The principal eigenvalue of some nth order linear boundary value problems

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Abstract

The purpose of this paper is to present a procedure for the estimation of the smallest eigenvalues and their associated eigenfunctions of nth order linear boundary value problems with homogeneous boundary conditions defined in terms of quasi-derivatives. The procedure is based on the iterative application of the equivalent integral operator to functions of a cone and the calculation of the Collatz–Wielandt numbers of such functions. Some results on the sign of the Green functions of the boundary value problems are also provided.

Keywords: Eigenvalue; Boundary value problem; Quasi-derivatives; Green function; Cone theory; Collatz–Wielandt numbers

1 Introduction

Let L be a disconjugate linear differential operator of nth order on an interval $[a, b]$ which, according to a well-known theorem of Pólya [1], can be factored as a product of operators of first order as

$$L_0 y = \rho_0 y,$$
$$L_i y = \rho_i (L_{i-1} y)', \quad i = 1, \ldots, n, \qquad (1)$$
$$L y = L_n y,$$

where $\rho_i > 0$, $\rho_0 \rho_1 \cdots \rho_n = 1$, and $\rho_i \in C^{n-i}[a, b]$. Following Elias notation we will call $L_0 y, L_1 y, \ldots, L_n y$ the quasi-derivatives of y.

In this paper we will tackle the eigenvalue problem

$$L y + q(x) y = \lambda \sum_{l=0}^{m} p_l(x) L_l y,$$
$$L_i y(a) = 0, \quad i \in \alpha, \qquad L_j y(b) = 0, \quad j \in \beta, \qquad (2)$$

where $0 \le m \le n - 1$, α is the ordered set $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, β is the ordered set $\{\beta_1, \beta_2, \ldots, \beta_{n-k}\}$, $1 \le k \le n - 1$, both $\alpha_k, \beta_{n-k} < n$ and $p_l, q \in C[a, b]$ such that $(-1)^{n-k} q \le 0$ a.e. on $[a, b]$ and p_l have signs to be determined.

A special case of these homogeneous boundary conditions, which will be very relevant in what follows, is that in which the number of boundary conditions at a and b set on quasi-derivatives of order lower than t is greater or equal than t for $t = 1, \ldots, n$. Elias denoted these conditions by *poised* in [2], term that we will use in the rest of the manuscript. Poised boundary conditions imply that $\lambda = 0$ is not an eigenvalue of (2) as per [2, Lemma 10.3].

The purpose of this paper is to provide an iterative procedure to:

1. Calculate the smallest or *principal* eigenvalue of problem (2) when the boundary conditions are poised (note that the principal label refers to the positivity of the associated eigenfunction).
2. Calculate the second smallest eigenvalue of problem (2) when the boundary conditions are not poised and $q(x) \equiv 0$.
3. Estimate the eigenfunction y of problem (2) associated with such eigenvalues.

We will work on the equivalent integral eigenvalue problem

$$\lambda My = y, \tag{3}$$

where M is the operator $C[a,b] \to C^n[a,b]$ defined by

$$My = \int_a^b G(x,t) \sum_{l=0}^m p_l(t) L_l y(t)\, dt, \quad x \in [a,b], \tag{4}$$

and $G(x,t)$ is the Green function of the problem

$$Ly + q(x)y = 0; \qquad L_i y(a) = 0, \quad i \in \alpha; \qquad L_i y(b) = 0, \quad i \in \beta, \tag{5}$$

which exists as long as the boundary conditions are poised and problem (5) does not have an extremal point in $[a,b]$ (see [2, Theorem 4.16]).

The eigenvalue problem (2) has been studied thoroughly in the literature, references [3–6] being excellent examples. A good summary can also be found in [2, Chap. 10]. However, there do not seem to exist many algorithms for the calculation of the eigenvalues, as far as the authors are aware.

As for practical applications, in [7, Appendix D] one can find multiple examples of differential problems in the theory of fluid dynamics which lead to representations like (2). In particular, the pluriharmonic equation $\Delta^N y + \lambda p y = 0$, $x \in \mathbb{R}^q$, $\Delta = \sum_{i=1}^q \frac{\partial^2}{\partial x^2}$, if $p(x) = p(t)$ with $t = (\sum_{i=1}^q x_i^2)^{\frac{1}{2}}$, can be represented as $(t^{1-q}\frac{d}{dt}t^{q-1}\frac{d}{dt})^N y + \lambda p(t) y = 0$.

The calculation of the smallest eigenvalue of (2) is also relevant to proving the existence of solutions of nonlinear boundary value problems of the type $Ly + p(x)g(y) = 0$, in particular by comparing that eigenvalue with the quotient $\frac{g(y)}{y}$ for different values of y, especially when $y \to 0^+$ and when $y \to +\infty$. This approach was started by Erbe [8] for symmetric kernels and extended by Webb and Lan [9–11] and many others, [12] being a recent example.

The procedure is based on the iterative calculation of M^j (that is, the composition of M with itself $j - 1$ times, $\underbrace{M \circ M \circ \cdots \circ M}_{j \text{ times } M}$) on functions u of a cone and the determination of the so-called Collatz–Wielandt numbers of the resulting functions.

For self-completeness, let us recall that, given a Banach space B, a cone $P \subset B$ is a nonempty closed set defined by the conditions:

1. If $u \in P$ and $-u \in P$, then $u = 0$.
2. If $u, v \in P$, then $cu + dv \in P$ for any real numbers $c, d \geq 0$.

A cone in a Banach space B allows defining a partial ordering in the Banach space by setting $u \leq v$ if and only if $v - u \in P$. We will say that the operator M is u_0-positive if there exists $u_0 \in P$ such that for any $u \in P \backslash \{0\}$ one can find positive constants ϵ_1, ϵ_2 such that $\epsilon_1 u_0 \leq Mu \leq \epsilon_2 u_0$. A cone P is reproducing if $B = P - P$ and total if $B = \overline{P - P}$. We will denote by int$\{P\}$ the interior of the cone P, if it exists.

