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Common Best Proximity Point Results for T-GKT Cyclic ϕ-Contraction Mappings in Partial Metric Spaces with Some Applications

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Abstract: The aim of this paper is to derive some common best proximity point results in partial metric spaces defining a new class of symmetric mappings, which is a generalization of cyclic ϕ-contraction mappings. With the help of these symmetric mappings, the characterization of completeness of metric spaces given by Cobzas (2016) is extended here for partial metric spaces. The existence of a solution to the Fredholm integral equation is also obtained here via a fixed-point formulation for such mappings.

Keywords: common best proximity point; cyclic ϕ-contraction; partial metric space

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1. Introduction

The study of best proximity point theory has attracted the attention of several researchers over the last few decades. Extensive study is continuing in this area by different researchers with a view to applying the proximity point theory in various practical fields, viz., variational inequality problems, dynamical programming problems, solutions to differential and integral equations, matrix equations, etc. Recently, in [1], Isik et al. discussed the ϕ-best proximity point for (F, ϕ)-proximal contraction mappings demonstrating its application in a variational inequality. In [2], Usurelu et al. discussed the best proximity point of (EP)-operators, presenting a qualitative analysis with solid computational numerical simulation. All such recent developments suggest the significant importance of the study of proximity point analysis in different spaces.

The origin of the concept of proximity theory goes back to the classical best approximation theorem introduced by Fan [3] in 1969. After that, many extensions of Fan's results were derived in different directions [2,4,5]. In 2011, Basha et al. [6] showed the existence of common best proximity points in metric spaces. Various interesting works on common fixed point and common best proximity point theory in metric spaces can be found in the literature [7–9].

Motivated by these works, in this paper, we define a new class of mappings that are termed T-GKT cyclic ϕ-contraction mappings. We see that this new class of mappings is also symmetric in nature. Using such mappings, we establish some common best proximity point results in partial metric spaces and show some applications. The paper is arranged in the following sections.

In Section 2, the preliminary definitions and results are presented from the literature to obtain main results. In Section 3, the best proximity point results of the above-mentioned newly defined mappings are established in both a partial metric space and a metric space. In Section 4, one of the obtained results is employed to show the characterization of completeness of metric space and another result is applied to show the existence of a solution to the Fredholm integral equation. Finally, a conclusion is drawn in Section 5.

2. Preliminaries and Definitions

In this section, we present the basic definitions and results that are required to obtain the main results.

Definition 1 ([10]). *Let X be a non-empty set and $p : X \times X \longrightarrow [0, \infty)$ be a mapping such that for all $x, y, z \in X$*

$(P1)$ $p(x, x) = p(y, y) = p(x, y)$ *if and only if* $x = y$,
$(P2)$ $p(x, x) \leq p(x, y)$,
$(P3)$ $p(x, y) = p(y, x)$,
$(P4)$ $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.
Then p is said to be a partial metric and the pair (X, p) is called a partial metric space.

For example, on the set $\mathbb{R}$ of real numbers, $p_1(x, y) = 1 + |x - y|$, $p_2(x, y) = max\{|x|, |y|\}$, $p_3(x, y) = \begin{cases} 1, & x = y \\ l > 1, & x \neq y \end{cases}$; for all $x, y \in X$, are partial metrics.

Each partial metric p on a non-empty set X generates a T_0-topology λ_p on X with the family of open p-balls,

$$\{B_p(x, \epsilon) : x \in X : \epsilon > 0\},$$

where $B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}$, for all $x \in X$ and $\epsilon > 0$, as a base of λ_p [10,11].

Remark 1. *In a partial metric space (X, p) the limit of a sequence need not be unique. However, if $\{x_n\}$ and $\{y_n\}$ are sequences in a partial metric space (X, p) such that $x_n \to x$ and $y_n \to y$ then $p(x_n, y_n)$ need not converge to $p(x, y)$, i.e., p need not be continuous [11].*

Definition 2 ([10]). *In a partial metric space (X, p),*

(i) *a sequence $\{x_n\}$ is said to be convergent to a point x in X if and only if $\lim_{n \to \infty} p(x_n, x) = p(x, x)$,*
(ii) *a sequence $\{x_n\}$ is called a Cauchy sequence if and only if $\lim_{n,m \to \infty} p(x_n, x_m)$ exists and is finite.*

A partial metric space (X, p) is called complete if and only if every Cauchy sequence $\{x_n\}$ in X is convergent, with respect to the topology λ_p, to a point $x \in X$ such that

$$p(x, x) = \lim_{n,m \to \infty} p(x_n, x_m).$$

Definition 3 ([10]). *For a partial metric space (X, p), a mapping $d_p : X \times X \longrightarrow [0, \infty)$ defined by*

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y), \text{ for all } x, y \in X,$$

is a metric and it is called the induced metric.

