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New fourth and sixth-order classes of iterative methods for solving systems of nonlinear equations and their stability analysis

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Abstract In this paper, a two-step class of fourth-order iterative methods for solving systems of nonlinear equations is presented. We further extend the two-step class to establish a new sixth-order family which requires only one additional functional evaluation. The convergence analysis of the proposed classes is provided under several mild conditions. A complete dynamical analysis is made, by using real multidimensional discrete dynamics, in order to select the most stable elements of both families of fourth and sixth-order of convergence. To get this aim, a novel tool based on the existence of critical points has been used, the parameter line. The analytical discussion of the work is upheld by performing numerical experiments on some application oriented problems. Finally on the basis of numerical results, it has been concluded that our methods are comparable with the existing ones of similar nature in terms of order and efficiency and also that the stability results provide the most efficient member of each class of iterative schemes.

Keywords Systems of nonlinear equations · Order of convergence · Multipoint iterative methods · Stability analysis.

1 Introduction

Systems of nonlinear equations are of immense importance for applications in many areas of science and engineering. For a given nonlinear system, $G(X) = 0$, where $G : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, we are interested to find a vector $X^* = (x_1^*, x_2^*, \cdots, x_n^*)^T$ such that $G(X^*) = 0$, where $G(X) = (g_1(X), g_2(X), \ldots, g_n(X))^T$ is a Fréchet differentiable function and $X = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$. The classical Newton's method is the most basic procedure to solve systems of nonlinear equations. It is given by

$$X^{(k+1)} = X^{(k)} - \{G'(X^{(k)})\}^{-1} G(X^{(k)}),\ k = 0, 1, \ldots \tag{1}$$

where $\{G'(X^{(k)})\}^{-1}$ is the inverse of first order Fréchet derivative of the function G evaluated in $(X^{(k)})$. Assuming that the function G is continuously differentiable and the initial approximation is good enough, then this method converges quadratically. In literature, there are a variety of higher-order methods which improve the order of convergence of Newton's scheme. For example, several authors have developed third-order methods [1–4] each requiring evaluation of one G, two G' and two matrix

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inversion per iteration. Cordero and Torregrosa [5] derived two more third-order methods, one of which requires one G and three G' whereas the other requires one G and four G' evaluations and two matrix inversions. Another third-order method by Darvishi and Barati [6] utilizes two G, two G' and two matrix inversions per iteration. Darvishi and Barati [7] and Potra and Pták [8] have proposed third-order methods that require two G, one G' and one matrix inversion. Babajee *et al.* [9] have presented fourth-order method which consumes one G, two G' and two matrix inversions per iteration. The fourth-order method by Cordero *et al.* [10] is developed using two evaluations of the function and the Jacobian and one matrix inversion, whereas the authors in [11] propose another fourth-order method utilizing three G, one G' and one matrix inversion per iteration. Another fifth-order method proposed by Cordero *et al.* [12] require three evaluations of the function and only one Jacobian evaluation, with the solution of three linear systems with the same matrix of coefficients per iteration.

In pursuit of faster algorithms, researchers have also developed fifth and sixth-order methods for example [13–16]. In [14], Narang *et al.* extended the existing Babajee's fourth-order scheme [17] to solve systems of nonlinear equations and developed a sixth-order convergent family of Chebyshev-Halley type methods. Their scheme requires two G , two G' evaluations and the solution of two linear systems per iteration. One can notice that while the researchers are making an attempt to improve the order of convergence of an iterative method, it mostly leads to increase in the computational cost per step. The computational cost is especially high if the method involves the use of second order Fréchet derivative $G''(X)$. This is a major limitation of the higher-order methods. Thus, while developing new iterative methods, we should try to keep the computational cost low. With this intention, we have made an attempt to develop a family of three-step sixth-order family of methods requiring two G, two G' and one matrix inversion per iteration. This family of methods are compared to be more efficient than existing methods. These have been found to be effective in solving particularly large-scale systems of nonlinear equations.

The outline of the paper is as follows. In Section 2, a parametric family of fourth-order methods are presented along with their convergence analysis. In Section 3, we present a class of new sixth-order schemes by adding a step to the fourth-order family and its convergence analysis. Section 4 is devoted to the analysis of the real dynamics of the proposed classes and the selection of their most stable elements. In Section 5, we test the consistency of convergence behaviour of the methods and examine the theoretical results with help of various numerical experiments. Finally, Section 6 contains some conclusions.

2 Development of fourth-order scheme

In this section, we extended the two-point fourth-order Chebyshev-Halley type methods proposed by Behl and Kanwar [18] for solving systems of nonlinear equations $G(X) = 0$. For this purpose, we write the generalized form as:

$$\begin{aligned} Y^{(k)} &= X^{(k)} - \frac{2}{3}\Gamma(X^{(k)}), \\ X^{(k+1)} &= X^{(k)} - \eta(X^{(k)})^{-1}\mu(X^{(k)})\Gamma(X^{(k)}), \end{aligned} \tag{2}$$

where

$$\begin{aligned} \Gamma(X^{(k)}) &= \{G'(X^{(k)})\}^{-1}G(X^{(k)}), \\ \mu(X^{(k)}) &= A_1 I + 3A_2\Omega(X^{(k)}) + 9A_3\Big(\Omega(X^{(k)})\Big)^2 - 27A_4\Big(\Omega(X^{(k)})\Big)^3, \\ \eta(X^{(k)}) &= 2\Big(A_5 I - 3A_6\Omega(X^{(k)}) + 9A_7\Big(\Omega(X^{(k)})\Big)^2 - 27A_8\Big(\Omega(X^{(k)})\Big)^3, \\ \Omega(X^{(k)}) &= \{G'(X^{(k)})\}^{-1}G'(Y^{(k)}), \end{aligned}$$

and

$$\begin{array}{ll} A_1 = 54a^3 - 135a^2 + 102a - 23, & A_2 = -54a^3 + 135a^2 - 112a + 29, \\ A_3 = 18a^3 - 45a^2 + 38a - 11, & A_4 = (a-1)^2(2a-1), \\ A_5 = 27a^3 - 54a^2 + 33a - 4, & A_6 = 27a^3 - 54a^2 + 35a - 6, \\ A_7 = 9a^3 - 18a^2 + 11a - 2, & A_8 = a(a-1)^2, \end{array}$$

being a a free disposable parameter and I the identity matrix of size $n \times n$. This class is denoted by PM_4.

2.1 Convergence analysis

In order to analyze the convergence of the proposed class (2), we need some tools and procedures introduced in [19]. Let $G : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be sufficiently differentiable in an open neighborhood D. The l-th derivative of G at $u \in \mathbb{R}^n$, $l \geq 1$, is the l-linear function $G^{(l)}(u) : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}^n$ such that $G^{(l)}(u)(v_1, \ldots, v_l) \in \mathbb{R}^n$. It is easy to observe that

1. $G^{(l)}(u)(v_1, \ldots, v_{l-1}, v_l) \in \mathcal{L}(\mathbb{R}^n)$

2. $G^{(l)}(u)(v_{\sigma(1)},\dots,v_{\sigma(l)}) = G^{(l)}(u)(v_1,\dots,v_l)$, for all permutations σ of $\{1,2,\dots,l\}$.

From the above properties, we can use the following notation:

(a) $G^{(l)}(u)(v_1,\dots,v_l) = G^{(l)}(u)v_1,\dots,v_l$
(b) $G^{(l)}(u)v^{l-1}G^{(p)}v^p = G^{(l)}(u)G^{(p)}(u)v^{p+l-1}$.

On the other side, for $(X^*+h)\in\mathbb{R}^n$ lying in a neighborhood of a solution X^* of $G(X)=0$, we can apply Taylor's series expansion and assuming that the Jacobian matrix $G'(X^*)$ is non-singular, we have

$$G(X^*+h) = G'(X^*)\Big[h+\sum_{l=2}^{p-1}C_lh^l\Big]+O(h^p), \tag{3}$$

where $C_l = \frac{1}{l!}\{G'(X^*)\}^{-1}G^{(l)}(X^*)$ for $l\geq 2$. It is clear that $C_lh^l\in\mathbb{R}^n$, since $G^{(l)}(X^*)\in\mathcal{L}(\mathbb{R}^n\times\cdots\times\mathbb{R}^n,\mathbb{R}^n)$ and $\{G'(X^*)\}^{-1}\in\mathcal{L}(\mathbb{R}^n)$. In addition, G' can be expressed as

$$G'(X^*+h) = G'(X^*)\Big[I+\sum_{l=2}^{p-2}lC_lh^{l-1}\Big]+O(h^{p-1}), \tag{4}$$

where I is the identity matrix. Therefore, $lC_lh^{l-1}\in\mathbb{R}^n$. From equation (4), we have

$$\{G'(X^*+h)\}^{-1} = \{G'(X^*)\}^{-1}\Big(I+C_2^*h+C_3^*h^2+C_4^*h^3+C_5^*h^4+\dots\Big)+O(h^p), \tag{5}$$

where

$$\begin{aligned}
C_2^* =& -2C_2,\\
C_3^* =& 4C_2^2-3C_3,\\
C_4^* =& -4C_4+6C_2C_3+6C_3C_2-8C_2^3,\\
&\vdots
\end{aligned}$$

Let $e^{(k)} = X^{(k)}-X^*$ be the error at k^{th} iteration. The equation $e^{(k+1)} = M(e^{(k)})^\rho + O((e^{(k)})^{\rho+1})$, where M is a ρ-linear function $M\in\mathcal{L}(\mathbb{R}^n\times\cdots\times\mathbb{R}^n,\mathbb{R}^n)$, is called error equation, being ρ its convergence order.

