190} & \frac{3}{190} \end{bmatrix}, \; A^{\textcircled{\dagger}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \; \textit{and } A^{d,\dagger} = \begin{bmatrix} 1 & \frac{39}{11} & \frac{6}{11} & -\frac{15}{11} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In addition, the CMP inverse $A^{c,\dagger}$ and the WG inverse $A^{\textcircled{w}}$ are

$$A^{c,\dagger} = \begin{bmatrix} \frac{10}{19} & \frac{390}{209} & \frac{60}{209} & -\frac{150}{209} \\ 0 & 0 & 0 & 0 \\ \frac{9}{19} & \frac{351}{209} & \frac{54}{209} & -\frac{135}{209} \\ -\frac{3}{19} & -\frac{117}{209} & -\frac{18}{209} & \frac{45}{209} \end{bmatrix} \quad \textit{and} \quad A^{\textcircled{w}} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Finally, the MPCEP inverse $A^{\dagger,\textcircled{\dagger}}$, and the WC inverse $A^{\textcircled{w},\dagger}$ are

$$A^{\dagger,\textcircled{\dagger}} = \begin{bmatrix} \frac{10}{19} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{9}{19} & 0 & 0 & 0 \\ -\frac{3}{19} & 0 & 0 & 0 \end{bmatrix} \quad \textit{and} \quad A^{\textcircled{w},\dagger} = \begin{bmatrix} 1 & \frac{9}{11} & \frac{9}{11} & -\frac{6}{11} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In (14) we used three conditions to define the WC inverse. The next theorem provides a characterization of the WC inverse requiring only two conditions and from a more geometrical point of view.

Theorem 3.11. *Let $A \in \mathbb{C}^{n\times n}$ with $Ind(A) = k$ and C as in Definition 3.1. The system of conditions*

$$AX = CA^{\dagger} \quad \textit{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^k), \tag{17}$$

is consistent and it has the unique solution $X = A^{\textcircled{w},\dagger}$.

Proof. Let $X = A^{\textcircled{w},\dagger}$. Clearly, from (14) we obtain $AX = CA^{\dagger}$. On the other hand, according to Remark 3.7 (b), we have $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$. So, we deduce that $A^{\textcircled{w},\dagger}$ satisfies the two conditions in (17). In order to show that system (17) has a unique solution, assume that both X_1 and X_2 satisfy (17), that is, $AX_1 = CA^{\dagger} = AX_2$, $\mathcal{R}(X_1) \subseteq \mathcal{R}(A^k)$, and $\mathcal{R}(X_2) \subseteq \mathcal{R}(A^k)$. Since $A(X_1 - X_2) = 0$, we obtain $\mathcal{R}(X_1 - X_2) \subseteq \mathcal{N}(A) \subseteq \mathcal{N}(A^k)$. We also get $\mathcal{R}(X_1 - X_2) \subseteq \mathcal{R}(A^k)$. Therefore, $\mathcal{R}(X_1 - X_2) \subseteq \mathcal{N}(A^k) \cap \mathcal{R}(A^k) = \{0\}$ because A has index k. Thus, $X_1 = X_2$. □

Once again, we confirm that the WC inverse is a more general concept than that of the core inverse in light of Theorem 3.11 and Remark 3.7 (a). Clearly, the systems (1) and (17) (replacing X with $A^{\textcircled{w},\dagger}$ in the second) coincide provided that $A \in \mathbb{C}_n^{\mathrm{CM}}$.

Next, we present a representation for WC inverses by using the core EP decomposition.

Theorem 3.12. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix written as in (7). Then*

$$A^{\textcircled{w},\dagger} = U \begin{bmatrix} T^{-1} & T^{-2}SP_N \\ 0 & 0 \end{bmatrix} U^*.$$

Proof. From (7) and the expressions of $A^\dagger$ and Δ given in Theorem 2.2 (a), we have

$$\begin{aligned} AA^\dagger &= U \begin{bmatrix} [TT^* + S(I_{n-t} - Q_N)S^*]\Delta & -[TT^* + S(I_{n-t} - Q_N)S^*]\Delta SN^\dagger + SN^\dagger \\ 0 & P_N \end{bmatrix} U^* \\ &= U \begin{bmatrix} I_t & 0 \\ 0 & P_N \end{bmatrix} U^*. \end{aligned}$$

Consequently, from (15) and Theorem 2.2 (e) we obtain

$$\begin{aligned} A^{\textcircled{w},\dagger} &= A^{\textcircled{w}}(AA^\dagger) \\ &= U \begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_t & 0 \\ 0 & P_N \end{bmatrix} U^* \\ &= U \begin{bmatrix} T^{-1} & T^{-2}SP_N \\ 0 & 0 \end{bmatrix} U^*. \end{aligned}$$

□

The following theorem shows the power of the above canonical form and allows us to give other characterizations of the WC inverse.

Theorem 3.13. *Let $A \in \mathbb{C}^{n\times n}$ with Ind$(A) = k$ and C as in Definition 3.1. The system of equations*

$$XAX = X, \quad XA = A^{\textcircled{w}}A, \quad \text{and} \quad C^kX = C^kA^\dagger, \tag{18}$$

is consistent and $X = A^{\textcircled{w},\dagger}$ is its unique solution.