Lemma 1 ([10]). *For a partial metric space (X, p),*

(i) *a sequence $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, d_p),*
(ii) *(X, p) is complete if and only if (X, d_p) is complete.*

Given two non-empty subsets A and B of a partial metric space (X, p), Zhang et al. [12] defined the distance p between A and B as

$$p(A, B) = inf\{p(x, y) : x \in A, y \in B\}.$$

Definition 4 ([12]). *Let A, B be two subsets of a partial metric space (X, p) and consider a mapping $S : A \longrightarrow B$. A point $x \in A$ is called a best proximity point of S if $p(x, Sx) = p(A, B)$.*

Example 1. *Consider the partial metric space (X, p) with $X = \mathbb{R}^+$ and $p(x, y) = max\{x, y\}$, for all $x, y \in X$. For $A = [2, 4]$ and $B = [0, 1]$, define $S : A \longrightarrow B$ by*

$$S(x) = \frac{x-2}{2}.$$

Then, $p(2, S(2)) = p(2, 0) = 2 = p(A, B)$, i.e., 2 is a best proximity point of S in A.

In 2011, Basha et al. [6] discussed common best proximity points in a metric space. Later in 2014, Zhang etal. [12] studied best proximity points in a partial metric space. In a similar manner, here we define the common best proximity point in a partial metric space as follows.

Definition 5. *Let A and B be two non-empty subsets of a partial metric space (X, p). For two mappings S,T : $A \longrightarrow B$, a point $x \in A$ is said to be a common best proximity point of S and T if*

$$p(x, Tx) = p(x, Sx) = p(A, B).$$

Example 2. *Consider two subsets $A = [1, 2]$, $B = [0, \frac{1}{2}]$ of $X = \mathbb{R}^+$ with partial metric $p(x, y) = max\{x, y\}$. Define $S, T : A \longrightarrow B$ by*

$$Sx = \frac{1}{1+x} \text{ and } Tx = \frac{1}{1+2x} \text{ for all } x \in A.$$

Then, $p(1, S1) = p(1, T1) = p(A, B) = 1$, i.e., 1 is a common best proximity point of S, T.

Definition 6 ([13])**.** *A mapping $S : A \cup B \longrightarrow A \cup B$ is cyclic if $S(A) \subseteq B$, $S(B) \subseteq A$.*

Example 3. *Consider the partial metric space (X, p) with $X = \mathbf{R}^2$ and the partial metric*

$$p((x_1, x_2), (y_1, y_2)) = 1 + max\{|x_1 - y_1|, |x_2 - y_2|\}, \text{ for all } (x_1, x_2) = (y_1, y_2) \in X.$$

Let A, B be two non-empty subsets of X such that $A = \{(x, 1) : 0 \leq x \leq 1\}$ and $B = \{(x, -1) : 0 \leq x \leq 1\}$. Suppose $S : A \cup B \longrightarrow A \cup B$ be defined by

$$S(x_1, x_2) = \begin{cases} (x_1, 1), & (x_1, x_2) \in B; \\ (x_1, -1), & (x_1, x_2) \in A. \end{cases}$$

Then S is a cyclic mapping on $A \cup B$.

3. Main Results

In [7], Basha et al. defined generalized cyclic contraction and showed the existence of common best proximity points in a metric space. Generalizing this notion, we introduce the following type of mappings in a partial metric space.

Definition 7. *Let A and B be two-non empty subsets of a partial metric space (X, p) and $\phi : [0, \infty) \longrightarrow [0, \infty)$ be a strictly increasing continuous mapping. For a self-mapping T on $A \cup B$, a cyclic mapping $S : A \cup B \longrightarrow A \cup B$ is said to be a T-generalized Kannan-type (GKT) cyclic ϕ-contraction on $A \cup B$ with respect to p if the following conditions are satisfied:*

(T1) $p(Tx, Ty) \geq \phi(p(Tx, Ty)) - \phi(p(A, B)) \geq 0,$

(T2) $p(Sx, Sy) \leq (1-\lambda)[p(Tx, Ty) + \phi(p(A, B)) - \phi(p(Tx, Ty))] + \frac{\lambda'}{2}[p(Tx, Sx) + p(Ty, Sy)];$

for all $x \in A, y \in B$ and for some $\lambda, \lambda' \in [0, 1)$.

Remark 2. *If $T = I$, then S reduces to a generalized Kannan-type (GKT) cyclic ϕ-contraction mapping, which was defined in [14]. Moreover, for $T = I$ and $\lambda' = 0$, this mapping reduces* *to a Banach contraction mapping. In addition, every cyclic Kannan contraction mapping is a T-generalized Kannan-type (GKT) cyclic ϕ-contraction taking T and ϕ as the identity mappings. Considering contraction mappings, there are many interesting results in the literature with different applications. In [15], Mlaiki et al. established some fixed-point results of the contraction mappings in a C^*-algebra valued partial b-metric space and applied these results to show the existence of a solution to the Fredholm integral equation. Since our newly defined mapping is a type of generalized contraction mapping, in the context of [15], fixed-point and proximity-point results can be investigated for mappings as defined in 7 in the case of C^*-algebra valued partial metric as well as b-metric spaces with different practical applications.*

Remark 3. *As another remark, we also emphasize the symmetric nature of the above defined type of mappings on the set $A \cup B$.*