Theorem 1 *Let $G:D\subseteq\mathbb{R}^n\to\mathbb{R}^n$ be sufficiently differentiable in an open neighborhood D of X^* which is solution of the system $G(X)=0$. Consider that initial guess $X^{(0)}$ is sufficiently close to the required solution X^* and $G'(X)$ is continuous and nonsingular in X^*. Then, the proposed iterative scheme* (2) *has local fourth-order of convergence for all $a\in\mathbb{R}$.*

Proof Let $e^{(k)} = X^{(k)}-X^*$ be the error at kth iteration. With the help of Taylor's series expansion, one can obtain the following expansions of the function $G(X^{(k)})$ and its first-order derivative $G'(X^{(k)})$ around the point X^*.

$$\begin{aligned}
G(X^{(k)}) = G'(X^*)\Big(&e^{(k)}+C_2(e^{(k)})^2+C_3(e^{(k)})^3+C_4(e^{(k)})^4+C_5(e^{(k)})^5\\
&+C_6(e^{(k)})^6+O((e^{(k)})^7)\Big).
\end{aligned} \tag{6}$$

$$\begin{aligned}
G'(X^{(k)}) = G'(X^*)\Big(&I+2C_2e^{(k)}+3C_3(e^{(k)})^2+4C_4(e^{(k)})^3+5C_5(e^{(k)})^4\\
&+6C_6(e^{(k)})^5+O((e^{(k)})^6)\Big),
\end{aligned} \tag{7}$$

where $C_j = \frac{1}{j!}\{G'(X^*)\}^{-1}G^{(j)}(X^*)$ for $j=2,3,4\dots$.

From (7), one can obtain the expression of the inverse

$$\begin{aligned}
\{G'(X^{(k)})\}^{-1} = \{G'(X^*)\}^{-1}\Big(&I-2C_2e^{(k)}+\Big(4C_2^2-3C_3\Big)(e^{(k)})^2\\
&+\Big(-8C_2^3+12C_3C_2-4C_4\Big)(e^{(k)})^3\\
&+\Big(16C_2^4-36C_3C_2^2+16C_4C_2+9C_3^2-5C_5\Big)(e^{(k)})^4+O\Big((e^{(k)})^5\Big)\Big).
\end{aligned} \tag{8}$$

By using expressions (6) and (8), one gets

$$
\begin{aligned}
\{G'(X^{(k)})\}^{-1}G(X^{(k)}) &= e^{(k)} - C_2(e^{(k)})^2 + 2\Big(C_2^2 - C_3\Big)(e^{(k)})^3 \\
&+ \Big(4C_2C_3 + 3C_3C_2 - 4C_2^3 - 3C_4\Big)(e^{(k)})^4 \\
&+ \Big(- 4C_5 + 6C_2C_4 - 8C_2^2C_3 + 6C_3^2 + 4C_4C_2 - 6C_2C_3C_2 \\
&+ 8C_2^4 - 6C_3C_2^2\Big)(e^{(k)})^5 - \Big(C_6 - 2C_2C_5 - 5C_5C_2 + 4C_2^2C_4 \\
&+ 8C_4C_2^2 - 3C_3C_4 - 4C_4C_3 + 6C_2C_3^2 + 6C_3C_2C_3 - 8C_2^3C_3 \\
&+ 8C_2C_4C_2 - 12C_2^2C_3C_2 + 9C_3^2C_2 - 12C_2C_3C_2^2 + 16C_2^5 \\
&- 12C_3C_2^3\Big)(e^{(k)})^6 + O((e^{(k)})^7).
\end{aligned}
\tag{9}
$$

The first-step of proposed scheme (2) can be rewritten as:

$$
Y^{(k)} - X^* = X^{(k)} - X^* - \frac{2}{3}\{G'(X^{(k)})\}^{-1}G(X^{(k)}).
\tag{10}
$$

In view of (9), equation (10) yields as:

$$
\begin{aligned}
Y^{(k)} - X^* &= \frac{1}{3}e^{(k)} + \frac{2}{3}C_2(e^{(k)})^2 - \frac{2}{3}\Big(2C_2^2 - 2C_3\Big)(e^{(k)})^3 \\
&- \frac{2}{3}\Big(- 4C_2^3 + 7C_3C_2 - 3C_4\Big)(e^{(k)})^4 \\
&- \frac{2}{3}\Big(8C_2^4 - 20C_3C_2^2 + 10C_4C_2 + 6C_3^2 - 4C_5\Big)(e^{(k)})^5 \\
&- \frac{2}{3}\Big(- 16C_2^5 + 52C_3C_2^3 - 28C_4C_2^2 - 33C_3^2C_2 + 13C_5C_2 \\
&+ 17C_3C_4 - 5C_6\Big)(e^{(k)})^6 + O\Big((e^{(k)})^7\Big).
\end{aligned}
\tag{11}
$$

Further, the Taylor's series expansions of $G(Y^{(k)})$ and $G'(Y^{(k)})$ around the point X^* are given as:

$$
\begin{aligned}
G(Y^{(k)}) &= G'(X^*)\Big[(Y^{(k)} - X^*) + C_2(Y^{(k)} - X^*)^2 + C_3(Y^{(k)} - X^*)^3 + O((Y^{(k)} - X^*)^4)\Big], \\
&= G'(X^*)\Big[\frac{1}{3}e^{(k)} + \frac{7}{9}C_2(e^{(k)})^2 - \frac{1}{27}\Big(24C_2^2 - 37C_3\Big)(e^{(k)})^3 + \frac{1}{81}\Big(180C_2^3 \\
&- 288C_3C_2 + 163C_4\Big)(e^{(k)})^4 - \frac{1}{243}\Big(1296C_2^4 - 2916C_3C_2^2 + 1272C_4C_2 \\
&+ 864C_3^2 - 649C_5\Big)(e^{(k)})^5 + \frac{1}{729}\Big(9072C_2^5 - 26352C_3C_2^3 + 12384C_4C_2^2 \\
&+ 15552C_3^2C_2 - 4992C_5C_2 - 7632C_3C_4 + 2431C_6\Big)(e^{(k)})^6 + O\Big((e^{(k)})^7\Big)\Big],
\end{aligned}
\tag{12}
$$

and

$$
\begin{aligned}
G'(Y^{(k)}) &= G'(X^*)\Big(I + \frac{2}{3}C_2e^{(k)} + \frac{1}{3}\Big(4C_2^2 + C_3\Big)(e^{(k)})^2 + \Big(- \frac{8}{3}C_2^3 + 4C_3C_2 + \frac{4C_4}{27}\Big)(e^{(k)})^3 \\
&+ \frac{1}{81}\Big(432C_2^4 - 864C_3C_2^2 + 396C_4C_2 + 216C_3^2 + 5C_5\Big)(e^{(k)})^4 + O\Big((e^{(k)})^5\Big)\Big).
\end{aligned}
\tag{13}
$$

Also, by using equations (8) and (13), one can have

$$
\begin{aligned}
\Omega(X^{(k)}) &= I - \frac{4C_2}{3}e^{(k)} + \Big(4C_2^2 - \frac{8C_3}{3}\Big)(e^{(k)})^2 - \frac{8}{27}\Big(36C_2^3 - 45C_3C_2 + 13C_4\Big)(e^{(k)})^3 \\
&+ \frac{4}{81}\Big(540C_2^4 - 999C_3C_2^2 + 363C_4C_2 + 216C_3^2 - 100C_5\Big)(e^{(k)})^4 + O\Big((e^{(k)})^5\Big),
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
(\Omega(X^{(k)}))^2 &= I - \frac{8C_2}{3}e^{(k)} + \frac{8}{9}\Big(11C_2^2 - 6C_3\Big)(e^{(k)})^2 - \frac{16}{27}\Big(54C_2^3 - 57C_3C_2 \\
&+ 13C_4\Big)(e^{(k)})^3 + \frac{8}{81}\Big(990C_2^4 - 1575C_3C_2^2 + 467C_4C_2 + 288C_3^2 \\
&- 100C_5\Big)(e^{(k)})^4 + O\Big((e^{(k)})^5\Big),
\end{aligned}
\tag{15}
$$

and

$$\begin{aligned}(\Omega(X^{(k)}))^3 = I - 4C_2e^{(k)} + \Big(\frac{52C_2^2}{3} - 8C_3\Big)(e^{(k)})^2 - \frac{8}{27}\Big(224C_2^3 - 207C_3C_2 \\ + 39C_4\Big)(e^{(k)})^3 + \frac{4}{27}\Big(1584C_2^4 - 2247C_3C_2^2 + 571C_4C_2 + 360C_3^2 \\ - 100C_5\Big)(e^{(k)})^4 + O\Big((e^{(k)})^5\Big).\end{aligned} \tag{16}$$