Following Forster–Nagy definition [13], if $u \in P \backslash \{0\}$, the upper and lower Collatz–Wielandt numbers are defined, respectively, as

$$\overline{r}(M, u) = \inf\{w \in \mathbb{R} : Mu \leq wu\}, \qquad \underline{r}(M, u) = \sup\{w \in \mathbb{R} : wu \leq Mu\}. \tag{6}$$

They are called upper and lower Collatz–Wielandt numbers as they extend the estimates for the spectral radius of a nonnegative matrix given by L. Collatz [14] and H. Wielandt [15].

The properties of $\underline{r}(M, u)$ and $\overline{r}(M, u)$ and their relationship with the spectral radius of the operator M have been studied by several authors, starting with Marek [16, 17], Forster and Nagy [13], who corrected some previous mistakes from Marek, and Marek again [18]. The concept has been extended to multiple types of operators, Banach spaces and cones. The references [19–22] include a good account of recent results.

Although Chang [20] already proposed the use of iterative Collatz–Wielandt numbers (what he called the *power method*) to obtain converging upper and lower bounds of the principal eigenvalue of some boundary value problems, it was Webb [23, Sect. 6] who first applied to a boundary value problem of type (3)–(4) by taking into consideration the following result by Marek [18, Theorem 5.2].

Theorem 1 *Let P be a normal cone in a Banach space. Let M be a compact linear operator u_0-positive in P. Then, for any $u \in P \backslash \{0\}$, the sequences of Collatz–Wielandt numbers $\underline{r}(M, M^j u)$ and $\overline{r}(M, M^j u)$ converge to $r(M)$.*

Webb's paper [23] left open the question of how to determine the u_0-positivity of M in the general case. His article included another iterative method to bound and calculate the principal eigenvalue which had a slower convergence rate and was based on the existence of certain bounds for the Green function used as a kernel in (4).

In [24] the authors proposed an iterative approach to estimate the principal eigenvalue and associated eigenfunction of an nth order linear boundary value problem, which in essence coincides with that of Webb. The u_0-positivity of the associated operator M was proved by using sign results of the derivatives of the corresponding Green function. This paper follows a similar modus operandi.

The organization of the paper is as follows. Section 2 elaborates on the sign properties of the quasi-derivatives of the Green functions of (5). Section 3 uses them to show that, when the boundary conditions are poised, there is a cone P for which M fulfils the conditions for the Collatz–Wielandt numbers to converge, yielding a procedure for the estimation of the principal eigenvalue and eigenfunction of (2). It also shows how to adapt the process to cope with non-poised boundary conditions. In Sect. 4 some practical considerations for the calculation of the Collatz–Wielandt numbers of this problem are presented. Section 5

gives an example of how to apply the previous theory to calculate the principal eigenvalue of a boundary value problem. Finally, Sect. 6 discusses some conclusions.

2 The sign of the quasi-derivatives of the Green function

In this section we study the signs of the quasi-derivatives of the Green function of the problem

$$Ly + q(x)y = 0, \qquad L_i y(a) = 0, \quad i \in \alpha; \qquad L_i y(b) = 0, \quad i \in \beta; \tag{7}$$

with $q \in C[a,b]$ such that either $q \equiv 0$ or $(-1)^{n-k}q < 0$ a.e. on $[a,b]$, provided that the boundary conditions are poised and such a problem does not have an extremal point in $[a,b]$. From nomenclature perspective we will assume that the quasi-derivative $L_i G(x,t)$ applies only to the variable x of $G(x,t)$.

We will also need the following definitions.

Definition 1 Fixed $t \in [a,b]$,

- A *zero component* is a closed subinterval $I \subset [a,b]$ where a quasi-derivative of $G(x,t)$ is identically zero. If a quasi-derivative has several zero components, there must be subintervals of $[a,b]$ of positive measure separating them. Otherwise they will be considered the same zero component.
- $z_i[a,b]$ is the number of isolated zeroes or zero components of the ith quasi-derivative of $G(x,t)$ on $[a,b]$ for $i = 0,\ldots,n-2$.
- $z_i(a,b)$ is the number of isolated zeroes or zero components of the ith quasi-derivative of $G(x,t)$ *entirely lying* on (a,b) for $i = 0,\ldots,n-2$.
- $Z_i\{\alpha,\beta\}$ is the number of homogeneous boundary conditions defined in $\{\alpha,\beta\}$ which are lower than or equal to i.
- $E_i[a,b]$ is the excess of isolated zeroes or zero components of the ith quasi-derivative of $G(x,t)$ on $[a,b]$ not due to the boundary conditions and Rolle's theorem which, for reasons that will become clear later, we will define as

$$E_i[a,b] = z_i[a,b] - Z_i\{\alpha,\beta\} + i, \quad i = 0,\ldots,n-2. \tag{8}$$

- $m(\alpha,i)$ is the number of quasi-derivatives of order equal to or higher than i which the boundary conditions α do not specify to vanish at a.
- $n(\beta,i)$ is the number of quasi-derivatives of order higher than i which the boundary conditions β do specify to vanish at b.
- $i_{\max}$ is the lowest quasi-derivative such that either $\{i_{\max},\ldots,n-1\} \subset \alpha$ or $\{i_{\max},\ldots,n-1\} \subset \beta$, with $i_{\max} = n$ if there are no boundary conditions on the $(n-1)$th quasi-derivative.
- $i(\alpha,j)$ is the lowest quasi-derivative higher than or equal to j such that in all quasi-derivatives between the jth and the $(i(\alpha,j)-1)$th one there are boundary conditions set at a. If there is no boundary condition set on the jth quasi-derivative at a, we will say $i(\alpha,j) = j$.
- $i(\beta,j)$ is the lowest quasi-derivative higher than or equal to j such that in all quasi-derivatives between the jth and the $(i(\beta,j)-1)$th one there are boundary conditions set at b. If there is no boundary condition set on the jth quasi-derivative at b, we will say $i(\beta,j) = j$.