Example 4. *Consider the partial metric space (X, p) with $X = \mathbb{R}^+$ and the partial metric $p(x, y) = max\{x, y\}$. Let $\phi(x) = \frac{x}{2}$, $A = [1, 2]$, and $B = [3, 5]$. The mappings $S, T : A \cup B \longrightarrow A \cup B$ are defined by*

$$Sx = \begin{cases} \frac{2x+5}{2}, & x \in [1,2]; \\ \frac{x-1}{2}, & x \in [3,5], \end{cases} \text{ and } Tx = \begin{cases} 5, & x \in [1,2]; \\ 2, & x \in [3,5]. \end{cases}$$

Then S is a T-GKT cyclic ϕ-contraction for $\lambda = \frac{3}{10}$ and $\lambda' = \frac{3}{5}$.

Example 5. *Consider $X = \{0, 1, 2, 3, 4\}$ with $p : X \times X \longrightarrow [0, \infty)$ defined by:*

$$p(0,0) = p(1,1) = p(2,2) = p(3,3) = p(4,4) = 0,$$

$$p(1,2) = p(2,1) = \frac{1}{15}, \; p(1,3) = p(3,1) = \frac{1}{12},$$

$$p(1,4) = p(4,1) = \frac{1}{11}, \; p(3,4) = p(4,3) = \frac{1}{10},$$

$$p(2,3) = p(3,2) = \frac{1}{14}, \; p(2,4) = p(4,2) = \frac{1}{13},$$

$$p(0,1) = p(1,0) = \frac{1}{9}, \; p(0,2) = p(2,0) = \frac{1}{8},$$

$$p(0,3) = p(3,0) = \frac{1}{7}, \; p(0,4) = p(4,0) = \frac{1}{6}.$$

Let $A = \{0, 1, 3\}$ and $B = \{2, 4\}$. Define $S, T : A \cup B \longrightarrow A \cup B$ by

$$Sx = \begin{cases} 2, & x \in A; \\ 1, & x \in B, \end{cases}$$

and

$$Tx = \begin{cases} 4, & x \in A; \\ 0, & x \in B. \end{cases}$$

Then S is a T-GKT cyclic ϕ-contraction for $\lambda = \frac{1}{5}$ and $\lambda' = \frac{3}{5}$ with $\phi(x) = 1 + x$. Considering a metric space (X, d), we can define the following version of T-GKT cyclic ϕ-contraction with respect to d.

Definition 8. *Let A and B be two non-empty subsets of a metric space (X, d) and $\phi : [0, \infty) \longrightarrow [0, \infty)$ be a strictly increasing continuous mapping. For a self-mapping T on $A \cup B$, a cyclic mapping $S : A \cup B \longrightarrow A \cup B$ is said to be a T-GKT cyclic ϕ-contraction on $A \cup B$ with respect to d if the following conditions are satisfied :*

(T1) $d(Tx, Ty) \geq \phi(d(Tx, Ty)) - \phi(d(A, B)) \geq 0,$

(T2) $d(Sx, Sy) \leq (1-\lambda)[d(Tx, Ty) + \phi(d(A, B)) - \phi(d(Tx, Ty))]$
$+ \frac{\lambda'}{2}[d(Tx, Sx) + d(Ty, Sy)]$; *for all* $x \in A, y \in B$ *and for some* $\lambda, \lambda' \in [0, 1)$.

Example 6. *Consider* $X = \mathbb{R}$ *with* $d(x, y) = |x - y|$ *for all* $x, y \in X$. *Let* $\phi(x) = \frac{x^2}{1+x}$, $A = [1, 3]$ *and* $B = [4, 6]$. *Let the mappings* $S, T : A \cup B \longrightarrow A \cup B$ *be defined by*

$$Sx = \begin{cases} 4, & x \in A; \\ 3, & x \in B, \end{cases} \text{ and } Tx = \begin{cases} 6, & x \in A; \\ 1, & x \in B. \end{cases}$$

Then S *is a* T*-GKT cyclic* ϕ*-contraction for* $\lambda = \lambda' = \frac{1}{2}$.

In all the subsequent results, we consider the partial metric p to be continuous, i.e., if $x_n \to x$ and $y_n \to y$ then $p(x_n, y_n) \to p(x, y)$.

In the following theorem, we prove the existence of a common best proximity point for a T-GKT cyclic ϕ-contraction in a partial metric space under adequate conditions. Here we consider that S and T commute, i.e., $STx = TSx$ for all $x \in A \cup B$.

Theorem 1. *Let* (X, p) *be a complete partial metric space and* A, B *be two non-empty subsets of* X *with* B *closed in* (X, p). *Let* T *be a self mapping on* $A \cup B$, $\phi : [0, \infty) \longrightarrow [0, \infty)$ *be a strictly increasing continuous mapping and* S *be a* T*-GKT cyclic* ϕ*-contraction mapping on* $A \cup B$ *with respect to* p *for some* $\lambda = \lambda' \in [0, 1)$. *If* B *is sequentially compact with respect to the induced metric* d_p, *and the following conditions are satisfied:*

(i) $S(A) \subseteq T(A) \subseteq B$,

(ii) S *and* T *commute and* T *is continuous,*

then there exists a common best proximity point of S *and* T *in* B.