In view of equations (9), (14), (15) and (16), the second-step of scheme (2), one gets the following error equation

$$\begin{aligned}X^{(k+1)} - X^* = \frac{1}{9}\Big(-72a^2C_2^3 + 144aC_2^3 - 63C_2^3 - 9C_3C_2 + C_4\Big)(e^{(k)})^4 \\ + \frac{2}{27}\Big(216a^3C_2^4 - 648a^2C_3C_2^2 - 648aC_2^4 + 1296aC_3C_2^2 \\ + 378C_2^4 - 540C_3C_2^2 - 30C_4C_2 - 27C_3^2 + 4C_5\Big)(e^{(k)})^5 + O\Big((e^{(k)})^6\Big).\end{aligned} \tag{17}$$

This implies that the scheme (2) achieves fourth-order convergence. This completes the proof. □

3 The sixth-order scheme and its convergence analysis

Now, we propose the following parametric family of sixth-order iterative scheme based on (2) by introducing an additional step requiring only one new function evaluation as follows:

$$\begin{aligned}Y^{(k)} &= X^{(k)} - \frac{2}{3}\Gamma(X^{(k)}),\\ Z^{(k)} &= X^{(k)} - \eta(X^{(k)})^{-1}\mu(X^{(k)})\Gamma(X^{(k)}),\\ X^{k+1} &= Z^{(k)} - \Big(pI + \Omega(X^{(k)})\Big(qI + r\Omega(X^{(k)})\Big)\Big)\{G'(X^{(k)})\}^{-1}G(Z^{(k)}),\end{aligned} \tag{18}$$

where p, q, r are free disposable parameters, $\Gamma(X^{(k)})$, $\mu(X^{(k)})$, $\eta(X^{(k)})$, and $A_i's$ (for $i = 1, 2, \dots, 8$) are defined as previously in scheme (2). In the remain of the manuscript, this family will be denoted by PM_6.

In the following result, we establish the local order of convergence of class (18).

Theorem 2 *Let $G : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be sufficiently differentiable in an open neighborhood D of X^* which is a solution of $G(X) = 0$. Consider that initial guess $X^{(0)}$ is sufficiently close to the required zero X^* and $G'(X)$ is continuous and nonsingular in X^*. Then, for all $a \in \mathbb{R}$ and $r \in \mathbb{R}$, the local order of convergence of the sequence $\{X^{(k)}\}$ generated by the proposed iterative scheme* (18) *is six for $p = r + \frac{5}{2}$ and $q = -\frac{3}{2} - 2r$.*

Proof Using the error equation (17), expand the function $G(Z^{(k)})$ around the point X^*, one obtains

$$G(Z^{(k)}) = G'(X^*)\Big[(Z^{(k)} - X^*) + C_2(Z^{(k)} - X^*)^2 + C_3(Z^{(k)} - X^*)^3 + O((Z^{(k)} - X^*)^4)\Big]. \tag{19}$$

Pre-multiply the (19) with (8) and after simplification, one can have

$$\begin{aligned}\{G'(X^{(k)})\}^{-1}G(Z^{(k)}) = \frac{1}{9}\Big(-72a^2C_2^3 + 144aC_2^3 - 63C_2^3 - 9C_3C_2 + C_4\Big)(e^{(k)})^4 \\ + \frac{2}{27}\Big(216a^3C_2^4 + 216a^2C_2^4 - 648a^2C_3C_2^2 - 1080aC_2^4 \\ + 1296aC_3C_2^2 + 567C_2^4 - 513C_3C_2^2 - 33C_4C_2 - 27C_3^2 + 4C_5\Big)(e^{(k)})^5 \\ + O\Big((e^{(k)})^6\Big).\end{aligned} \tag{20}$$

In view of equations (14) and (20), one can have

$$\begin{aligned}&\Big(pI + \Omega(X^{(k)})\Big(qI + r\Omega(X^{(k)})\Big)\Big)\{G'(X^{(k)})\}^{-1}G(Z^{(k)}) = \\ &(p + q + r) - \frac{4}{3}\Big(C_2q + 2C_2r\Big)e^{(k)} + \frac{4}{9}\Big(9C_2^2q - 6C_3q + 22C_2^2r - 12C_3r\Big)(e^{(k)})^2 \\ &- \frac{8}{27}\Big(36C_2^3q - 45C_3C_2q + 13C_4q + 108C_2^3r - 114C_3C_2r + 26C_4r\Big)(e^{(k)})^3 \\ &+ \frac{4}{81}\Big(540C_2^4q - 999C_3C_2^2q + 363C_4C_2q + 216C_3^2q - 100C_5q + 1980C_2^4r \\ &- 3150C_3C_2^2r + 934C_4C_2r + 576C_3^2r - 200C_5r\Big)(e^{(k)})^4 + O\Big((e^{(k)})^5\Big).\end{aligned} \tag{21}$$

Substituting (21) in the last step of method (18), one obtains

$$\begin{aligned}
X^{(k+1)} - X^* &= \frac{1}{9}\Big(9(8(a-2)a+7)C_2^3 + 9C_3C_2 - C_4\Big)(p+q+r-1)(e^{(k)})^4 \\
&+ \frac{2}{27}\Big(-216a^3C_2^4 + 648(a-2)aC_3C_2^2 + 27C_3^2 - 4C_5\Big)(p+q+r-1) \\
&+ \frac{2}{27}\Big(-216a^2C_2^4p - 360a^2C_2^4q - 504a^2C_2^4r + 1080aC_2^4p + 1368aC_2^4q \\
&+ 1656aC_2^4r - 648aC_2^4 - 63C_2^4(9p+11q+13r-6) + C_4C_2(33p \\
&+ 35q + 37r - 30) + 513C_3C_2^2p + 495C_3C_2^2q + 477C_3C_2^2r - 540C_3C_2^2\Big)(e^{(k)})^5 \\
&+ \frac{1}{81}\Big(5184a^4C_2^5 - 10368a^3C_3C_2^3 + 5616(a-2)aC_4C_2^2 + 7776(a-2)aC_3^2C_2 \\
&- 42C_6\Big)(p+q+r-1) + \frac{1}{81}\Big(-6480a^3C_2^5p - 4752a^3C_2^5q - 3024a^3C_2^5r \\
&+ 9072a^3C_2^5 + 288a^2C_2^5(45p+60q+79r-36) - 5832a^2C_3C_2^3p \\
&- 12744a^2C_3C_2^3q - 19656a^2C_3C_2^3r - 3888a^2C_3C_2^3 + 432aC_3C_2^3(99p \\
&+ 131q + 163r - 54) - 27216aC_2^5p - 41040aC_2^5q - 57168aC_2^5r \\
&+ 14256aC_2^5 + 2C_5C_2(159p + 175q + 191r - 135) + 3C_3C_4(207p \\
&+ 215q + 223r - 198) + 13770C_2^5p - 22275C_3C_2^3p + 4230C_4C_2^2p \\
&+ 5751C_3^2C_2p + 20574C_2^5q - 27567C_3C_2^3q + 3930C_4C_2^2q + 5319C_3^2C_2q \\
&+ 28386C_2^5r - 32715C_3C_2^3r + 3614C_4C_2^2r + 4887C_3^2C_2r - 6966C_2^5 \\
&+ 14418C_3C_2^3 - 4626C_4C_2^2 - 6318C_3^2C_2\Big)(e^{(k)})^6 + O\Big((e^{(k)})^7\Big).
\end{aligned} \tag{22}$$

To achieve the sixth-order of convergence, the coefficients of $(e^{(k)})^4$ and $(e^{(k)})^5$ must be equal to zero, which reduces to the following linear system:

$$\begin{aligned}
p + q + r &= 1, \\
3p + 5q + 7r &= 0.
\end{aligned} \tag{23}$$

Now, solving the above system of equations, one gets $p = r + \frac{5}{2}$ and $q = -\frac{3}{2} - 2r$. Furthermore, (22) reduces to the following final error equation

$$\begin{aligned}
X^{(k+1)} - X^* &= \frac{1}{81}\Big(9(8(a-2)a+7)C_2^3 + 9C_3C_2 - C_4\Big)\Big(2C_2^2(8r-27) + 9C_3\Big)(e^{(k)})^6 \\
&+ O\Big((e^{(k)})^7\Big).
\end{aligned} \tag{24}$$

This completes the proof of Theorem 2. □

Finally, using the conditions of the Theorem 2, we obtain the sixth-order scheme as follows:

$$\begin{aligned}
Y^{(k)} &= X^{(k)} - \frac{2}{3}\Gamma(X^{(k)}), \\
Z^{(k)} &= X^{(k)} - \eta(X^{(k)})^{-1}\mu(X^{(k)})\Gamma(X^{(k)}), \\
X^{k+1} &= Z^{(k)} - \Big((r+\frac{5}{2})I + \Omega(X^{(k)})\Big((-\frac{3}{2} - 2r)I + r\Omega(X^{(k)})\Big)\Big)\{G'(X^{(k)})\}^{-1}G(Z^{(k)}),
\end{aligned} \tag{25}$$

where $a, r \in \mathbb{R}$.