Proof. Let A be written as in (7) and let X be partitioned as

$$X = U \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} U^*,$$

according to the size of blocks in A. From (10), direct calculations show that the equation $XA = A^{\textcircled{w}}A$ is satisfied if and only if $X_1 = T^{-1}$ and $X_2N = T^{-2}SN$, $X_3 = 0$ and $X_4N = 0$. Therefore,

$$X = U \begin{bmatrix} T^{-1} & X_2 \\ 0 & X_4 \end{bmatrix} U^*,$$

and so $XAX = X$ is equivalent to

$$\begin{bmatrix} T^{-1} & X_2 + (T^{-1}S + T^{-2}SN)X_4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T^{-1} & X_2 \\ 0 & X_4 \end{bmatrix}$$

from where $X_4 = 0$. Hence,

$$X = U \begin{bmatrix} T^{-1} & X_2 \\ 0 & 0 \end{bmatrix} U^*.$$

On the other hand, by Proposition 3.3 (d) and Theorem 2.2 (a) we have that $C^k X = C^k A^\dagger$ is equivalent to

(i) $T^{k-1} = T^k T^* \Delta + T^{k-1} S(I_{n-t} - Q_N) S^* \Delta$,

(ii) $T^k X_2 = -T^k T^* \Delta S N^\dagger + T^{k-1} S N^\dagger - T^{k-1} S(I_{n-t} - Q_N) S^* \Delta S N^\dagger + T^{k-2} S P_N$.

Hence, by substituting (i) in (ii) we get

$$\begin{aligned} T^k X_2 &= -[T^k T^* \Delta + T^{k-1} S(I_{n-t} - Q_N) S^* \Delta] S N^\dagger + T^{k-1} S N^\dagger + T^{k-2} S P_N \\ &= -T^{k-1} S N^\dagger + T^{k-1} S N^\dagger + T^{k-2} S P_N \\ &= T^{k-2} S P_N. \end{aligned}$$

Now, the nonsingularity of T implies $X_2 = T^{-2} S P_N$, from where

$$X = U \begin{bmatrix} T^{-1} & T^{-2} S P_N \\ 0 & 0 \end{bmatrix} U^*.$$

Hence, from Theorem 3.12 we derive $X = A^{ⓦ,\dagger}$. □

Remark 3.14. *For $A \in \mathbb{C}^{n\times n}$ and C as in Definition 3.1, we notice that:*

(a) conditions in (18) are similar to those used to define the DMP inverse; we only have to change $A^{ⓦ}$ and C^k with A^d and A^k in (3), respectively.

(b) $AXA = C$. In fact, it immediately follows by premultiplying by A both sides of the equation $XA = A^{ⓦ} A$ in (18).

The next result shows that the role of A (close to X) in (14) can be changed with that of C. It also provides a 2-condition algebraic characterization for WC inverses.

Theorem 3.15. *Let $A \in \mathbb{C}^{n\times n}$ and C as in Definition 3.1. Then the following statements are equivalent:*

(a) X is the WC inverse of A, that is, $X = A^{Ⓦ,\dagger}$,

(b) $XCX = X$, $CX = CA^\dagger$, and $XC = A^dC$,

(c) $AX = CA^\dagger$ and $AX^2 = X$.

Proof. $(a) \Rightarrow (b)$ Let $X = A^{Ⓦ,\dagger}$. According to Proposition 3.3 (a) and (b) we have $XCX = A^d(CA^\dagger C)A^dCA^\dagger = A^d(CA^dC)A^\dagger = A^dCA^\dagger = X$, $CX = (CA^dC)A^\dagger = CA^\dagger$, and $XC = A^d(CA^\dagger C) = A^dC$.

$(b) \Rightarrow (c)$ By hypothesis, $X = X(CX) = (XC)A^\dagger = A^dCA^\dagger$. From Theorem 3.4, we get $AX = CA^\dagger$. Finally, the equation $AX^2 = X$ follows from an easy computation by using Theorem 3.12.

$(c) \Rightarrow (a)$ Since $AX^2 = X$, [13, Lemma 4.1] yields $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$. So, Theorem 3.11 assures that $X = A^{Ⓦ,\dagger}$. □

Let $A \in \mathbb{C}^{n\times n}$ of rank r. Let T be a subspace of $\mathbb{C}^n$ of dimension $s \leq r$, and let S be a subspace of $\mathbb{C}^n$ of dimension $n - s$. It is well known that A has an outer inverse X such that $\mathcal{R}(X) = T$ and $\mathcal{N}(X) = S$ if and only if $AT \oplus S = \mathbb{C}^n$, in which case X is unique and is denoted by $A^{(2)}_{T,S}$ [1, Theorem 14, p. 72].