Lemma 1 $E_i[a,b]$ *satisfies* $E_i[a,b] \geq E_{i-1}[a,b] \geq 0$ *for* $i = 1,\ldots,n-2$.

Proof From the definition of $E_i[a,b]$ it is clear that

$$z_i[a,b] = E_i[a,b] + Z_i\{\alpha,\beta\} - i, \quad i = 0,\ldots,n-2. \tag{9}$$

$G(x,t)$ cannot be identically zero on $[a,b]$ as otherwise $L_{n-1}G(x,t)$ cannot have a discontinuity jump at $x = t$. Therefore, it cannot have a single zero component covering $[a,b]$. This implies $z_0[a,b] = E_0[a,b] + Z_0\{\alpha,\beta\} \geq Z_0\{\alpha,\beta\}$, so $E_0[a,b] \geq 0$. Next, by Rolle's theorem

$$z_i(a,b) \geq z_{i-1}[a,b] - 1,$$

and

$$z_i[a,b] \geq Z_i\{\alpha,\beta\} - Z_{i-1}\{\alpha,\beta\} + z_{i-1}[a,b] - 1.$$

From here and (9) one has

$$z_i(a,b) \geq E_{i-1}[a,b] + Z_{i-1}\{\alpha,\beta\} - i \tag{10}$$

and

$$z_i[a,b] \geq Z_i\{\alpha,\beta\} - Z_{i-1}\{\alpha,\beta\} + E_{i-1}[a,b] + Z_{i-1}\{\alpha,\beta\} - i + 1 - 1$$
$$= E_{i-1}[a,b] + Z_i\{\alpha,\beta\} - i, \tag{11}$$

which together with (9) prove the statement. □

Remark 1 Note that Lemma 1 refers to quasi-derivatives and boundary conditions, not requiring any particular condition on $q(x)$ of (7). This could take any values as long as the associated Green function exists.

Theorem 2 *Fixed* $t \in [a,b]$, *if the boundary conditions* $\{\alpha,\beta\}$ *are poised, then* $E_i[a,b] = 0$ *and* $z_i[a,b] \geq 1$ *for* $i = 0,\ldots,n-2$. *In addition,*

$$(-1)^{m(\alpha,i)}L_iG(a,t) > 0, \quad i \notin \alpha; \qquad (-1)^{n(\beta,i)}L_iG(b,t) > 0, \quad i \notin \beta; \tag{12}$$

and if $Z_{i-1}\{\alpha,\beta\} = i$, *then*

$$(-1)^{m(\alpha,i)}L_iG(x,t) > 0, \quad x \in (a,b). \tag{13}$$

Proof If the boundary conditions are poised, then $Z_i\{\alpha,\beta\} \geq i + 1$ for $i = 0,\ldots,n-1$. This and (9) lead to $z_i[a,b] \geq E_i[a,b] + 1 \geq 1$, $i = 0,\ldots,n-2$.

　Next, from [2, Theorem 4.16] one knows that

$$(-1)^{n-k}G(x,t) > 0, \quad x \in (a,b), \tag{14}$$

which, given that $(-1)^{n-k}q(x) < 0$ a.e. on $[a,b]$ by hypothesis, implies

$$L_n G(x,t) = LG(x,t) = -q(x)G(x,t) > 0, \quad \text{a.e. for } x \in (a,b). \tag{15}$$

This means that $L_{n-1}G(x,t)$ is increasing on $[a,b]$, including at the discontinuity point $x = t$.

Let us assume that any $E_i[a,b] \neq 0$ with $0 \le i \le n-2$. By Lemma 1, $E_{n-2}[a,b] \ge 1$ and there can be two cases:

- Either there is no boundary condition at $L_{n-1}G(x,t)$, which, given that the total number of boundary conditions is n, implies that $Z_{n-2}\{\alpha,\beta\} = n$ and therefore $z_{n-2}[a,b] = E_{n-2}[a,b] + 2 \ge 3$. This is impossible since by Rolle's theorem $L_{n-1}G(x,t)$ must have at least two change of signs, which is not compatible with $L_{n-1}G(x,t)$ being increasing on $[a,b]$.
- Or there is one boundary condition at $L_{n-1}G(x,t)$. This implies in turn that $Z_{n-2}\{\alpha,\beta\} = n-1$ and therefore $z_{n-2}[a,b] = E_{n-2}[a,b] + 1 \ge 2$. By Rolle's theorem $L_{n-1}G(x,t)$ must have at least one change of sign, but this is also impossible since $L_{n-1}G(x,t)$ is increasing on $[a,b]$ and is set to 0 at one of the extremes.

The previous monotonicity argument can be applied to show that $L_{n-1}G(x,t)$ can only have either one zero at a or b, set by the boundary conditions, or a single zero on (a,b), or one change of sign at $x = t$, the last two cases as a result of Rolle's theorem.

As for the sign of $L_iG(x,t)$, if $L_iG(a,t) = 0$, then obviously $L_iG(x,t)L_{i+1}G(x,t) > 0$ for $x \in (a, a+\delta)$. Else, since $z_i[a,b] \ge 1$ and $E_{i+1}[a,b] = 0$, there cannot be a zero of $L_{i+1}G(x,t)$ on (a, x_i), where x_i is the smallest zero of $L_iG(x,t)$. Accordingly,

$$-L_iG(x,t) = L_iG(x_i,t) - L_iG(x,t) = \int_x^{x_i} \frac{L_{i+1}G(s,t)}{\rho_{i+1}(s)}\, ds,$$

so $L_iG(x,t)L_{i+1}G(x,t) < 0$ for $x \in (a, a+\delta)$. From here and (14) one gets

$$(-1)^{m(\alpha,i)} L_iG(x,t) > 0, \quad x \in (a, a+\delta), i = 0, \dots, n-1. \tag{16}$$

In a similar way it is possible to prove that

$$(-1)^{n(\beta,i)} L_iG(x,t) > 0, \quad x \in (b-\delta, b], i \notin \beta, \tag{17}$$

and

$$(-1)^{n(\beta,i)} L_iG(x,t) < 0, \quad x \in (b-\delta, b), i \in \beta. \tag{18}$$

Inequalities (12) are a consequence of (16) and (17).