4 Stability analysis of class PM_4

From the early works of Fatou and Julia [20,21] at the end of XIX-th century about the qualitative performance of Newton's iterative method on quadratic polynomials (by means of the analysis of its associate rational function), discrete dynamics has shown to be a powerful tool for the study of the stability of iterative schemes for solving nonlinear problems. Nowadays, when a class of iterative methods depending on one or more parameters is designed, the analysis of its performance on quadratic or cubic polynomials allows us to select the most stable members of the class (see, for example, [33,22–28]).

These elements are also numerically checked in order to test their performance on other functions. Although there not exist any result showing that these behavior will be held, it is shown in practice that an unstable iterative method on low-degree polynomials will be also unstable for another nonlinear functions. Also, when an iterative scheme shows a clear stable performance (as Newton's method does) then this good behavior will remain, up to some extent, when it is applied on more complicated functions.

The most used qualitative techniques are those of real and complex discrete dynamics, when the iterative processes under analysis are scalar. That is, when the methods are designed for solving nonlinear equations. Nevertheless, when the scheme or family of iterative methods are constructed for solving systems of nonlinear equations, the multidimensional real dynamics is needed to study its stability. In these cases, a system of second-degree polynomials is used and the resulting rational function is also multidimensional.

Let us also remark that, when the iterative scheme uses the evaluation of Jacobian matrices on the previous iterate, the Scaling Theorem is satisfied (see, for example, [29,30]) and then the dynamics related to the rational function is equivalent, under conjugation, to that resulting from using any order second-degree polynomial system. Therefore, we work with the simplest nonlinear quadratic system.

In what follows, we use real multidimensional discrete dynamics tools to determine which elements of proposed classes of iterative methods PM_4 posses better performance, in terms of the dependence of their convergence on the initial estimations used. To get this aim, let us recall some concepts.

Let us denote by $R(x)$ the vectorial fixed-point rational function associated to an iterative method (or a parametric family of schemes) applied to n-variable polynomial system $p(x) = 0, p : \mathbb{R}^n \to \mathbb{R}^n$. Let us also remark that the majority of the definitions can be directly extended from those in complex dynamics (see, for example, [31,29] for more explanations).

In the following, we denote by $M^4(x,a) = (m_1^4(x,a), m_2^4(x,a), \dots, m_n^4(x,a))$, the fixed point function associated to (2) class, applied on a system of quadratic polynomials of size n, $p(x) = 0$, where:

$$p_i(x) = x_i^2 - 1,\ i = 1, 2, \dots, n. \tag{26}$$

Let us remark that our system has separated variables, so all the coordinate functions $m_j^4(x,a)$ are similar, with the difference of the index $j = 1, 2, \dots, n$. The expression of these coordinate functions are:

$$m_j^4(x,a) = \frac{1 - 6x_j^2 - 6x_j^4 - 6x_j^6 + x_j^8 + 2a^3 p_j(x)^3 \left(1 + x_j^2\right) - a^2 p_j(x)^3 \left(5 + 3x_j^2\right) - 2a\left(2 - 7x_j^2 + x_j^4 + 3x_j^6 + x_j^8\right)}{-8a^2 x_j p_j(x)^3 + 4a^3 x_j p_j(x)^3 - 8\left(x_j^3 + x_j^5\right) - 4a\left(x_j - 5x_j^3 + 3x_j^5 + x_j^7\right)}, \tag{27}$$

for $j = 1, 2, \dots, n$.

Let us remark that the components of this rational function can be highly simplified for specific values of parameter a; in particular, for $a = 1$, the resulting components are:

$$m_j^4(x,1) = \frac{1 + 6x_j^2 + x_j^4}{4\left(x_j + x_j^3\right)}, \tag{28}$$

for $j = 1, 2, \dots, n$.

A summary of the stability study of the fixed points of $M^4(x,a)$ appears in the following result.

Theorem 3 *Rational function $M^4(x,a)$ associated to the family of iterative methods (2) has 2^n superattracting fixed points whose components are roots of $p(x)$. It also has some real fixed points different from the roots whose components are found combining the roots of polynomial $l(t) = 1 - 4a + 5a^2 - 2a^3 + (-5 + 14a - 15a^2 + 6a^3)t^2 + (-3 - 8a + 15a^2 - 6a^3)t^4 + (-1 - 2a - 5a^2 + 2a^3)t^6$, depending on a, and the roots of $p(x)$:*

(a) If $a < \frac{1}{2}$, there exist two real roots of $l(t)$, denoted by $l_i(a)$, $i = 1, 2$, whose respective eigenvalues are greater than one (in absolute value). So, the strange fixed point expressed as $(l_{\sigma_1}(a), l_{\sigma_2}(a), \dots, l_{\sigma_n}(a))$ being $\sigma_i \in \{1, 2\}$, are repulsive. Moreover, if at least one of the components of the strange fixed point (but not all) are equal to 1 or -1, it will be a saddle fixed point.

(b) If $\frac{1}{2} \le a < c^*$, being $c^* \approx 2.90369$ the biggest real root of polynomial $-1 - 2t - 5t^2 + 2t^3$, then the roots of polynomial $l(t)$ are complex and there not exist any real strange fixed point.

(c) If $a > c^*$, then $l_1(\alpha)$ and $l_2(a)$ are real. Moreover, the strange fixed point $(l_{\sigma_1}(a), l_{\sigma_2}(a), \dots, l_{\sigma_n}(a))$ being $\sigma_i \in \{1, 2\}$, are attracting if $c^* < a < c^{**} \approx 3.864355$ and repulsive if $a > c^{**}$ (c^{**} is the greatest real root of polynomial $-167 + 958t - 2031t^2 + 2666t^3 - 3716t^4 + 4152t^5 - 2224t^6 + 352t^7$). Indeed, if at least one of the components of the strange fixed point (but not all) are equal to ± 1, it will be a saddle fixed point for $a > 3.864355$ and attracting in other cases.

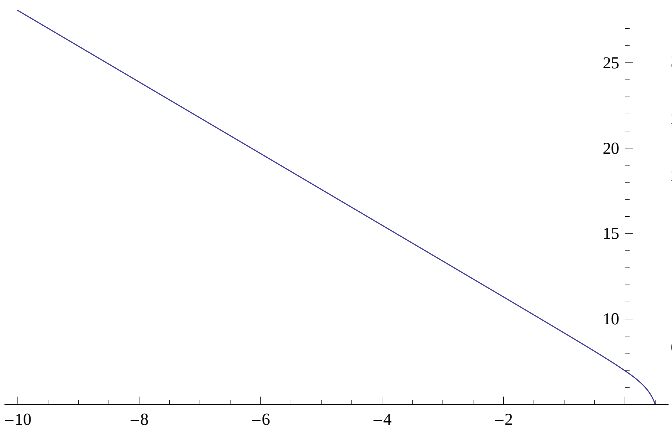
Proof Fixed points of $M^4(x,a)$ can be obtained by solving $m_j^4(x,a) = x_j, : -(x_j^2 - 1)(1 - 4a + 5a^2 - 2a^3 + (-5 + 14a - 15a^2 + 6a^3)x_j^2 + (-3 - 8a + 15a^2 - 6a^3)x_j^4 + (-1 - 2a - 5a^2 + 2a^3)x_j^6) = 0,\ \ j = 1, 2, \dots, n$, that is, the components of the fixed points are $x_j = \pm 1$ and also the roots of the polynomial $l(t)$, provided that $t \ne 0$.