Recall that the Moore-Penrose inverse, the Drazin inverse, and the group inverse are outer inverses of A with prescribed range and null space satisfying

$$A^\dagger = A^{(2)}_{\mathcal{R}(A^*),\mathcal{N}(A^*)}, \quad A^d = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^k)}, \quad \text{and} \quad A^\# = A^{(2)}_{\mathcal{R}(A),\mathcal{N}(A)}.$$

Also, representations with prescribed range and null space for other recent generalized inverses are known, such as for the core EP, DMP, and CMP inverses [13, Theorem 3.2]

$$A^{\textcircled{\dagger}} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*)}, \quad A^{d,\dagger} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^kA^\dagger)}, \quad \text{and} \quad A^{c,\dagger} = A^{(2)}_{\mathcal{R}(A^\dagger A^k),\mathcal{N}(A^kA^\dagger)}. \tag{19}$$

In the following result we give a new representation of the WC inverse as an outer inverse with prescribed range and null space.

Theorem 3.16. *Let $A \in \mathbb{C}^{n\times n}$ with $Ind(A) = k$. Then*

$$A^{Ⓦ,\dagger} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^2A^\dagger)}.$$

Proof. By Theorem 3.4 and Remark 3.7 (b) we have that $A^{Ⓦ,\dagger}$ is an outer inverse of A with $\mathcal{R}(A^{Ⓦ,\dagger}) = \mathcal{R}(A^k)$. On the other hand, we are going to prove that $\mathcal{N}(A^{Ⓦ,\dagger}) = \mathcal{N}\left((A^k)^*A^2A^\dagger\right)$ holds. In fact, by Theorem 3.4 and Proposition 3.3 (e) we get

$$\mathcal{N}(A^{Ⓦ,\dagger}) = \mathcal{N}(AA^{Ⓦ,\dagger}) = \mathcal{N}(CA^\dagger) = \mathcal{N}(A^{\textcircled{\dagger}}A^2A^\dagger).$$

Then $x \in \mathcal{N}(A^{ⓦ,\dagger})$ if and only if $A^2 A^\dagger x \in \mathcal{N}(A^{\textcircled{\dagger}}) = \mathcal{N}((A^k)^*)$, where the last equality is due to the first representation in (19). Therefore, $x \in \mathcal{N}(A^{ⓦ,\dagger})$ if and only if $x \in \mathcal{N}((A^k)^* A^2 A^\dagger)$. □

Corollary 3.17. *[24, Remark 9, p. 301] Let $A \in \mathbb{C}_n^{\mathrm{CM}}$. Then $A^{\textcircled{\#}} = A^{(2)}_{\mathcal{R}(A),\mathcal{N}(A^*)}$.*

Proof. It is evident that $\mathcal{N}(A^*) = \mathcal{N}(A^\dagger) \subseteq \mathcal{N}(A^* A^2 A^\dagger)$. For the opposite inclusion, if $x \in \mathcal{N}(A^* A^2 A^\dagger)$ then $A^* A^2 A^\dagger x = 0$, and thus $A^2 A^\dagger x = AA^\dagger A^2 A^\dagger x = (A^\dagger)^* A^* A^2 A^\dagger x = 0$, from where $A^\dagger x \in \mathcal{N}(A^2) = \mathcal{N}(A)$ because A has index at most one, and thus, $x \in \mathcal{N}(AA^\dagger) = \mathcal{N}(A^\dagger)$. This completes the proof. □

Next, we present another representation of the WC inverse by using the Hartwig-Spindelböck decomposition.

Theorem 3.18. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix written as in (11). Then*

$$A^{ⓦ,\dagger} = U \begin{bmatrix} (\Sigma K)^{ⓦ} & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

Proof. By Theorem 2.4 (a), we obtain that

$$AA^\dagger = U \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U^*. \tag{20}$$

On the other hand, by [27, Theorem 3.8], we know that $B^{ⓦ} = (B^{\textcircled{\dagger}})^2 B$, for all square matrices B. Thus, Theorem 2.4 (b) implies

$$A^{ⓦ} = U \begin{bmatrix} ((\Sigma K)^{\textcircled{\dagger}})^2 \Sigma K & ((\Sigma K)^{\textcircled{\dagger}})^2 \Sigma L \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} (\Sigma K)^{ⓦ} & ((\Sigma K)^{\textcircled{\dagger}})^2 \Sigma L \\ 0 & 0 \end{bmatrix} U^*. \tag{21}$$

Finally, from (15) we have $A^{ⓦ,\dagger} = A^{ⓦ} AA^\dagger$, and so the assertion follows directly from (20) and (21). □

4 Properties of the WC inverse

The next results state some representations and properties that the WC inverse inherits from the core inverse.

Theorem 4.1. *Let $A \in \mathbb{C}^{n\times n}$ with $Ind(A) = k$ and C as in Definition 3.1. Then*

(a) $A^{ⓦ,\dagger} = (AA^{\textcircled{\dagger}}A)^{\#} P_A$,

(b) $A^{ⓦ,\dagger} = (A^{\textcircled{\dagger}})^2 A P_A = (A^2)^{\textcircled{\dagger}} A P_A$,

(c) $A^{\text{ⓦ},\dagger} = A^k (A^{k+2})^{\oplus} A P_A$,

(d) $A^{\text{ⓦ},\dagger} = (A^2 P_{A^k})^\dagger A P_A$,

(e) $A^{\text{ⓦ},\dagger}$ *is a reflexive inverse of* C,

(f) $\text{rk}(A^{\text{ⓦ},\dagger}) = \text{rk}(A^d) = \text{rk}(A^k)$.