Last, if $Z_{i-1}\{\alpha,\beta\} = i$, then $z_i[a,b] = Z_i\{\alpha,\beta\} - Z_{i-1}\{\alpha,\beta\}$, that is, the only zeroes of $L_iG(x,t)$ happen at a or b and $L_iG(x,t)$ does not change sign on (a,b). This and (16) yield (13). □

Let us turn our attention to the Green function of (7) for the case $q \equiv 0$, that is,

$$Ly = 0; \qquad L_iy(a) = 0, \quad i \in \alpha; \qquad L_iy(b) = 0, \quad i \in \beta. \tag{19}$$

For this problem we can obtain a result similar to Theorem 2.

Theorem 3 *Fixed $t \in [a,b]$, if the boundary conditions $\{\alpha, \beta\}$ are poised, then $z_i[a,b] \geq 1$ and $E_i[a,b] = 0$ for $i = 0, \ldots, n-2$.*

In addition

$$(-1)^{m(\alpha,i)} L_i G(a,t) > 0, \quad i \notin \alpha; \qquad (-1)^{n(\beta,i)} L_i G(b,t) > 0, \quad i \notin \beta; \tag{20}$$

and if $Z_{i-1}\{\alpha, \beta\} = i$ with $i < i_{\max}$, then

$$(-1)^{m(\alpha,i)} L_i G(x,t) > 0, \quad x \in (a,b). \tag{21}$$

Proof If the boundary conditions $\{\alpha, \beta\}$ are poised, then the Green function exists and one can reason as in Theorem 2 to obtain $z_i[a,b] \geq 1$ for $i = 0, \ldots, n-2$.

Next, let us assume that $i_{\max} < n$ and $i_{\max} \in \alpha$. From the boundary conditions and (19) one has $L_i G(x,t) = 0$ for $x \in [a,t)$, $i = i_{\max}, \ldots, n-1$. It cannot happen that there is $x_i \in (t,b]$ such that $L_i G(x_i,t) = 0$ for any $i \geq i_{\max}$ since otherwise, by Rolle's theorem, there should be a zero of $L_{n-1}G(x,t)$ on (t,b), which is impossible since $L_n G = 0$ on that subinterval and $L_{n-1}G(x,t)$ has a discontinuity at $x = t$. That implies that $(-1)^{n(\beta,i)} L_i G(x,t) = L_i G(x,t) > 0$ for $i = i_{\max}, \ldots, n-1$ and $x \in (t,b]$, $E_{n-2}[a,b] = 0$ and, according to Lemma 1, $E_i[a,b] = 0$ for $i = 0, \ldots, n-2$. In addition, $L_{i_{\max}-1}G(x,t)$ must be constant on $[a,t]$ and monotonic increasing on $(t,b]$.

If the zero of $L_{i_{\max}-1}G(x,t)$ (there must be at least one!) is at b and is due to the boundary conditions, then $L_{i_{\max}-1}G(x,t) < 0$ for $x \in [a,b)$ or, in other words, $-(-1)^{n(\beta,i_{\max}-1)} \times L_{i_{\max}-1}G(x,t) = (-1)^{m(\alpha,i_{\max}-1)} L_{i_{\max}-1}G(x,t) > 0$ for $x \in [a,b)$. Otherwise, as the definition of $i_{\max}$ prevents that a boundary condition on $L_{i_{\max}-1}G(x,t)$ is set at a, and the boundary conditions are poised, from Lemma 1, in particular (10), the number of isolated zeroes or zero components of $L_{i_{\max}-1}G(x,t)$ entirely lying on (a,b) is

$$z_{i_{\max}-1}(a,b) = E_{i_{\max}-1}[a,b] + i_{\max} - (i_{\max} - 1) = 1.$$

The subinterval $[a,t]$ cannot be that one, since it is not entirely within (a,b), so that zero must be in (t,b) and therefore $L_{i_{\max}-1}G(x,t) \neq 0$ for $x \in [a,t]$. In particular $(-1)^{n(\beta,i_{\max}-1)} \times L_{i_{\max}-1}G(b,t) = L_{i_{\max}-1}G(b,t) > 0$ and $(-1)^{m(\alpha,i_{\max}-1)} L_{i_{\max}-1}G(x,t) = -L_{i_{\max}-1}G(x,t) > 0$ for $x \in [a, a+\delta]$. In both cases, since $z_i[a,b] \geq 1$ for $i = 0, \ldots, n-2$, one can reason as in Theorem 2 to get to (20) and (21).

A similar result is obtained if $i_{\max} < n$ and $i_{\max} \in \beta$.

Last, let us assume $i_{\max} = n$, that is, no boundary condition set at a or b for $L_{n-1}G$. From (9), $z_{n-2}[a,b] = E_{n-2}[a,b] + n - (n-2) \geq 2$. From Rolle's theorem there must be at least a change of sign of $L_{n-1}G(x,t)$ in (a,b). As $L_n G = LG = 0$ in each subinterval, that change of sign is only possible if $(-1)^{m(\alpha,n-1)} L_{n-1}G(x,t) = -L_{n-1}G(x,t) > 0$ on $[a,t)$ and $(-1)^{n(\beta,n-1)} L_{n-1}G(x,t) = L_{n-1}G(x,t) > 0$ on $(t,b]$. There cannot be any other zero or change of sign, so $E_{n-2}[a,b] = 0$ and, from Lemma 1, $E_i[a,b] = 0$ for $i = 0, \ldots, n-2$. As before, one can reason as in Theorem 2 to get to (20) and (21). $\qquad\square$

3 The calculation of the principal eigenvalue

Let us first consider the eigenvalue problem

$$Ly + q(x)y = \lambda \sum_{l=0}^{m} p_l(x) L_l y; \qquad L_i y(a) = 0, \quad i \in \alpha; \qquad L_i y(b) = 0, \quad i \in \beta, \tag{22}$$

with $\{\alpha, \beta\}$ being poised boundary conditions, $(-1)^{n-k}q < 0$ a.e. on $[a,b]$ and $q, p_l \in C[a,b]$.