Let us denote the roots of $l(t)$ as $l_i(a)$, $i = 1, 2, \dots, 6$. It can be checked that at most two of the roots of $l(t)$ are real, depending on the value of parameter a. The stability of the fixed points of $M^4(x,a)$ is given by the absolute value of the eigenvalues of the associated Jacobian matrix evaluated at the fixed points. Due to the nature of the polynomial system, these eigenvalues coincide with the coordinate function of the rational operator:

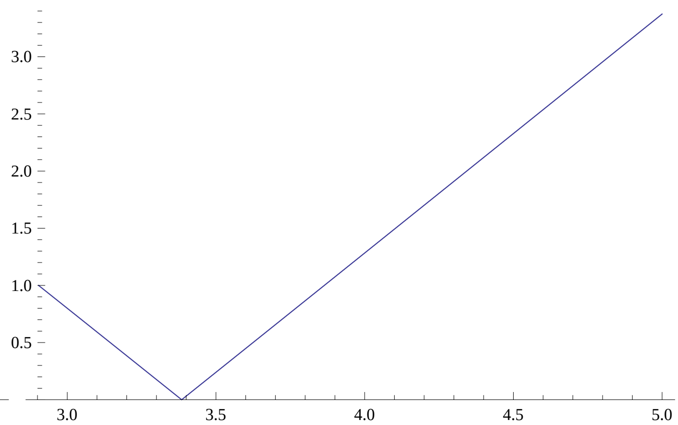
$$Eig_j(l_j(a),\dots,l_j(a)) = (\mu_j(a))^3 \left[\frac{2a^6(\mu_j(a))^4 - a^5(\mu_j(a))^3\alpha_1(a) + 2a^4(\mu_j(a))^2\alpha_2(a)}{4l_j(a)^2(2a^2(\mu_j(a))^3 - a^3(\mu_j(a))^3 + 2\beta_1(a) + a\beta_2(a))^2} + \frac{-2l_j(a)^2\alpha_3(a) + 2a^2\alpha_4(a) - a\alpha_5(a) + 2a^3\alpha_6(a)}{4l_j(a)^2(2a^2(\mu_j(a))^3 - a^3(\mu_j(a))^3 + 2\beta_1(a) + a\beta_2(a))^2}\right],$$

being $\mu_j(a) = -1 + l_j(a)^2$, $\alpha_1(a) = -9 + 7l_j(a)^2$, $\alpha_2(a) = 8 - 17l_j(a)^2 + l_j(a)^4$, $\alpha_3(a) = 3 + 8l_j(a)^2 + 3l_j(a)^4$, $\alpha_4(a) = 3 - 32l_j(a)^2 - 5l_j(a)^4 + 18l_j(a)^6$, $\alpha_5(a) = 1 - 30l_j(a)^2 - 42l_j(a)^4 + 6l_j(a)^6 + l_j(a)^8$, $\alpha_6(a) = -7 + 40l_j(a)^2 - 33l_j(a)^4 - 4l_j(a)^6 + 4l_j(a)^8$, $\beta_1(a) = l_j(a)^2 + l_j(a)^4$ and $\beta_2(a) = 1 - 5l_j(a)^2 + 3l_j(a)^4 + l_j(a)^6$.

By taking into account the absolute value of this eigenvalues in the intervals where different fixed points are real, we state that those fixed points whose components are ± 1 are superattracting. Roots of $l(t)$ are real for $a > c^*$, being $c^* \approx 2.90369$ the biggest real root of polynomial $-1 - 2t - 5t^2 + 2t^3$. Then, it can be checked that the respective eigenvalues are bigger than one in absolute value, except in case $c^* < a < c^{**} \approx 3.864355$, as c^{**} is the greatest real root of polynomial $-167 + 958t - 2031t^2 + 2666t^3 - 3716t^4 + 4152t^5 - 2224t^6 + 352t^7$, where they are lower than one. Therefore, combinations among the roots of $l(t)$ can give rise to attracting or repulsive strange fixed points depending on the value of a. Moreover, all the fixed points whose components are ± 1 and real $l_j(\alpha)$ are classified as saddle. In Figure 1, some of the eigenvalues are plotted; if $a < \frac{1}{2}$, the eigenvalues of the Jacobian matrix evaluated at all the real strange fixed points have the same performance (see Figure 1a), being higher than one (so, points are repulsive). Moreover, for $a > c^*$, the possibility of attracting strange fixed point is deduced (see Figure 1b), as the respective eigenvalues are lower than one when $c^* < a < c^{**} \approx 3.864355$. □



(a) $Eig_j(l_1(a),\dots,l_1(a),a)$ for $a < \frac{1}{2}$



(b) $Eig_j(l_1(a),\dots,l_1(a),a)$ for $a > c^*$

Fig. 1: Eigenvalues associated to the fixed points

Once the existence of strange fixed points is studied and their stability is determined, it is necessary to analyze if it is possible to get any other attracting behavior, as attracting periodic orbits or even strange attractors. This can be made through the orbits of the free critical points, if they exist.

4.1 Bifurcation Analysis of Free Critical Points

Now, we analyze the Jacobian matrix $M^{4'}(x,a)$ of the rational function under analysis and its critical points. Let us recall that, in this context, the critical point are the solutions of $\det\left(M^{4'}(x,a)\right) = 0$. When the critical point is not a solution of $p(x) = 0$, it is called free critical point.

Theorem 4 *The free critical points of operator $M^4(x,a)$, denoted by $(cr_{\sigma_1}(a), cr_{\sigma_n}(a),\dots,cr_{\sigma_n}(a))$, $\sigma_i \in \{1,2,\dots,m\}$ $m \le 6$, are those making zero all the components of the Jacobian matrix, for $j = 1,2,\dots,n$, whose components are different from those of the roots of $p(x)$, that is:*

(a) If $a \le k^* \approx -0.744644$, $1-\sqrt{2} < a < 0$, $\frac{1}{4}\left(4-\sqrt{2}\right) < a < 0.710821$, $1.18627 < a < \frac{1}{4}\left(4+\sqrt{2}\right)$, $\frac{1}{4}\left(4+\sqrt{2}\right) < a < 1.88923$ or $a > 1+\sqrt{2}$ (where k^* is the lowest root of polynomial $1-2t-3t^2+2t^3$) then $cr_1(a) = -\sqrt{s^*}$, $cr_2(a) = \sqrt{s^*}$,

$cr_3(a) = -\sqrt{s^{**}}$, $cr_4(a) = \sqrt{s^{**}}$ are the different components of the free critical points, being s^* and s^{**} the two positive roots of polynomial $z(s) = -a+6a^2-14a^3+16a^4-9a^5+2a^6+(-6+30a-64a^2+80a^3-66a^4+34a^5-8a^6)s+(-16+42a-10a^2-66a^3+86a^4-48a^5+12a^6)s^2+(-6-6a+36a^2-8a^3-38a^4+30a^5-8a^6)s^3+(-a+8a^3+2a^4-7a^5+2a^6)s^4$.

(b) If $k^* < a < -0.578202$, $-0.464045 < a < -0.429149$, $-0.429149 < a < 1-\sqrt{2}$ or $\frac{1}{2} < a < \frac{1}{4}\left(4-\sqrt{2}\right)$, then $cr_1(a)$, $cr_2(a)$, $cr_3(a)$, $cr_4(a)$, $cr_5(a) = -\sqrt{s^{***}}$ and $cr_6(a) = \sqrt{s^{***}}$ are the different components of the free critical points, being s^*, s^{**} and s^{***} the three positive roots of polynomial $z(s)$ in this interval.

(c) If $a = -0.578202$ or $a = -0.464045$, the free critical points have as components $cr_1(a)$, $cr_3(a)$, $cr_4(a)$ and $cr_5(a)$.

(d) If $-0.578202 < a < -0.464045$ or $0.355416 < a < \dfrac{1}{2}$ or $1.88923 < a < 1+\sqrt{2}$, then $cr_1(a)$ and $cr_2(a)$ are the only components of the free critical points.

(e) If $a = -0.429149$, then $cr_1(a)$, $cr_3(a)$, $cr_4(a)$, and $cr_6(a)$ are the different components of the free critical points.

(f) If $0 \le a \le 0.355416$ or $0.710821 < a < 1.18627$, then there are no free critical points.

(g) If $a = \dfrac{1}{2}$, then the components of the free critical points are ± -2.54246, the only real roots of polynomial $-3-6t^2+t^4$.

(h) If $a = \frac{1}{4}\left(4-\sqrt{2}\right)$ or $a = \frac{1}{4}\left(4+\sqrt{2}\right)$, then the components of the free critical points are $cr_2(a)$ and $cr_3(a)$.

(i) If $a = 0.710821$ or $a = 1.18627$, then the components of the free critical points are $cr_1(a)$ and $cr_3(a)$.