Proof. Items (a)-(d) are direct consequences of (15) and [27, Theorem 3.8 and Theorem 3.9].
(e) We must prove that $CA^{\text{ⓦ},\dagger}C = C$ and $A^{\text{ⓦ},\dagger}CA^{\text{ⓦ},\dagger} = A^{\text{ⓦ},\dagger}$. In fact, by parts (a) and (b) of Proposition 3.3 we have

$$CA^{\text{ⓦ},\dagger}C = CA^d(CA^\dagger C) = CA^dC = C.$$

The other equality can be proved similarly.
(f) It is well known that $\text{rk}(A^d) = \text{rk}(A^k)$. On the other hand, $\text{rk}(A^{\text{ⓦ},\dagger}) = \text{rk}(A^k)$ is a direct consequence of Remark 3.7 (b).

This completes the proof. □

Notice that item (d) in Theorem 4.1 allows us to obtain the WC inverse only by means of the Moore-Penrose inverse, which is included in every computational package.

Theorem 4.2. *Let* $A \in \mathbb{C}^{n\times n}$ *with Ind*$(A) = k$ *and* C *as in Definition 3.1. Then*

(a) $AA^{\text{ⓦ},\dagger}$ *is the oblique projector onto the column space of* A^k *along the null space of* $(A^k)^*A^2A^\dagger$ *satisfying* $AA^{\text{ⓦ},\dagger} = CA^{\text{ⓦ},\dagger}$;

(b) $A^{\text{ⓦ},\dagger}A$ *is the oblique projector onto the column space of* A^k *along the null space of* $(A^k)^*A^2$ *satisfying* $A^{\text{ⓦ},\dagger}A = A^{\text{ⓦ},\dagger}C$;

Proof. Since, by definition, $A^{\text{ⓦ},\dagger}$ is an outer inverse of A, we obtain that $AA^{\text{ⓦ},\dagger}$ and $A^{\text{ⓦ},\dagger}A$ are idempotents and $\mathcal{N}(AA^{\text{ⓦ},\dagger}) = \mathcal{N}(A^{\text{ⓦ},\dagger})$ and $\mathcal{R}(A^{\text{ⓦ},\dagger}A) = \mathcal{R}(A^{\text{ⓦ},\dagger})$. Therefore, Theorem 3.16 implies $\mathcal{N}(AA^{\text{ⓦ},\dagger}) = \mathcal{N}\left((A^k)^*A^2A^\dagger\right)$ and $\mathcal{R}(A^{\text{ⓦ},\dagger}A) = \mathcal{R}(A^k)$.
(a) According to Remark 3.7 (b) we have $\mathcal{R}(AA^{\text{ⓦ},\dagger}) = A\mathcal{R}(A^{\text{ⓦ},\dagger}) = A\mathcal{R}(A^k) = \mathcal{R}(A^{k+1}) = \mathcal{R}(A^k)$. On the other hand, by the definition of the WC inverse and Proposition 3.3 (b) we obtain $AA^{\text{ⓦ},\dagger} = CA^\dagger = CA^dCA^\dagger = CA^{\text{ⓦ},\dagger}$.
(b) First, we are going to prove that $\mathcal{N}(A^{\text{ⓦ},\dagger}A) = \mathcal{N}((A^k)^*A^2)$ holds. In fact, $x \in \mathcal{N}(A^{\text{ⓦ},\dagger}A)$ if and only if $Ax \in \mathcal{N}(A^{\text{ⓦ},\dagger}) = \mathcal{N}\left((A^k)^*A^2A^\dagger\right)$. Therefore, $x \in \mathcal{N}(A^{\text{ⓦ},\dagger}A)$ if and only if $x \in \mathcal{N}((A^k)^*A^2A^\dagger A) = \mathcal{N}((A^k)^*A^2)$.
Finally, by the definition of the WC inverse and Proposition 3.3 (a) we get $A^{\text{ⓦ},\dagger}A = A^dC = A^dCA^\dagger C = A^{\text{ⓦ},\dagger}C$. □

5 Weak core matrix

In this section, we investigate a new class of matrices that is more general than that known as weak group matrices (that is, $AA^{ⓦ} = A^{ⓦ}A$ or equivalently $A^{ⓦ} = A^d$). Using the notion of weak core inverses we introduce and study this new class of matrices by extending the results given in [28].

As we will see in this Section 5 and in Section 6, extensions of the above equivalence can be done in two directions. Firstly, inspired by the equality $A^{ⓦ} = A^d$, we start with Definition 5.1. In Section 6, we will find equivalent conditions that extend the equality $AA^{ⓦ} = A^{ⓦ}A$ by giving characterizations of $AA^{ⓦ,\dagger} = A^{ⓦ,\dagger}A$.