Problem (22) is equivalent to the integral eigenvalue problem $\lambda My = y$, with M defined as

$$My = \int_a^b G(x,t) \sum_{l=0}^m p_l(t)L_l y(t)\, dt, \quad x \in [a,b], \tag{23}$$

and $G(x,t)$ being the Green function of problem (7). We will show that the problem is compliant with the conditions presented in [24] for certain Banach spaces and cones. Thus, let S be the set of the indices l such that $Z_{l-1}\{\alpha, \beta\} = l$, $0 \leq l \leq m$, where we assume $Z_{-1}\{\alpha, \beta\} = 0$. If $m = 0$, we will define the Banach space B as

$$B = \{y \in PC[a,b]\}, \tag{24}$$

and if $m > 0$ as

$$\begin{aligned} B = \{&L_{m-1}y \in C[a,b] : L_m y \in PC[a,b], \\ &L_i y(a) = L_j y(b) = 0, i \in \alpha, j \in \beta, i,j < m\}, \end{aligned} \tag{25}$$

in all cases the associated norm being

$$\|y\| = \max\{\sup\{|L_i y(x)|, x \in [a,b]\}, i = 0,\dots,m\}. \tag{26}$$

We will define the cone P by

$$P = \{y \in B : (-1)^{m(\alpha,l)}L_l y(x) \geq 0, x \in [a,b], l \in S\}. \tag{27}$$

In a similar manner, we will define the Banach space $\mathcal{B}$ as

$$\mathcal{B} = \{L_{n-1}y \in C[a,b] : L_n y \in PC[a,b], L_i y(a) = L_j y(b) = 0, i \in \alpha, j \in \beta\}, \tag{28}$$

with the associated norm

$$\|y\| = \max\{\sup\{|L_i y(x)|, x \in [a,b]\}, i = 0,\dots,n\}, \tag{29}$$

and the cone $\mathcal{P}$ as

$$\mathcal{P} = \{y \in \mathcal{B} : (-1)^{m(\alpha,l)}L_l y(x) \geq 0, x \in [a,b], l \in S\}. \tag{30}$$

The cone $\mathcal{P}$ is solid and its interior is defined by

$$\begin{aligned} \mathrm{int}\{\mathcal{P}\} = \{&y \in \mathcal{B} : (-1)^{m(\alpha,l)}L_l y(x) > 0, x \in (a,b), l \in S; \\ &(-1)^{m(\alpha,l)}L_{i(\alpha,l)}y(a) > 0, (-1)^{m(\alpha,l)-i(\beta,l)+l}L_{i(\beta,l)}y(b) > 0, l \in S\}. \end{aligned} \tag{31}$$

These cones are the tools for proving the next theorem.

Theorem 4 *Let us suppose that both $i(\alpha, m), i(\beta, m) < n$. Let us also suppose that $\{\alpha, \beta\}$ are poised,*

$$(-1)^{m(\alpha,l)} p_l(x) \geq 0, \quad x \in [a,b], l \in S; \qquad p_l(x) \equiv 0, \quad l \notin S, \tag{32}$$

and that there exists an index $s \in S$ such that $(-1)^{m(\alpha,s)} p_s(x) > 0$ a.e. on $[a,b]$.

Then there exists a principal eigenvalue λ_0 of problem (22) *which satisfies*

$$\underline{r}(M, M^j u) \leq \frac{1}{\lambda_0} \leq \overline{r}(M, M^j u), \quad j = 0, 1, \ldots, \tag{33}$$

and

$$\lim_{j \to \infty} \underline{r}(M, M^j u) = \lim_{j \to \infty} \overline{r}(M, M^j u) = \frac{1}{\lambda_0}, \tag{34}$$

and its associated eigenfunction v satisfies $v \in P \setminus \{0\}$ and

$$\lim_{j \to \infty} \lambda_0^j M^j u = f(u) v \tag{35}$$

for any $u \in P \setminus \{0\}$, where $f(u)$ is a nonzero linear functional dependent on u.

Proof The assertions are a consequence of [24, Theorems 6 and 7], which in turn require that $\mathcal{P}$ is reproducing (this is a consequence of the cone being solid and [25, Lemma 1.1]), M is compact (this can be easily proved using Arzelà–Ascoli theorem), and that $M\{P \setminus \{0\}\} \subset \text{int}\{\mathcal{P}\}$. To prove the last assertion, let us assume that $f \in P \setminus \{0\}$ and let us check if Mf satisfies (31). From the definition of M one has

$$L_i M f(x) = \int_a^b L_i G(x,t) \sum_{l=0}^m p_l(t) L_l f(t) \, dt, \quad x \in [a,b], 0 \leq i \leq n-1. \tag{36}$$