Proof The components of the Jacobian matrix $M^{4'}(x,a)$ different from zero are

$$\begin{aligned}\frac{\partial m_j^4(x,a)}{\partial x_j} &= (p_j(x))^3\left[\frac{2a^6(p_j(x))^4-a^5(p_j(x))^3\left(-9+7x_j^2\right)+2a^4(p_j(x))^2\left(8-17x_j^2+x_j^4\right)}{4x_j^2\left(2a^2(p_j(x))^3-a^3(p_j(x))^3+2\left(x_j^2+x_j^4\right)+a\left(1-5x_j^2+3x_j^4+x_j^6\right)\right)^2}\right.\\ &\quad+\frac{-2x_j^2\left(3+8x_j^2+3x_j^4\right)+2a^2\left(3-32x_j^2-5x_j^4+18x_j^6\right)}{4x_j^2\left(2a^2(p_j(x))^3-a^3(p_j(x))^3+2\left(x_j^2+x_j^4\right)+a\left(1-5x_j^2+3x_j^4+x_j^6\right)\right)^2}\\ &\quad\left.+\frac{-a\left(1-30x_j^2-42x_j^4+6x_j^6+x_j^8\right)+2a^3\left(-7+40x_j^2-33x_j^4-4x_j^6+4x_j^8\right)}{4x_j^2\left(2a^2(p_j(x))^3-a^3(p_j(x))^3+2\left(x_j^2+x_j^4\right)+a\left(1-5x_j^2+3x_j^4+x_j^6\right)\right)^2}\right],\quad j=1,2.\end{aligned}$$

Therefore, it is straightforward that the roots of $-a+6a^2-14a^3+16a^4-9a^5+2a^6+(-6+30a-64a^2+80a^3-66a^4+34a^5-8a^6)t^2+(-16+42a-10a^2-66a^3+86a^4-48a^5+12a^6)t^4+(-6-6a+36a^2-8a^3-38a^4+30a^5-8a^6)t^6+(-a+8a^3+2a^4-7a^5+2a^6)t^8$, when they are real, are the components of the critical points. A simple change $s=t^2$ yields to solve $z(s)$ in order to define the free critical points. □

Let us remark that the case $a=1$, where rational function $M^4(x,a)$ is simplified, has no free critical points. The importance of this knowledge is in a classical result from Julia and Fatou (see, for example, [32]), stating that in the immediate basin of attraction of any attracting point (fixed or periodic) there exist at least one critical point. So, the existence of these free critical points states the possibility of another attracting behavior different from that of the roots and their absence means that no other behavior is possible than convergence to the roots. We can check this performance by plotting the dynamical planes of $M^4(x,a)$ for different values of a in these areas where there not exist free critical points.

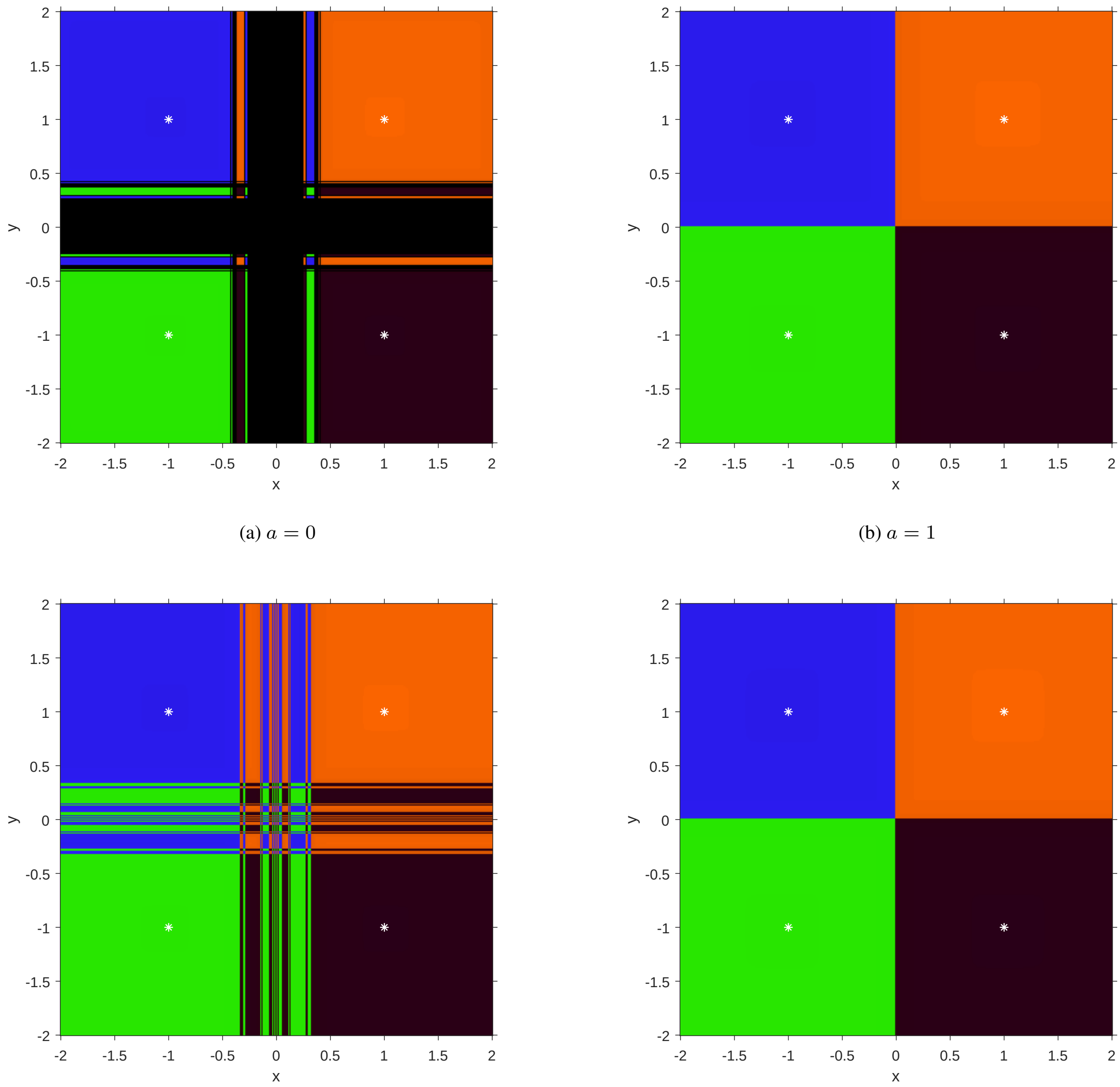
Pictures in Figure 2 have been obtained using the routines appearing in [33] in the following way: it has been used a mesh with 400×400 points, with 80 as maximum number of iterations, being the tolerance of the stopping criterium 10^{-3}. We have painted a point with different colors depending on where it converges to. This color is brighter when the number of iterations used is lower; moreover, it is colored in black if it reaches the maximum number of iterations without converging to any of the roots.

Figures 2a and 2c correspond to values of a in the interval $0\le a\le 0.355416$; it can be observed that the basins of attraction of the roots have an infinite number of connected components and in case of $a=0$, there is a basin of attraction of infinity (in black in the figure), where all the initial estimations diverge. On the contrary, in Figures 2b and 2d there exist only one connected component of each basin of attraction and there are not divergent behavior; they correspond to values of a in the interval $0.710821<a<1.18627$. They have a performance as stable as Newton's method but with fourth-order of convergence.

Some examples of unstable behavior can be observed in Figure 3: for $a=3$ where many attracting strange fixed points appear (see Figure 3a), combining in pairs $\{\pm1,\pm5.076757\}$. These points are marked with white stars and all of them have their own basin of attraction, although not all of them have a color assigned. In Figure 3b, corresponding to $a=3.864355$, the pairs obtained combining ±1.86676 are non-hyperbolic strange fixed points (the associate Jacobian matrix have eigenvalues equal to one) and they can behave as attracting points. So, the wideness of the basins of attraction of the roots is quite reduced.