Definition 5.1. *A matrix $A \in \mathbb{C}^{n\times n}$ is called a weak core matrix (or, in short, WC matrix) if $A^{ⓦ,\dagger} = A^{d,\dagger}$.*

The set of all $n \times n$ WC matrices is denoted by $\mathbb{C}_n^{\mathrm{WC}}$, that is

$$\mathbb{C}_n^{\mathrm{WC}} = \{A \in \mathbb{C}^{n\times n} : A^{ⓦ,\dagger} = A^{d,\dagger}\}.$$

The following proposition shows that the class $\mathbb{C}_n^{\mathrm{WC}}$ contains at least all matrices having index at most 2.

Proposition 5.2. *Let $A \in \mathbb{C}^{n\times n}$ with Ind$(A) = k$. If $k \leq 2$ then $A \in \mathbb{C}_n^{\mathrm{WC}}$.*

Proof. If $k = 0$, that is, A is nonsingular, clearly $A^{ⓦ,\dagger} = A^{d,\dagger} = A^{-1}$, and so $A \in \mathbb{C}_n^{\mathrm{WC}}$. Otherwise, the equality $A^{ⓦ,\dagger} = A^{d,\dagger}$ is a direct consequence of Theorem 3.18 and [16, Theorem 2.5, Lemma 2.8]. □

The next theorem offers a characterization of WC matrices by using the core EP decomposition.

Theorem 5.3. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix written as in (7). Then $A \in \mathbb{C}_n^{\mathrm{WC}}$ if and only if $SN^2 = 0$.*

Proof. First of all, we observe that the matrix N in (7) satisfies $N = 0$ and $N^2 = 0$ for $k = 1$ and $k = 2$, respectively. Then, the result clearly holds for these two cases considering Proposition 5.2. Assume $k \geq 3$. From Theorem 3.12 and Theorem 2.2 (c) we obtain that $A^{ⓦ,\dagger} = A^{d,\dagger}$ is equivalent to $T^{-2}SP_N = T^{-(k+1)}\widetilde{T}P_N$ which, in turn, is valid if and only if $T^{k-1}SN = \widetilde{T}N$ holds. Now, we are going to prove that $T^{k-1}SN = \widetilde{T}N$ is equivalent to $SN^2 = 0$, for which it is worth recalling that

$$\widetilde{T} = SN^{k-1} + TSN^{k-2} + T^2SN^{k-3} + \cdots + T^{k-3}SN^2 + T^{k-2}SN + T^{k-1}S. \tag{22}$$

It is obvious that $SN^2 = 0$ implies $T^{k-1}SN = \widetilde{T}N$. In order to establish the opposite implication, we assume that $T^{k-1}SN = \widetilde{T}N$ holds. From (22) and using that $N^k = 0$ and T is nonsingular, we get

$$SN^{k-1} + TSN^{k-2} + T^2SN^{k-3} + \cdots + T^{k-2}SN^2 = 0. \tag{23}$$

Postmultiplying by N^{k-3} both sides of (23), we have $SN^{k-1} = 0$. If $k = 3$, we get $SN^2 = 0$ as desired. Otherwise, it follows from (23) that

$$SN^{k-2} + TSN^{k-3} + \cdots + T^{k-3}SN^2 = 0.$$

Multiplying by N^{k-4} on the right of the equation above and applying $SN^{k-1} = 0$ we have $SN^{k-2} = 0$. Following in this way, we arrive at $SN^{k-3} = \cdots = SN^2 = 0$. $\square$

It is well known that $\mathbb{C}_n^{\text{EP}} \subseteq \mathbb{C}_n^{\text{CM}}$. Recently, the relationship $\mathbb{C}_n^{\text{CM}} \subseteq \mathbb{C}_n^{\text{WG}}$ was proved in [28, Theorem 4.7]. Next, we obtain a relationship between WG matrices $\mathbb{C}_n^{\text{WG}}$ and WC matrices $\mathbb{C}_n^{\text{WC}}$. Before giving our next result, we recall the following one.

Lemma 5.4. *[27, Corollary 3.12] Let $A \in \mathbb{C}^{n\times n}$ be a matrix written as in (7). Then $A \in \mathbb{C}_n^{\text{WG}}$ if and only if $SN = 0$.*

Theorem 5.5. *It verifies that $\mathbb{C}_n^{\text{WG}} \subseteq \mathbb{C}_n^{\text{WC}}$.*

Proof. It follows directly from Remark 3.7 (a), Proposition 5.2, Theorem 5.3, and Lemma 5.4. $\square$

The following example shows that the class $\mathbb{C}_n^{\text{WG}}$ is a proper subset of $\mathbb{C}_n^{\text{WC}}$.