The definition of P implies, using Rolle's theorem as in Lemma 1 and Theorem 2, that all quasi-derivatives of f of order $l \in S$ have at least a zero in $[a,b]$, but perhaps the quasi-derivative of mth order. Therefore, if any of them was identically zero, all of them would vanish identically. From here, the fact that $Z_{l-1}\{\alpha, \beta\} = l$ for $l \in S$, (13), (30), (32), and (36) it follows that

$$(-1)^{m(\alpha,l)} L_l M f(x) > 0, \quad x \in (a,b), l \in S. \tag{37}$$

In a similar manner, according to (12) and (13), $n(\beta, l) = m(\alpha, l)$ for $l \in S \setminus (S \cap \beta)$. From this, (12), (30), (32), and (36), one has

$$(-1)^{m(\alpha,l)} L_l M f(a) > 0, \quad l \notin \alpha, l \in S, \tag{38}$$

and

$$(-1)^{m(\alpha,l)} L_l M f(b) > 0, \quad l \notin \beta, l \in S. \tag{39}$$

Next, from the definitions of $m(\alpha, l)$, $i(\alpha, l)$, $n(\beta, l)$, and $i(\beta, l)$ it follows

$$m\big(\alpha, i(\alpha, l)\big) = m(\alpha, l), \qquad n\big(\beta, i(\beta, l)\big) = n(\beta, l) - i(\beta, l) + l + 1,$$

which in combination with the hypothesis $i(\alpha, m), i(\beta, m) < n$, (12), (30), (32), (36), and $(-1)^{n(\beta, j)+1} = (-1)^{m(\alpha, j)}$ for $j \in \beta$, yields

$$(-1)^{m(\alpha, l)} L_{i(\alpha, l)} Mf(a) > 0, \qquad (-1)^{m(\alpha, l) - i(\beta, l) + l} L_{i(\beta, l)} Mf(b) > 0 \tag{40}$$

for $l \in S$. This completes the proof. □

Remark 2 Theorem 4 shows that, selecting any $u \in P \backslash \{0\}$, the calculation of $\underline{r}(M, M^j u)$ and $\overline{r}(M, M^j u)$ yields lower and upper bounds for the eigenvalue $\frac{1}{\lambda_0}$ which converge to this as the iteration index j grows, and that $M^j u$ also converges in norm to the corresponding eigenfunction v. Such a calculation requires the comparison of $\frac{\partial^l M^j u}{\partial x^l}$ and $\frac{\partial^l M^{j+1} u}{\partial x^l}$ for all $l \in S$. This is an aspect different from [24], where the comparison was restricted to the partial derivative of a certain order, but it is the price to pay for allowing a wider amount of different quasi-derivatives in the right-hand side of (2).

For the case $q(x) \equiv 0$, namely for the problem

$$Ly = \lambda \sum_{l=0}^{m} p_l(x) L_l y; \qquad L_i y(a) = 0, \quad i \in \alpha; \qquad L_i y(b) = 0, \quad i \in \beta, \tag{41}$$

with $\{\alpha, \beta\}$ being poised boundary conditions and $p_j \in C[a, b]$, it is possible to obtain a result like Theorem 4.

Theorem 5 *Let us suppose that both $i(\alpha, m), i(\beta, m) \leq i_{\max} - 1$. Let us also suppose that $\{\alpha, \beta\}$ are poised,*

$$(-1)^{m(\alpha, l)} p_l(x) \geq 0, \quad x \in [a, b], l \in S; \qquad p_l(x) \equiv 0, \quad l \notin S, \tag{42}$$

and there exists an index $s \in S$ such that $(-1)^{m(\alpha, s)} p_s(x) > 0$ a.e. on $[a, b]$. Then there exists a principal eigenvalue λ_0 of problem (41) which satisfies

$$\underline{r}\big(M, M^j u\big) \leq \frac{1}{\lambda_0} \leq \overline{r}\big(M, M^j u\big), \quad j = 0, 1, \dots, \tag{43}$$

and

$$\lim_{j \to \infty} \underline{r}\big(M, M^j u\big) = \lim_{j \to \infty} \overline{r}\big(M, M^j u\big) = \frac{1}{\lambda_0}, \tag{44}$$

and its associated eigenfunction v satisfies $v \in P \backslash \{0\}$ and

$$\lim_{j \to \infty} \lambda_0^j M^j u = f(u) v \tag{45}$$

for any $u \in P \backslash \{0\}$, where $f(u)$ is a nonzero linear functional dependent on u.

Proof The proof is exactly the same as that of Theorem 4 by considering inequalities (20)–(21) instead of (12)–(13).   □

When $q(x) \equiv 0$ and $m = 0$, problem (22) becomes

$$Ly = \lambda p y; \qquad L_i y(a) = 0, \quad i \in \alpha; \qquad L_i y(b) = 0, \quad i \in \beta, \tag{46}$$

with $p \in C[a, b]$ such that $(-1)^{n-k} p > 0$ on $[a, b]$. If, in addition, the boundary conditions $\{\alpha, \beta\}$ are not poised, the principal eigenvalue is exactly 0 due to [2, Lemma 10.3]). However, it is possible to bound and estimate the next smallest eigenvalue (and associated eigenfunction) following a similar procedure as that of Theorem 5 and taking into consideration the next theorem.

Theorem 6 *Any problem of type* (46) *with non-poised boundary conditions and a nonzero eigenvalue is equivalent to another problem of type* (46) *with poised boundary conditions.*

Proof According to [2, Lemma 4.13], for any boundary conditions $\{\alpha, \beta\}$ there exists an index s such that the boundary conditions are s-poised, that is, there are at least i conditions imposed on the first i quasi-derivatives in the sequence

$$L_s y, L_{s+1} y, \ldots, L_{n-1} y, L_0 y, \ldots, L_{s-1} y,$$

for $i = 1, \ldots, n$. Such an index s, according to [2, Equation (4.10)] and the definition of $Z_i\{\alpha, \beta\}$, satisfies

$$Z_{s-1}\{\alpha, \beta\} - s = \min_j \left(Z_j\{\alpha, \beta\} - j - 1 \right) \tag{47}$$

for $j = 0, \ldots, n - 1$.