5 Stability analysis of class PM_6

In order to analyze the performance of class PM_6 on the polynomial system $p(x)=0$, we fix the value $a=1$ that has shown to be the best in the study of the stability of class PM_4 (although close values of a in $]0.710821, 1.18627[$ would



(a) $a=0$ (b) $a=1$ (c) $a=0.2$ (d) $a=0.8$

Fig. 2: Stable dynamical planes of $M^4(x,a)$

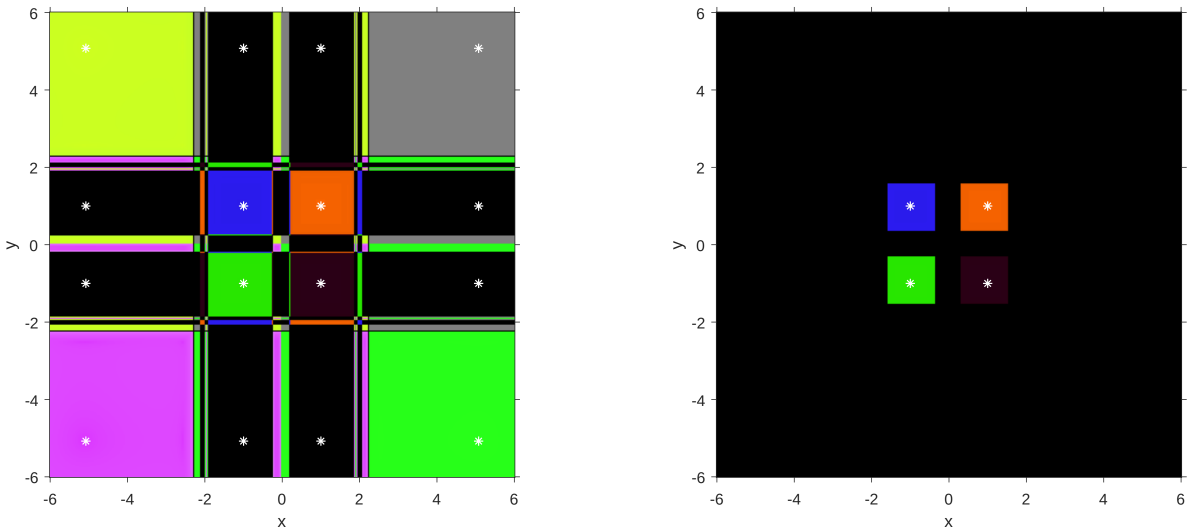
lead to very similar results). Therefore, the third step of PM_6 depend only on parameter r. Let us denote by $M^6(x,r) = (m_1^6(x,r), m_2^6(x,r), \dots, m_n^6(x,r))$, the fixed point function associated to (18) class on $p(x)$. Coordinate functions $m_j^6(x,a)$ can be expressed as:

$$m_j^6(x,r) = \frac{-2r\left(-1+x_j^2\right)^5 + 3\left(1-7x_j^2+34x_j^4+90x_j^6+125x_j^8+13x_j^{10}\right)}{192x_j^5\left(1+x_j^2\right)^2}, \tag{29}$$

for $j=1,2,\dots,n$.

Let us remark that the components of this rational function can be highly simplified for specific values of parameter r; in particular, for $r=\dfrac{39}{2}$, the resulting components are:

$$m_j^6\left(x,\frac{39}{2}\right) = \frac{7-36x_j^2+82x_j^4-20x_j^6+95x_j^8}{32x_j^5\left(1+x_j^2\right)^2}, \tag{30}$$



(a) $a = 3$ (b) $a = 3.864355$

Fig. 3: Unstable dynamical planes of $M^4(x, a)$

for $j = 1, 2, \ldots, n$.

In the following result, we summarize the information about the stability analysis of the fixed points of $M^6(x, r)$. The proof is similar to that of Theorem 3, so it is omitted.

Theorem 5 *Rational function $M^6(x, a)$ associated to the family of iterative methods* (18) *has 2^n superattracting fixed points whose components are roots of $p(x)$. It has also several real strange fixed points whose components are found combining the roots of polynomial*

$$h(t) = 3 + 2r + (-18 - 8r)t^2 + (84 + 12r)t^4 + (162 - 8r)t^6 + (153 + 2r)t^8,$$

depending on r, and the roots of $p(x)$:

(a) If $r < -\frac{153}{2}$, four roots of $h(t)$, denoted by $h_i(r)$, $i = 1, 2, 3, 4$ are real, being their associate eigenvalues greater than one (in absolute value). So, the strange fixed point expressed as $(h_{\sigma_1}(r), h_{\sigma_2}(r), \ldots, h_{\sigma_n}(r))$ being $\sigma_i \in \{1, 2, 3, 4\}$, are repulsive. Moreover, if at least one of the components of the strange fixed point (but not all) are equal to 1 or -1, it will be a saddle fixed point.

(b) If $-\frac{153}{2} \leq r < -\dfrac{3}{2}$, only $h_1(r)$ and $h_2(r)$ are real. The absolute value of the eigenvalues of their associate Jacobian matrix are all greater than one. So, the strange fixed point expressed as $(h_{\sigma_1}(r), h_{\sigma_2}(r), \ldots, h_{\sigma_n}(r))$ being $\sigma_i \in \{1, 2\}$, are repulsive. However, if at least one of the components of the strange fixed point (but not all) are equal to 1 or -1, it is a saddle fixed point.

(c) If $r = -\dfrac{3}{2}$ or $r > \frac{3\left(-1237+41\sqrt{41}\right)}{2048} \approx -1.42745$, then all the roots of polynomial $h(t)$ are complex and there not exist any real strange fixed point.

(d) If $-\frac{3}{2} < r < \frac{3\left(-1237+41\sqrt{41}\right)}{2048}$, then $l_i(r)$, $i = 1, 2, 3, 4$, are real. Regarding the stability, strange fixed points $(h_{\sigma_1}(r), h_{\sigma_2}(r), \ldots, h_{\sigma_n}(r$ with $\sigma_i \in \{2, 3\}$ are attracting for $r \in]-1.427649, -1.427449[$; in any other case, the strange fixed points are unstable repelling (if $\sigma_i \in \{1, 4\}$) or saddle.

(e) If $r = \frac{3\left(-1237+41\sqrt{41}\right)}{2048}$, then $l_1(r)$ and $l_3(r)$ are real and the strange fixed point $(l_{\sigma_1}(r), l_{\sigma_2}(r), \ldots, l_{\sigma_n}(r))$ being $\sigma_i \in \{1, 3\}$ are parabolic.

From this result, it can be concluded that $r = -\dfrac{3}{2}$ or $r > \dfrac{3\left(-1237 + 41\sqrt{41}\right)}{2048} \approx -1.42745$ are the best areas for avoiding the existence of strange fixed points; however, the only area with attracting strange fixed points is extremely small, $]-1.427649, -1.427449[$. So, the existence of attracting periodic orbits would be the only undesirable behavior to avoid in practice. To get this aim, we study $M^{6'}(x, r)$ and its critical points.

Theorem 6 *There exist 2^n real free critical points of $M^6(x,r)$, only if $-\frac{3}{2} \leq r < \frac{39}{2}$. In this case, their only components are* $\pm\sqrt{\frac{12+14r}{39-2r} + \sqrt{\frac{729+696r+176r^2}{(39-2r)^2}}}$.

Proof The eigenvalues of the Jacobian matrix $M^{6\prime}(x,r)$ are

$$Eig_j(x,r) - \frac{\left(-1+x_j^2\right)^4 \left(15+24x_j^2-39x_j^4+2r\left(5+14x_j^2+x_j^4\right)\right)}{192x_j^6\left(1+x_j^2\right)^3}, \quad j=1,2.$$

Then, to get the free critical points it is only necessary to find the real roots of $15+10r+(24+28r)t+(-39+2r)t^2$, being $t = x_j^2$. □

As there exist at least one critical point in each basin of attraction (see [32]), the orbits of these points give us relevant information about the stability of the rational function and, therefore, of the iterative method involved. We present in Figure 4 real parametric lines showing these orbits (see Theorem 6) for $n=2$. In each one of these pictures, a different free critical point is used as starting point of each member of the family of iterative schemes, using $-\frac{3}{2} \leq r < \frac{39}{2}$ to ensure the existence of real critical points.

To plot these parameter lines, a mesh of 500×500 points is made in the region $[0,1] \times]-\frac{3}{2}, \frac{39}{2}[$. We use $[0,1]$ to fatten the interval where r is defined, allowing a better visualization. So, the color corresponding to each value of r is red if the corresponding critical point converges to one of the roots of the polynomial system, blue in case of divergence and black in other cases. This color is also assigned to all the values of $[0,1]$ with the same value of the parameter. The maximum number of iterations used is 200 and the tolerance for the error estimation is 10^{-3}, when the iterates tend to a fixed point.