Example 5.6. *Let*

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 5 & 2 & 6 \\ 0 & -2 & -1 & -3 \end{bmatrix}.$$

It is easy to see that $\operatorname{Ind}(A) = 3$. *Since $T = 1$, $S = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, and $N = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ 0 & -2 & -3 \end{bmatrix}$, we have $SN = \begin{bmatrix} 1 & 1 & 3 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ and $SN^2 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$.*

The next (proper) inclusions collect the information about EP matrices, core matrices, weak group matrices, and weak matrices

$$\mathbb{C}_n^{\text{EP}} \subsetneq \mathbb{C}_n^{\text{CM}} \subsetneq \mathbb{C}_n^{\text{WG}} \subsetneq \mathbb{C}_n^{\text{WC}}.$$

6 Further characterizations of the WC inverse

In the previous section we found necessary and sufficient conditions under which the WC inverse coincides with the DMP inverse. These equivalences allowed us to define the class of WC matrices that is wider than that of WG matrices. Now, it is of interest to inquire when the WC inverse coincides with another generalized inverse known in the literature.

Let the core EP decomposition of A be as in (7). A straightforward computation shows that

$$A^k = U \begin{bmatrix} T^k & \widetilde{T} \\ 0 & 0 \end{bmatrix} U^*, \tag{24}$$

where $\widetilde{T}$ is defined as in Theorem 2.2.

Theorem 6.1. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix with $Ind(A) = k$ written as in (7). Then*

(a) $A^{\text{ⓦ},\dagger} = A^\dagger$ if and only if $S = 0$ and $N = 0$.

(b) $A^{\text{ⓦ},\dagger} = A^d$ if and only if $\widetilde{T} = T^{k-1}SP_N$ (or equivalently $\widetilde{T}(I_{n-t} - P_N) = 0$ and $SN^2 = 0$).

(c) $A^{\text{ⓦ},\dagger} = A^{\text{†}\!\!\!\bigcirc}$ if and only if $SN = 0$.

(d) $A^{\text{ⓦ},\dagger} = A^{\text{ⓦ}}$ if and only if $S(I_{n-t} - P_N) = 0$.

(e) $A^{\text{ⓦ},\dagger} = A^{c,\dagger}$ if and only if $S(I_{n-t} - Q_N) = 0$ and $SN^2 = 0$.

(f) $A^{\text{ⓦ},\dagger} = A^{\dagger,\text{†}\!\!\!\bigcirc}$ if and only if $S(I_{n-t} - Q_N) = 0$ and $SN = 0$.

Proof. (a) According to Theorem 3.12 and Theorem 2.2 (a) we have that $A^{\text{ⓦ},\dagger} = A^\dagger$ if and only if the following conditions simultaneously hold:

(i) $T^{-1} = T^*\Delta$,

(ii) $T^{-2}SP_N = -T^*\Delta SN^\dagger$,

(iii) $0 = (I_{n-t} - Q_N)S^*\Delta$,

(iv) $0 = N^\dagger - (I_{n-t} - Q_N)S^*\Delta SN^\dagger$.

Now, we will show that (i)-(iv) hold if and only $S = 0$ and $N = 0$. In fact, (iii) and (iv) yield $N^\dagger = 0$, whence $N = 0$. Thus, (iii) and the nonsingularity of Δ imply $S^* = 0$, and so $S = 0$. Conversely, if $S = 0$ and $N = 0$, clearly (i)-(iv) are true.

(b) From Theorem 3.12 and Theorem 2.2 (b) we have that $A^{\text{ⓦ},\dagger} = A^d$ is equivalent to

$$\widetilde{T} = T^{k-1}SP_N. \tag{25}$$

Next, we are going to prove that (25) holds if and only if $\widetilde{T}(I_{n-t} - P_N) = 0$ and $SN^2 = 0$. Firstly, we assume that (25) holds. Postmultiplying by $I_{n-t} - P_N$ both sides of (25), we get $\widetilde{T}(I_{n-t} - P_N) = 0$. In order to prove the equality $SN^2 = 0$, we only consider the case $k \geq 3$; otherwise, it satisfies $N^2 = 0$. Thus, postmultiplying by N both sides of (25) we derive the equation (23). Now, the same reasoning as in the proof of Theorem 5.3 yields $SN^2 = 0$. Conversely, if $\widetilde{T}(I_{n-t} - P_N) = 0$ and $SN^2 = 0$, clearly $\widetilde{T} = \widetilde{T}P_N = T^{k-1}SP_N$.

(c) From Theorem 3.12 and (8), $A^{\textcircled{w},\dagger} = A^{\textcircled{\dagger}}$ if and only if $T^{-2}SP_N = 0$ if and only if $SN = 0$.

(d) It is an immediate consequence of Theorem 3.12 and Theorem 2.2 (e).

(e) From Theorem 3.12 and Theorem 2.2 (d) it is easy to see that $A^{\textcircled{w},\dagger} = A^{c,\dagger}$ if and only if the following conditions simultaneously hold:

(i) $T^{-1} = T^*\Delta$,

(ii) $T^{-2}SP_N = T^*\Delta T^{-k}\widetilde{T}P_N$,

(iii) $(I_{n-t} - Q_N)S^*\Delta = 0$,

(iv) $(I_{n-t} - Q_N)S^*\Delta T^{-k}\widetilde{T}P_N = 0$.