Let us assume that the eigenvalue λ is not zero. As in [2, Theorem 5.5] we can permute the quasi-derivatives of y in a cyclic order. Thus let $z = L_s y$ and

$$N_0 z = r_0 z, \qquad N_i z = r_i (N_{i-1} z)', \quad i = 1, \ldots, n,$$

with

$$r_i = \begin{cases} 1, & i = 0, n, \\ \rho_{i+s}, & i = 1, \ldots, n - s - 1, \\ \rho_0 \rho_n / |p|, & i = n - s, \\ \rho_{i+s-n}, & i = n - s + 1, \ldots, n - 1. \end{cases}$$

As $|p| = (-1)^{n-k} p$, this means in practice

$$N_i z = \begin{cases} L_{i+s} y, & i = 0, \ldots, n - s - 1, \\ (-1)^{n-k} \lambda L_{i+s-n} y, & i = n - s, \ldots, n - 1, \\ (-1)^{n-k} \lambda \rho_s^{-1} L_s y, & i = n. \end{cases}$$

This implies that problem (46) can be transformed into the problem

$$N_n z = \lambda (-1)^{n-k} \rho_s^{-1} z,$$

$$N_i y(a) = 0, \quad i \in \{\alpha_1 - s, \dots, \alpha_k - s \ (\mathrm{mod}\, n)\},$$

$$N_i y(b) = 0, \quad i \in \{\beta_1 - s, \dots, \beta_{n-k} - s \ (\mathrm{mod}\, n)\},$$

whose boundary conditions are poised, thus being compliant with the hypotheses of Theorem 5. We can apply Theorem 5, calculate z and λ, and obtain y from $N_{n-s}z$, ρ_0 and λ. □

Remark 3 It is worth highlighting that [24] did not focus on $\underline{r}(M, M^j u)$ and $\overline{r}(M, M^j u)$ as the only lower and upper bounds of the principal eigenvalue, but it extended those properties to $\underline{r}(M^j, M^i u)$ and $\overline{r}(M^j, M^i u)$ for different values of i and j, including $i = 0$.

4 Practical considerations for calculating Collatz–Wielandt numbers

The determination of the Collatz–Wielandt numbers $\underline{r}(M, M^j u)$ and $\overline{r}(M, M^j u)$ requires the comparison of several quasi-derivatives of $M^j u$ and $M^{j+1}u$ (those of order $l \in S$) across the interval $[a, b]$. When $M^j u$ is calculated numerically, this is not really possible. However, in [24, Sect. 2.4] several mechanisms were shared which allow reducing the comparison of functions to one point (in some cases) or to a finite set of points. These mechanisms [24, Theorems 14 and 15] are also applicable to this case.

For the concrete case $m = 0$, if the number of boundary conditions set on the quasi-derivative L_0 is one and, in the case $q \equiv 0$, $i_{\max} > 1$, it is straightforward to show that one can reduce the comparison of $M^j u$ and u to a single point (a or b, since $M^j u$ is monotonic and u can be picked up so that it is monotonic), as proposed in [24, Theorem 14]. However, this is not possible if there are two boundary conditions on the quasi-derivative L_0.

For this latter case, nevertheless, it is worth considering the following theorem.

Theorem 7 *Let u be a solution of $Lu = f$ with L as in (1), poised homogeneous boundary conditions $\{\alpha, \beta\}$ and $f \in PC[a, b]$ such that $f \geq 0$ on $[a, b]$ and $f > 0$ on a subset of $[a, b]$ of positive measure. Then the number of zeroes of $L_i u$ on (a, b) is exactly $Z_{i-1}\{\alpha, \beta\} - i$ for $i < i_{\max}$.*

Proof Extending the definitions of $E_i[a, b]$, $Z_i\{\alpha, \beta\}$, and $z_i(a, b)$ from $G(x, t)$ to u, and following the same steps of Lemma 1, one gets to $E_{n-1}[a, b] \geq E_i[a, b] \geq 0$ for $i = 0, \dots, n - 2$. If $E_{n-1}[a, b] > 0$, by Rolle's theorem there must be a change of sign of $Lu = L_n u = f$ in (a, b), which contradicts the hypothesis $f \geq 0$. Therefore $E_{n-1}[a, b] = E_i[a, b] = 0$ and $z_i(a, b) = Z_{i-1}\{\alpha, \beta\} - i$. □

As a result, if $(-1)^{n-k}z(x) \geq 0$, $x \in [a, b]$ with $(-1)^{n-k}z(x) > 0$ in a subset of $[a, b]$ of positive measure, and $Z_0\{\alpha, \beta\} = 2$, given that $(-1)^{n-k}p(x) > 0$ a.e. on $[a, b]$ and $i_{\max} > 1$ (the total number of boundary conditions is n and two of them are set on $L_0 z$), the previous theorem shows that the number of zeroes of $L_1 Mz$ on (a, b) is exactly one, so that $(-1)^{n-k}L_0 Mz$ has only one maximum on that interval. This allows extending [24, Theorem 14] to the case $Z_0\{\alpha, \beta\} = 2$ by means of the following theorems.

Theorem 8 *Let* $z(x) = (-1)^{n-k}$, $x \in [a,b]$. *If* $m = 0$ *and* $Z_0\{\alpha, \beta\} = 2$, *then the calculation of* $\bar{r}(M^j, z)$ *for* $j \geq 1$, *can be restricted to a single point, and these numbers satisfy* $\bar{r}(M^j, z) \geq \frac{1}{\lambda_0^j}$ *and* $\lim_{j \to \infty} (\bar{r}(M^j, z))^{\frac{1}{j}} = \frac{1}{\lambda_0}$.

Proof Since z is constant, in order to calculate $\bar{r}(M^j, z)$ it suffices to determine $\sup\{M^j z(x), x \in [a,b]\}$. The latter assertions are a result of [24, Theorems 7 and 8] and Theorem 5. □

Theorem 9 *Let* c_1, c_2 *be such that* $a < c_1 < c_2 < b$, *and let* z *be defined by*

$$
z(x) = \begin{cases} 0, & x \in [a, c_1), \\ (-1)^{n-k}, & x \in [c_1, c_2], \\ 0, & x \in (c_2, b]. \end{cases}
$$

If $m = 0$ *and* $Z_0\{\alpha, \beta\} = 2$, *then the calculation of* $\underline{r}(M^j, z)$ *for* $j \geq 1$ *can be restricted to the points* c_1 *and* c_2. *In addition,* $\underline{r}(M^j, z) \leq \frac{1}{\lambda_0^j}$ *and* $\lim_{j \to \infty} (\underline{r}(M^j, z))^{\frac{1}{j}} = \frac{1}{\lambda_0}$.