Proof. From (i), for $x_0 \in A$, there exists $x_1 \in A$ such that

$$S(x_0) = T(x_1).$$

Again, since $S(x_1) \in T(A)$, there exists $x_2 \in A$ such that

$$S(x_1) = T(x_2).$$

In this way, we obtain a sequence $\{x_n\}$ in A with

$$S(x_n) = T(x_{n+1}).$$

Now, B being sequentially compact in (X, d_p), there exists a convergent sub-sequence $\{Sx_{n_k}\}$ of $\{Sx_n\}$ in B. Clearly, $\{Sx_{n_k}\}$ is a Cauchy sequence in (X, d_p) and so by Lemma 1(i), $\{Sx_{n_k}\}$ is a Cauchy sequence in (X, p). (X, p) being complete, $\{Sx_{n_k}\}$ converges to some $y \in B$. Thus $Sx_{n_k} \to y$ and also $Tx_{n_k} \to y$ as $k \to \infty$.

By the continuity of T,

$$TSx_{n_k} \to Ty \text{ and } TTx_{n_k} \to Ty \text{ as } k \to \infty.$$

Since S and T commute, we have $\lim_{k\to\infty} TSx_{n_k} = \lim_{k\to\infty} STx_{n_k}$.
Now,

$$p(Sx_{n_k}, STx_{n_k}) \leq (1-\lambda)[p(Tx_{n_k}, TTx_{n_k}) + \phi(p(A, B)) - \phi(p(Tx_{n_k}, TTx_{n_k}))]$$
$$+ \frac{\lambda}{2}[p(Sx_{n_k}, Tx_{n_k}) + p(STx_{n_k}, TTx_{n_k})],$$

i.e.,

$$(1-\lambda)\phi(p(Tx_{n_k}, TTx_{n_k})) \leq (1-\lambda)[p(Tx_{n_k}, TTx_{n_k}) + \phi(p(A,B))] + \frac{\lambda}{2}[p(Sx_{n_k}, Tx_{n_k}) + p(TSx_{n_k}, TTx_{n_k})] - p(Sx_{n_k}, STx_{n_k})$$
$$\leq (1-\lambda)[p(Tx_{n_k}, TTx_{n_k}) + \phi(p(A,B))] - p(Sx_{n_k}, TSx_{n_k}) + \frac{\lambda}{2}[p(Sx_{n_k}, Tx_{n_k}) + p(TSx_{n_k}, Sx_{n_k}) + p(Sx_{n_k}, TTx_{n_k}) - p(Sx_{n_k}, Sx_{n_k})].$$

Taking $k \to \infty$ in the above inequality (since the limit exists by the continuity of p and ϕ),

$$(1-\lambda)\lim_{k\to\infty}\phi(p(Tx_{n_k}, TTx_{n_k})) \leq (1-\lambda)[p(y,Ty) + \phi(p(A,B))] - p(y,Ty) + \frac{\lambda}{2}[p(y,y) + p(Ty,y) + p(y,Ty) - p(y,y)]$$
$$\Rightarrow \lim_{k\to\infty}\phi(p(Tx_{n_k}, TTx_{n_k})) \leq \phi(p(A,B)).$$

Again, $\phi(p(A,B)) \leq \phi(p(Tx_{n_k}, TTx_{n_k}))$.
Therefore, $\lim_{k\to\infty} p(Tx_{n_k}, TTx_{n_k}) = p(y,Ty) = p(A,B)$.
Using Definition 7 we obtain,

$$p(Sx_{n_k}, Sy) \leq (1-\lambda)p(Tx_{n_k}, Ty) + \frac{\lambda}{2}[p(Sx_{n_k}, Tx_{n_k}) + p(Sy,Ty)]. \quad (1)$$

Taking $k \to \infty$ on both sides of (1) (the existence of the limit is again followed by the continuity of p),

$$p(y,Sy) \leq (1-\lambda)p(y,Ty) + \frac{\lambda}{2}[p(y,y) + p(Sy,Ty)]$$
$$\leq (1-\lambda)p(A,B) + \frac{\lambda}{2}[p(Ty,y) + p(Sy,y)]$$
$$\Rightarrow p(y,Sy) \leq p(A,B).$$

Therefore, $p(y,Sy) = p(A,B) = p(y,Ty)$. Hence, y is a common best proximity point of S and T. □

Let us exemplify the above theorem with the following example.