Hence, (iii) implies $S(I_{n-t} - Q_N) = 0$ since Δ is nonsingular. Also, $\Delta = (TT^*)^{-1}$, and by (ii) we get $T^{k-1}SN = \widetilde{T}N$ because T is nonsingular. As mentioned in part (b), to prove the equality $SN^2 = 0$, we only consider the case $k \geq 3$. Now, by using the expression of $\widetilde{T}$ we obtain (23). Now, proceeding as in part (b), we get $SN^2 = 0$. Conversely, since $S(I_{n-t} - Q_N) = 0$ we have $(I_{n-t} - Q_N)S^* = 0$ and so $\Delta = (TT^*)^{-1}$. Thus, (i), (iii), and (iv) hold. Furthermore, since by hypothesis $SN^2 = 0$, it is easy to check that (ii) is also true.

(f) It is an immediate consequence of Theorem 3.12 and Theorem 2.2 (f). $\square$

In [2, Theorem 3] the following equivalence was proved for index-one matrices:

$$AA^{\textcircled{\#}} = A^{\textcircled{\#}}A \quad \Longleftrightarrow \quad A^{\textcircled{\#}} = A^{\#}.$$

For A being a matrix of arbitrary index, in the following result we show that this statement remains valid when the superscripts $\textcircled{\#}$ and $\#$ are replaced with $\textcircled{w},\dagger$ and d, respectively.

Corollary 6.2. *Let $A \in \mathbb{C}^{n\times n}$ with $Ind(A) = k$. Then the following statements are equivalent:*

(a) $AA^{\textcircled{w},\dagger} = A^{\textcircled{w},\dagger}A$;

(b) $A^kA^{\textcircled{w},\dagger} = A^{\textcircled{w},\dagger}A^k$;

(c) $A^{\textcircled{w},\dagger} = A^d$.

Proof. Assume that $A \in \mathbb{C}^{n\times n}$ is written as in (7).

$(a) \Rightarrow (b)$ It is obvious.

$(b) \Rightarrow (c)$ By Theorem 3.12 and (24) we obtain $\widetilde{T} = T^{k-1}SP_N$. So, from Theorem 6.1 (b) we get $A^{\textcircled{w},\dagger} = A^d$.

$(c) \Rightarrow (a)$ It is evident. □

Corollary 6.3. *Let $A \in \mathbb{C}^{n\times n}$ with $Ind(A) = k$. Then $AA^{\textcircled{w},\dagger} = A^{\textcircled{w},\dagger}A$ if and only if $A \in \mathbb{C}_n^{\mathrm{WC}}$ and $A^kA^{d,\dagger} = A^{d,\dagger}A^k$.*

Proof. Let $A \in \mathbb{C}^{n\times n}$ be a matrix written as in (7). By [13, Theorem 3.13], we have that $A^kA^{d,\dagger} = A^{d,\dagger}A^k$ is equivalent to $\widetilde{T}(I_{n-t} - P_N) = 0$. So, Corollary 6.2, Theorem 6.1 (b), and Theorem 5.3 complete the proof. □

Remark 6.4. *In [13], the authors introduced k-DMP matrices by extending the concept of k-EP matrices (that is, $A^kA^\dagger = A^\dagger A^k$) studied in [17] and extended in [31]. We recall that a matrix $A \in \mathbb{C}^{n\times n}$ of index k is called a k-DMP matrix if A^k commutes with the DMP inverse $A^{d,\dagger}$ of A, that is,*

$$\mathbb{C}_n^{k,d} = \left\{A \in \mathbb{C}^{n\times n} : A^kA^{d,\dagger} = A^{d,\dagger}A^k\right\}.$$

From Corollary 6.3 we can deduce that $AA^{\textcircled{w},\dagger} = A^{\textcircled{w},\dagger}A$ is equivalent to $A^{\textcircled{w},\dagger} = A^{d,\dagger}$ whenever $A \in \mathbb{C}_n^{k,d}$.

Corollary 6.5. *Let $A \in \mathbb{C}^{n\times n}$. The following conditions are equivalent:*

(a) $AA^{\textcircled{w},\dagger} = A^{\textcircled{w},\dagger}A$,

(b) $\mathcal{N}\left((A^k)^*A^2A^\dagger\right) = \mathcal{N}((A^k)^*A^2)$*. In particular,* $\mathcal{N}((A^k)^*A^2)$ *is a* $A^\dagger$*-invariant subspace,*

(c) $CA^{\textcircled{w},\dagger} = A^{\textcircled{w},\dagger}C$.

Proof. It follows directly from Theorem 4.2. □

Corollary 6.6. *Let $A \in \mathbb{C}^{n\times n}$. Then*

(a) $A^{\textcircled{w},\dagger} = A^{d,\dagger} = A^d = A^{\textcircled{\dagger}} = A^{c,\dagger} = A^{\textcircled{w}} = A^{\dagger,\textcircled{\dagger}}$ *if and only if* $A^k \in \mathbb{C}_n^{\mathrm{EP}}$*, where* $Ind(A) = k$.