Proof According to Theorem 7, $(-1)^{n-k} M^j z(x)$ must have a single maximum on (a, b). Therefore, if $(-1)^{n-k} M^j z(c_1) \geq r^j$ and $(-1)^{n-k} M^j z(c_2) \geq r^j$, then $M^j z \geq r^j z$. The latter assertions are also a result of [24, Theorems 7 and 8] and Theorem 5. □

5 Example

Let us consider the problem

$$
Ly + \lambda \frac{1}{x} y = 0, \quad x \in [1, 2], \qquad L_1 y(1) = L_2 y(1) = L_3 y(1) = L_3 y(2) = 0, \tag{48}
$$

with $n = 4$ and

$$
\rho_0 = \frac{1}{x}, \qquad \rho_1 = \rho_2 = \rho_4 = 1, \qquad \rho_3 = x.
$$

The boundary conditions of (48) are not poised, so the smallest eigenvalue is $\lambda = 0$. We can apply Theorem 6 to transform the problem into a poised one and use the method of Theorem 5 to estimate the second smallest eigenvalue λ_0. To do so we must first identify the index s for which the problem is s-poised by means of equation (47).

Table 1 shows that the problem is 1, 2, and 3-poised, and we can select any of these indexes for our purposes. A close examination shows that $s = 3$ provides a problem whose Green function is easy to calculate, namely

$$
Nz + \lambda_0 \frac{1}{x} z = 0, \quad x \in [1, 2], \qquad N_0 z(1) = N_2 z(1) = N_3 z(1) = N_0 z(2) = 0, \tag{49}
$$

Table 1 Calculation of index s in the example

i	$Z_i\{\alpha, \beta\}$	$Z_i\{\alpha, \beta\} - i - 1$
0	0	−1
1	1	−1
2	2	−1
3	4	0

Table 2 Collatz–Wielandt numbers for $M^j z$ in the example

	$\underline{r}(M, M^j z)$	$\overline{r}(M, M^j z)$
$j = 0$	0.0000004	0.016373
$j = 1$	0.0059443	0.0074423
$j = 2$	0.0061166	0.0062013
$j = 3$	0.0061211	0.0061245
$j = 4$	0.0061213	0.0061214
$j = 5$	0.0061213	0.0061213

with

$$r_0 = r_1 = r_2 = r_3 = r_4 = 1.$$

Therefore we have to determine the Green function of the problem

$$\frac{\partial^4 G(x,t)}{\partial x^4} = 0, \quad x \in [1,2], \qquad G(1,t) = \frac{\partial^2 G(1,t)}{\partial x^2} = \frac{\partial^3 G(1,t)}{\partial x^3} = G(2,t) = 0, \tag{50}$$

which is

$$G(x,t) = \begin{cases} -\frac{(2-t)^3(x-1)}{6}, & x \in [1,t), \\ -\frac{(2-t)^3(x-1)}{6} + \frac{(x-t)^3}{6}, & x \in (t,2]. \end{cases} \tag{51}$$

Accordingly, the operator M can be calculated as

$$Mz = \frac{x-1}{6} \int_1^x \frac{(2-t)^3}{t} z(t)\, dt - \frac{x^3}{6} \int_1^x \frac{z(t)}{t}\, dt + \frac{x^2}{2} \int_1^x z(t)\, dt$$
$$- \frac{x}{2} \int_1^x t z(t)\, dt + \frac{1}{6} \int_1^x t^2 z(t)\, dt + \frac{x-1}{6} \int_x^2 \frac{(2-t)^3}{t} z(t)\, dt, \quad x \in [1,2]. \tag{52}$$

Note that the representation of Mz in (52), with the variable x outside of the integrals, reduces the memory usage of the algorithm in the case that Mz has to be calculated numerically, e.g., with Simpson's rule for the integrals, as it allows dealing only with vectors instead of matrices.

Starting with the function $z(x) = 1$, $x \in [1,2]$, it is possible to calculate $M^j z$ for different indexes j, in this case from $j = 1$ to $j = 5$, and determine the corresponding Collatz–Wielandt numbers, which are bounds of the inverse of the nonzero smallest eigenvalue of (48). These bounds can be seen in Table 2 and yield the estimation $\lambda_0 = 163.364$.

6 Conclusions

The procedure presented in this paper allows estimating the principal eigenvalue and associated eigenfunction of problems of type (2) as long as p and q satisfy some sign conditions and the boundary conditions are poised. It also allows estimating the second smallest eigenvalue and associated eigenfunction of that type of problems when $m = 0$ and $q \equiv 0$, regardless of the boundary conditions.

The main limitations it presents are related to the sign requirements of p and q, and also to the possible difficulties of finding the Green function $G(x,t)$. This mainly depends on the complexity of the differential operator L, but it is worth highlighting that in the

case $q \equiv 0$ this calculation can be done numerically in a quite straightforward manner as the Green function is just a solution of $L_n G(x, t) = 0$ subject to the boundary conditions $\{\alpha, \beta\}$ and with a discontinuity jump at $x = t$, and such a differential equation can be solved recursively by taking (1) into account.

The calculation of the Collatz–Wielandt numbers can be complex in some cases as they require comparing several quasi-derivatives of two functions throughout the interval $[a, b]$. Nevertheless, as Sect. 4 points out, there exist mechanisms to reduce the comparison to a finite set of points.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
PA proposed the problem. Both authors discussed and agreed on the mechanisms and tools to find a solution. PA focused on the properties of the Green functions and LJ on the nature of the cones to use. PA wrote the first version of the manuscript and LJ reviewed and corrected several points. All authors read and approved the final manuscript.

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