Example 7. *Consider the partial metric space (X,p) with $X = \mathbf{R}^2$ and the partial metric $p((x,x'),(y,y')) = 1 + max\{|x-y|, |x'-y'|\}$. Let $\phi(x) = x$, $A = \{(0,x) : 0 \leq x \leq 1\}$, and $B = \{(2,x) : 0 \leq x \leq 1\}$. Let $S,T : A \cup B \longrightarrow A \cup B$ be defined as*

$$S((x,y)) = \begin{cases} (2,\frac{x}{2}), & (x,y) \in A; \\ (0,\frac{x}{2}), & (x,y) \in B, \end{cases} \text{and} T((x,y)) = \begin{cases} (2,x), & (x,y) \in A; \\ (0,x); & (x,y) \in B. \end{cases}$$

It can be seen that S is a T-GKT cyclic ϕ-contraction for $\lambda = \lambda' = 0$. It can also be seen that all the conditions of Theorem 1 are satisfied. Therefore, S and T have a common best proximity point, which is clearly $(2,1)$ here.

It is easy to see that a similar result also holds in the case of a T-GKT cyclic ϕ-contraction with respect to d.

Theorem 2. *Let A, B be two non-empty subsets of a metric space (X,d) with B sequentially compact. Let T be a self-mapping on $A \cup B$, $\phi : [0,\infty) \longrightarrow [0,\infty)$ be a strictly increasing* *continuous mapping and S be a T-GKT cyclic ϕ-contraction mapping on $A \cup B$ with respect to d for some $\lambda = \lambda' \in [0,1)$. If the following conditions are satisfied:*

(i) $S(A) \subseteq T(A) \subseteq B$,

(ii) *S and T commute and T is continuous,*

then there exists a common best proximity point of S and T in B.

The continuity property of the mapping T and commutativity of S and T in the above results can be replaced by the following property, which we term the sequentially proximal equivalence property ((SPE) property) of S and T with respect to a partial metric p.

(*SPE*) property: For non-empty closed subsets A and B of a complete partial metric space (X,p), let, for some x in A and y in B, $p(x,y) = p(A,B)$. If there exists a sequence $\{x_n\}$ in A with

$$\lim_{n\to\infty} Sx_n = y = \lim_{n\to\infty} Tx_n,$$

and a sequence $\{y_n\}$ in B with

$$\lim_{n\to\infty} Sy_n = x = \lim_{n\to\infty} Ty_n,$$

then $p(Sx,x) = p(Tx,x)$ and $p(Sy,y) = p(Ty,y)$.

Example 8. *For $X = \mathbb{R}$ with partial metric $p(x,y) = 1 + |x-y|$ for all $x,y \in X$, let $A = [1,2]$, $B = [-2,-1]$. Define $S,T : A \cup B \longrightarrow A \cup B$ by*

$$S(x) = \begin{cases} -x, & x \in A; \\ 1, & x \in B, \end{cases}$$

$$T(y) = \begin{cases} \frac{-y-3}{4}, & y \in A; \\ -y, & y \in B. \end{cases}$$

It is easily seen that S, T are non-commutative.

Now, $p(A,B) = 3$. In addition, $p(x,y) = 3$ such that $x \in A$, $y \in B$ implies $x = 1$ and $y = -1$.
For a sequence $\{x_n\}$ in A, where $x_n = (1 + \frac{1}{n})$,

$$Sx_n = S(1+\frac{1}{n}) = -(1+\frac{1}{n}) \text{ and } Tx_n = T(1+\frac{1}{n}) = \frac{-1-\frac{1}{n}-3}{4}.$$

Therefore, $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = -1 = y$.

In addition, for a sequence $\{y_n\}$ in B with $y_n = -(1+\frac{1}{n})$,

$$Sy_n = S(-(1+\frac{1}{n})) = 1 \text{ and } Ty_n = T(-(1+\frac{1}{n})) = 1 + \frac{1}{n}.$$

Therefore, $\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Ty_n = 1 = x$.
Now, $p(Sx,x) = p(S(1),1) = 3$ and $p(Tx,x) = p(T(1),1) = 3$, *i.e.*, $p(Sx,x) = p(Tx,x)$. *In the same way, we obtain, $p(Sy,y) = p(Ty,y)$. Thus, S and T satisfy the (SPE) property with respect to p.*

Theorem 3. *Let (X,p) be a complete partial metric space and A, B be two non-empty closed subsets of (X,p). Let T be a self-mapping on $A \cup B$, $\phi : [0,\infty) \longrightarrow [0,\infty)$ be a strictly increasing continuous mapping and S be a T-GKT cyclic ϕ-contraction mapping on $A \cup B$ with respect to p for some $\lambda = \lambda' \in [0,1)$. If A, B are sequentially compact with respect to the induced metric d_p, and the following conditions are satisfied:*

(i) $S(A) \subseteq T(A) \subseteq B$ *and* $S(B) \subseteq T(B) \subseteq A$,

(ii) *S and T satisfy the* (SPE) *property with respect to* p,
then there exists a common best proximity point of S and T.

Proof. For $x_0 \in A$, as in Theorem 1, we can show that $\{Sx_{n_k}\}$ is a Cauchy sequence in (X, p) and so, for some $y \in B$, $Sx_{n_k} \to y$ and $Tx_{n_k} \to y$.

Similarly, taking a point $y_0 \in B$ and constructing the sequence $Sy_n = Ty_{n+1}$, for all $n \in \mathbb{N}$, we see that there exists some $x \in A$ with $Sy_{n_k} \longrightarrow x$ and $Ty_{n_k} \longrightarrow x$.