(b) $A^{\textcircled{w},\dagger} = A^\dagger$ *if and only if* $A \in \mathbb{C}_n^{\mathrm{EP}}$.

(c) $A^{\textcircled{w},\dagger} = A^{\textcircled{\dagger}}$ *if and only if* $A \in \mathbb{C}_n^{\mathrm{WG}}$.

Proof. Assume that $A \in \mathbb{C}^{n\times n}$ is written as in (7).

(a) If $A^k \in \mathbb{C}_n^{\mathrm{EP}}$, by [28, Theorem 2.3] we have $S = 0$, and by definition, $\widetilde{T} = 0$. Hence, Theorem 5.3 and Theorem 6.1 imply $A^{ⓦ,\dagger} = A^{d,\dagger} = A^d = A^{\textcircled{\dagger}} = A^{c,\dagger} = A^{ⓦ} = A^{\dagger,\textcircled{\dagger}}$. Conversely, we suppose that $A^{\textcircled{\dagger}} = A^{ⓦ}$ then by applying (8) and Theorem 2.2 (e) we have $S = 0$. Thus, again by [28, Theorem 2.3], we obtain that $A^k \in \mathbb{C}_n^{\mathrm{EP}}$.

(b) If $A^{ⓦ,\dagger} = A^\dagger$, then $S = 0$ and $N = 0$ by Theorem 6.1 (a). Now, by applying Theorem 2.2 (a) it is easy to check that $AA^\dagger = A^\dagger A$, that is, $A \in \mathbb{C}_n^{\mathrm{EP}}$. The converse follows from Remark 3.7 (a) and [2, Theorem 2 (iii)].

(c) It is an immediate consequence of Lemma 5.4 and Theorem 6.1 (c). □

7 The WC binary relation

This paper concludes with a remark that deals with matrix partial orders. Recall that a binary relation on a nonempty set which is reflexive and transitive is called a *pre-order*. A *partial order* is a pre-order that also satisfies the antisymmetric property.

As it was noted in [2, Section 3] when $A, B \in \mathbb{C}_n^{\mathrm{CM}}$, the binary relation

$$A \overset{\textcircled{\#}}{\leq} B \quad \text{if and only if} \quad A^{\textcircled{\#}}A = A^{\textcircled{\#}}B \text{ and } AA^{\textcircled{\#}} = BA^{\textcircled{\#}},$$

define a matrix partial ordering, which was called the *core ordering*.

In this section, we introduce the binary relation, called the WC relation, defined by

$$A \overset{ⓦ,\dagger}{\leq} B \quad \text{if and only if} \quad A^{ⓦ,\dagger}A = A^{ⓦ,\dagger}B \text{ and } AA^{ⓦ,\dagger} = BA^{ⓦ,\dagger},$$

where A and B are square matrices of the same size.

It is of interest to inquire whether the binary relation also becomes a matrix partial order or not.

The answer to this question is negative, and this can be confirmed by means of the following example.

Example 7.1. *Let*

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since

$$A^{ⓦ,\dagger} = B^{ⓦ,\dagger} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$A^{ⓦ,\dagger}A = A^{ⓦ,\dagger}B = B^{ⓦ,\dagger}B = B^{ⓦ,\dagger}A = B$ *and* $AA^{ⓦ,\dagger} = BA^{ⓦ,\dagger} = BB^{ⓦ,\dagger} = AB^{ⓦ,\dagger} = B^{ⓦ,\dagger}$. *Clearly,* $A \overset{ⓦ,\dagger}{\leq} B$ *and* $B \overset{ⓦ,\dagger}{\leq} A$ *hold. However, the WC relation is not antisymmetric because* $A \neq B$.

Furthermore, the WC relation is not a pre-order; the following example shows that it is not transitive.

Example 7.2. *Let*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 1 & 1 & 4 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

It is easy to see that $\mathrm{Ind}(A) = \mathrm{Ind}(B) = 2$, $\mathrm{Ind}(C) = 1$,

$$A^{ⓦ,\dagger} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad B^{ⓦ,\dagger} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Moreover,

$$A^{ⓦ,\dagger}A = A^{ⓦ,\dagger}B = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad AA^{ⓦ,\dagger} = BA^{ⓦ,\dagger} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$B^{ⓦ,\dagger}B = B^{ⓦ,\dagger}C = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad BB^{ⓦ,\dagger} = CB^{ⓦ,\dagger} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $A \overset{ⓦ,\dagger}{\leq} B$ *and* $B \overset{ⓦ,\dagger}{\leq} C$. *However, the inequality* $A \overset{ⓦ,\dagger}{\leq} C$ *is false since*

$$A^{ⓦ,\dagger}A \neq A^{ⓦ,\dagger}C = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, the relation $\overset{ⓦ,\dagger}{\leq}$ *is not transitive.*

We conclude this section by remarking that the WC relation is only reflexive. Consequently, we infer that, in general, the WC relation is neither a pre-order nor a partial order on $\mathbb{C}^{n\times n}$. Even more, the above examples and Proposition 5.2 allow us to derive that the WC relation is not a pre-order either on $\mathbb{C}_n^{\text{WC}}$.

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