Now,

$$\begin{aligned} p(Sx_{n_k}, Sy_{n_k}) &\le (1-\lambda)[p(Tx_{n_k}, Ty_{n_k}) + \phi(p(A,B)) - \phi(p(Tx_{n_k}, Ty_{n_k}))] + \\ &\quad \frac{\lambda}{2}[p(Tx_{n_k}, Sx_{n_k}) + p(Ty_{n_k}, Sy_{n_k})] \\ i.e., (1-\lambda)\phi(p(Tx_{n_k}, Ty_{n_k})) &\le (1-\lambda)[p(Tx_{n_k}, Ty_{n_k}) + \phi(p(A,B))] + \\ &\quad \frac{\lambda}{2}[p(Tx_{n_k}, Sx_{n_k}) + p(Ty_{n_k}, Sy_{n_k})] - p(Sx_{n_k}, Sy_{n_k}) \end{aligned} \tag{2}$$

Taking the limit as $k \to \infty$ on both sides of (2),

$$\begin{aligned} (1-\lambda)\lim_{k\to\infty}\phi(p(Tx_{n_k}, Ty_{n_k})) &\le (1-\lambda)[p(x,y) + \phi(p(A,B))] + \frac{\lambda}{2}[p(y,y) + p(x,x)] \\ &\quad - p(x,y) \\ &\le (1-\lambda)[p(x,y) + \phi(p(A,B))] + \lambda p(x,y) - p(x,y) \\ &\le (1-\lambda)\phi(p(A,B)), \end{aligned} \tag{3}$$

Thus,

$$\lim_{k\to\infty}\phi(p(Tx_{n_k}, Ty_{n_k})) = \phi(p(A,B)) \Rightarrow \lim_{n\to\infty} p(Tx_{n_k}, Ty_{n_k}) = p(x,y) = p(A,B).$$

Therefore, by the (SPE) property, $p(Sx,x) = p(Tx,x)$ and $p(Sy,y) = p(Ty,y)$.
Now,

$$\begin{aligned} p(Sx, Sy_{n_k}) &\le (1-\lambda)[p(Tx, Ty_{n_k}) + \phi(p(A,B)) - \phi(p(Tx, Ty_{n_k}))] + \\ &\quad \frac{\lambda}{2}[p(Tx, Sx) + p(Ty_{n_k}, Sy_{n_k})] \\ \Rightarrow (1-\lambda)\phi(p(Tx, Ty_{n_k})) &\le (1-\lambda)[p(Tx, Ty_{n_k}) + \phi(p(A,B))] + \\ &\quad \frac{\lambda}{2}[p(Tx, Sx) + p(Ty_{n_k}, Sy_{n_k})] - p(Sx, Sy_{n_k}) \end{aligned} \tag{4}$$

Taking the limit as $k \to \infty$ on both sides of (4),

$$\begin{aligned} (1-\lambda)\lim_{k\to\infty}\phi(p(Tx, Ty_{n_k})) &\le (1-\lambda)[p(Tx,x) + \phi(p(A,B))] + \frac{\lambda}{2}[p(Tx,Sx) + p(x,x)] - \\ &\quad p(Sx,x) \\ &\le (1-\lambda)[p(Tx,x) + \phi(p(A,B))] + \frac{\lambda}{2}[p(Tx,x) + p(Sx,x)] - \\ &\quad p(Sx,x) \\ &= (1-\lambda)[p(Tx,x) + \phi(p(A,B))] + \lambda p(Tx,x) - p(Tx,x), \\ &\qquad [\text{ since } p(Sx,x) = p(Tx,x)] \\ &= (1-\lambda)\phi(p(A,B)) \end{aligned} \tag{5}$$

Therefore, $p(Sx,x) = p(Tx,x) = p(A,B)$. Hence, x is a common best proximity point of S and T. In a similar way, y is also a common best proximity point of S and T. □

We note that a self-mapping $S : X \longrightarrow X$ can be viewed as a T-GKT cyclic ϕ-contraction self-mapping taking $A = B = X$ in Definitions 7 and 8. We now derive the following fixed-point result considering such T-GKT cyclic ϕ-contraction self-mapping, which will be used in the application part of the paper.

Theorem 4. *Let (X, p) be a complete partial metric space, $\phi : [0, \infty) \longrightarrow [0, \infty)$ be a strictly increasing continuous mapping, and S be a T-GKT cyclic ϕ-contraction self-mapping on X with $T = I$ (identity mapping on X) with respect to p for some $\lambda' = \lambda \in [0, 1)$. If (X, d_p) is also sequentially compact and S is continuous, then S has a fixed point in X.*

Proof. For $x_0 \in X$, we construct the sequence $\{x_n\}$ with

$$x_{n+1} = Sx_n, \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

As in Theorem 1, it is easy to show that $\{x_{n_k}\}$ is a Cauchy sequence in (X, p). Therefore, there exists some $x \in X$ such that $x_{n_k} \to x$.

Now,

$$\begin{aligned} p(x_{n_k}, Sx) &= p(Sx_{n_k-1}, Sx) \\ &\leq (1-\lambda)p(x_{n_k-1}, x) + \frac{\lambda}{2}[p(x_{n_k}, x_{n_k-1}) + p(Sx, x)]. \end{aligned} \tag{6}$$

Taking the limit as $k \to \infty$ on both sides of (6), we obtain,

$$\begin{aligned} p(x, Sx) &\leq (1-\lambda)p(x, x) + \frac{\lambda}{2}[p(x, x) + p(Sx, x)] \\ \text{i.e., } p(x, Sx) &\leq p(x, x). \end{aligned}$$

Therefore, $p(x, Sx) = p(x, x)$.

Similarly we obtain, $p(Sx, Sx) = p(x, x) = p(Sx, x)$, which implies that $Sx = x$, i.e., x is a fixed point of S. □

For a metric space (X, d), we have the following result.

Theorem 5. *Let (X, d) be a complete metric space, $\phi : [0, \infty) \longrightarrow [0, \infty)$ be a strictly increasing continuous mapping, S be a T-GKT cyclic ϕ-contraction self-mapping on X with $T = I$ with respect to d for some $\lambda' < \lambda \in (0, 1)$, and $\phi(x) \leq x$. Then S has a fixed point in X.*

Again, for $A = B = X$ and taking S, T as two self-mappings on X, Theorem 1 gives the following result.

Theorem 6. *Let (X, p) be a complete partial metric space, T be a self-mapping on X, $\phi : [0, \infty) \longrightarrow [0, \infty)$ be a strictly increasing continuous mapping and S be a T-GKT cyclic ϕ-contraction self-mapping on X with respect to p for some $\lambda' = \lambda \in [0, 1)$. If (X, d_p) is also sequentially compact and the following conditions are satisfied:*

(i) $S(X) \subseteq T(X)$,

(ii) *S and T commute and S is continuous,*

then there exists a common fixed point of S and T in X.

Proof. Proceeding as in Theorem 1 we obtain $p(y, Ty) = p(X, X) = p(y, Sy)$.

By the definition of a partial metric,

$$p(X, X) \leq p(y, y) \leq p(y, Ty) = p(X, X).$$

Similarly,

$$p(X, X) \leq p(Ty, Ty) \leq p(y, Ty) = p(X, X).$$

Therefore, $p(y, y) = p(Ty, Ty) = p(y, Ty)$, which implies that $Ty = y = Sy$. Hence, y is a common fixed point of S and T. □

4. Applications

4.1. Characterization of Completeness

Considering the contraction mapping, Hu [16] gave the proof of the characterization of completeness of metric spaces as follows.

Theorem 7 ([16])**.** *A metric space (X, d) is complete if and only if every contraction on closed subsets Y of X has a fixed point in Y.*

Cobzas [17] also showed the characterization of completeness of uniformly Lipschitz connected metric spaces as given below.

Theorem 8 ([17])**.** *A uniformly Lipschitz connected metric space (X, d) is complete if and only if it has the fixed point property for contractions.*

In line with the above results, next we show the completeness of partial metric spaces applying T-GKT cyclic ϕ-contraction mappings on the induced metric space (X, d_p).

Theorem 9. *Let (X, p) be a partial metric space, T be a self-mapping on X, and $\phi : [0, \infty) \longrightarrow [0, \infty)$ be a strictly increasing continuous mapping. If for every closed subset Y of the induced metric space (X, d_p), each T-GKT cyclic ϕ-contraction self-mapping on Y with respect to d_p with $T = I$ and $\phi(x) \leq x$ for all $x \in [0, \infty)$ has a fixed point in Y, then (X, p) is complete.*

Proof. We know that (X, p) is complete if and only if (X, d_p) is complete. Let $\{x_n\}$ be an arbitrary Cauchy sequence in (X, d_p). The proof will be complete if we can show that it has a convergent sub-sequence. Suppose, on the contrary, that $\{x_n\}$ does not have any convergent sub-sequence. Then

$$\beta(x_n) = inf\{d_p(x_n, x_m), m > n\} > 0 \text{ for all } n \in \mathbb{N}.$$

For $\lambda' \in (0, 1)$, we can construct $\{x_{n_k}\}$ such that

$$d_p(x_i, x_j) < \frac{\lambda'}{2}\beta(x_{n_{k-1}}) \text{ for all } i, j \geq n_k.$$

Therefore, $Y = \{x_{n_k} : k \in \mathbb{N}\}$ is a closed subset of (X, d_p).

Defining $S : Y \longrightarrow Y$ by $Sx_{n_k} = x_{n_{k+1}}$ for all $k \in \mathbb{N}$. It is clear that S does not have a fixed point.

Now, for $\phi(x) \leq x$,

$$\begin{aligned} d_p(Sx_{n_k}, Sx_{n_{k+1}}) &= d_p(x_{n_{k+1}}, x_{n_{k+2}}) \\ &< \frac{\lambda'}{2}\beta(x_{n_k}) \\ &\leq \frac{\lambda'}{2}d_p(x_{n_k}, x_{n_{k+1}}) \\ &= \frac{\lambda'}{2}d_p(Sx_{n_k}, x_{n_k}) \\ &\leq \frac{\lambda'}{2}[d_p(Sx_{n_k}, x_{n_k}) + d_p(Sx_{n_{k+1}}, x_{n_{k+1}})] \\ &\quad + (1-\lambda)[d_p(x_{n_k}, x_{n_{k+1}}) - \phi(d_p(x_{n_k}, x_{n_{k+1}}))]. \end{aligned}$$

Therefore, S is a T-GKT cyclic ϕ-contraction self-mapping on Y with $T = I$ and $\lambda' < \lambda$ where $\lambda, \lambda' \in [0, 1)$, having no fixed point. This is a contradiction. Hence (X, d_p) is complete, and so, by Lemma 1, (X, p) is complete. □

4.2. Existence of a Solution to the Fredholm Integral Equation

On the set $X = C([a,b],\mathbb{R})$ of all continuous real-valued functions defined on $[a,b]$, we take the metric d on X defined by $d(m,n) = \sup_{t\in[a,b]} |m(t) - n(t)|$ for all $m,n \in X$.

Consider the following Fredholm integral equation:

$$m(t) = f(t) + \int_a^b F(t,s,m(s))ds \tag{7}$$

for each $t \in [a,b]$, where $f : [a,b] \longrightarrow \mathbb{R}$ and $F : [a,b] \times [a,b] \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions.

Let $S : X \longrightarrow X$ be defined by

$$Sm(t) = f(t) + \int_a^b F(t,s,m(s))ds \tag{8}$$

for each $t \in [a,b]$, satisfying the condition

(A) $|F(t,s,m(s)) - F(t,s,n(s))| \leq \frac{1}{4(b-a)} \sup_{s\in[a,b]} |m(s) - n(s)|$, for all $m,n \in X$, $s \in [a,b]$.

Theorem 10. *If the operator F satisfies the condition (A), then Equation (7) has a solution.*

Proof. Let S be the self-mapping on X as defined in (8). We take ϕ as the identity mapping. The solution of the integral equation is exactly the fixed point of S. Therefore, the existence of a fixed point of S guarantees the existence of a solution to the integral Equation (7).

For arbitrary $m,n \in X$,

$$\begin{aligned}
d(Sm,Sn) &= \sup_{t\in[a,b]} |Sm(t) - Sn(t)| \\
&= \sup_{t\in[a,b]} |f(t) + \int_a^b F(t,s,m(s))ds - f(t) - \int_a^b F(t,s,n(s))ds| \\
&= \sup_{t\in[a,b]} |\int_a^b F(t,s,m(s))ds - \int_a^b F(t,s,n(s))ds| \\
&\leq \sup_{t\in[a,b]} \int_a^b |F(t,s,m(s)) - F(t,s,n(s))|ds \\
&\leq \sup_{t\in[a,b]} \int_a^b \frac{1}{4(b-a)} \sup_{s\in[a,b]} |m(s) - n(s)|ds \\
&= \sup_{t\in[a,b]} \int_a^b \frac{1}{4(b-a)} d(m,n)ds \\
&= \frac{1}{4(b-a)} d(m,n) \sup_{t\in[a,b]} \int_a^b ds \\
&= \frac{1}{4} d(m,n) \\
&\leq \frac{1}{4}[d(m,Sm) + d(Sm,Sn) + d(Sn,n)]
\end{aligned}$$

$$\begin{aligned}
\Rightarrow d(Sm,Sn) &\leq \frac{1}{3}[d(m,Sm) + d(Sn,n)] \\
&\leq (1-\lambda)[d(m,n) + \phi(d(X,X)) - \phi(d(m,n))] + \frac{\lambda'}{2}[d(Sm,m) + d(Sn,n)], \\
&\quad (\text{taking } \lambda' = \frac{2}{3} \text{ and } \lambda = \frac{3}{4})
\end{aligned}$$

Hence all the conditions of Theorem 5 are satisfied. Therefore, S has a fixed point, i.e., the integral Equation (7) has a solution. □

5. Conclusions

In this paper we have established some existence results of common best proximity points in partial metric spaces. Similar results are mentioned in the case of metric spaces. An obtained result has been applied to characterize the completeness of partial metric spaces. Moreover, the existence of a solution to the Fredholm integral equation is shown using the established result. There are many other important aspects and applications of proximity point theory as well as fixed point theory in different directions [18,19]. In 2020, Karapinar et al. [20] discussed the sufficient conditions for the existence and uniqueness of a solution for a coupled system of fractional hybrid differential equations. The application of our established results for solving fractional hybrid differential equations as well as nonlinear matrix equations provides scope for future discussion. In [21], Choudhury et al. obtained some best proximity point results and applied these in finding coupled best proximity points in partially ordered metric spaces. In view of this, the applicability of our results in the case of partially ordered metric spaces is another possible scope for future development in this area.

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