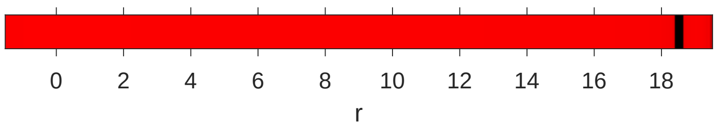


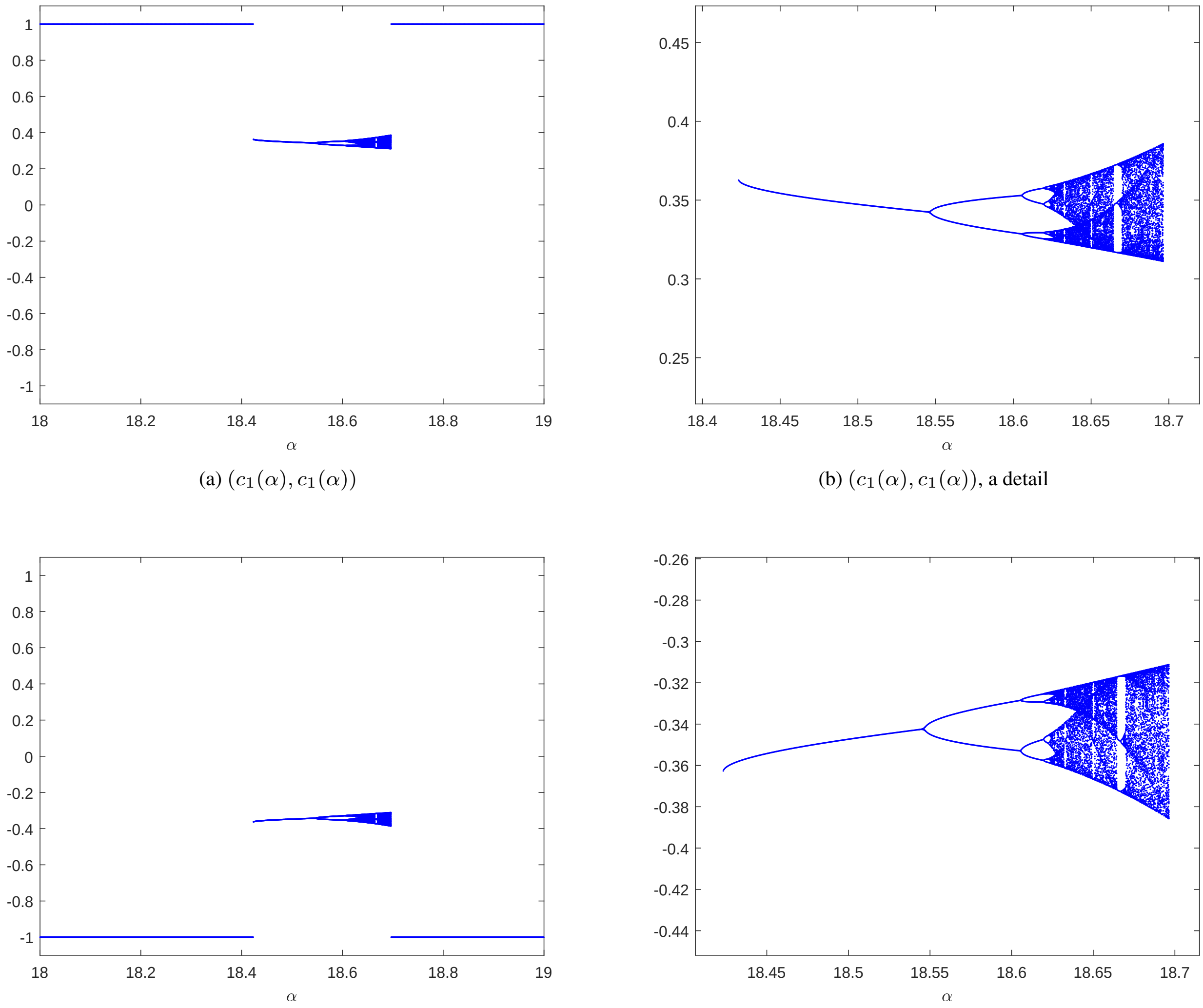
Fig. 4: Parameter line of $M^6(x,r)$ for $-\dfrac{3}{2} < r < \dfrac{39}{2}$

All the pairs of free critical points have the same global performance, so only the case $(cr_1(r), cr_1(r))$ is presented in Figure 4. In it, the parameter line is plotted for $-\dfrac{3}{2} < r < \dfrac{39}{2}$ as outside this interval all the components of the free critical points are complex. It is observed that there is convergence to the roots (red color) for almost all values of r, except a black small region around 18.5.

Bifurcation or Feigenbaum diagrams has been used in order to study the changes of each class of methods on $p(x)$ by using each of the free critical points of the rational function as a initial guess and noticing their performance for different ranges of the parameter. Therefore, different behavior can be observed after 1000 iterations by using a mesh with 3000 subintervals.

Figures 5 corresponds to the bifurcation diagrams in The black area of the parameter line for $-\dfrac{3}{2} < r < \dfrac{39}{2}$ (see Figure 4). In Figures 5a and 5c, it can be observed as there is a general convergence to one of the roots, but in a small interval around $r = 18.5$ a period-doubling cascade appears, including not only periodic but also chaotic behavior (blue regions). In them, there can be found strange attractors. To represent it, we plot in the (x_1, x_2)-space the orbit of $x^{(0)} = (0.35, 0.35)$ by $M^6\left((x_1,x_2), r\right)$, for $r = 18.659$, laying in the blue region. For each r, the number of different starting estimations is 1000 and, for each one, we do not plot the first 500 iterations, meanwhile following 400 are represented in blue color and the resting 100 in magenta. We observe it in Figure 6 as a neutral strange fixed point, after bifurcating in periodic orbits with increasing periods, falls in a chaotic behavior where orbits are dense in a small area of (x_1, x_2) space.

This unstable performance, as well as stable one can be checked by plotting the associated dynamical planes, where the value of the parameter is fixed and a mesh of initial estimations is used to see the performance of the methods. The first dynamical planes correspond to stable behavior, that is, there exist only convergence to the roots (see Figure 7). For values of r in $]-\dfrac{3}{2}, \dfrac{39}{2}[$, where there are free critical points, there are only one connected component per root in Figures 7a and 7b if values of red color in the parameter line (shown in Figure 4) are selected. Outside this interval, there are no free critical point and the performance is also stable, but there exist infinite connected components of each basin of attraction (Figures 7c and 7d).



(a) $(c_1(\alpha), c_1(\alpha))$

(b) $(c_1(\alpha), c_1(\alpha))$, a detail

(c) $(c_2(\alpha), c_2(\alpha))$

(d) $(c_2(\alpha), c_2(\alpha))$, a detail

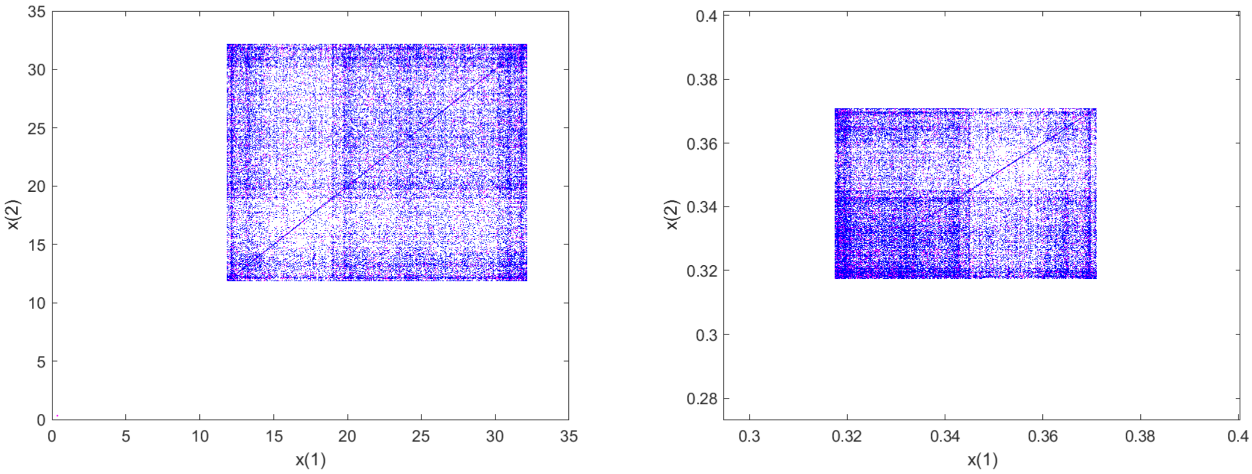
Fig. 5: Feigenbaum diagrams of $M^6(x, r)$ for $-\dfrac{3}{2} < r < \dfrac{39}{2}$

Regarding the unstable behavior, it is limited to values of r in the the black region of the parameter line (Figure 4). In Figures 8a and 8b, two parts of the strange attractor found for $r = 18.659$ can be observed, that was plotted in Figure 6; this dynamical plane have been obtained by avoiding the lines in the orbit of the initial estimation selected in the black area and plotting by yellow circles the elements of the orbit (this time 400 iterations have been used). Finally, in Figures 8c and 8d, the phase space for $r = 18.5$ is represented. In them, the 2-period orbits $\{(-0.3473, -0.3473), (-18.0176, -18.0176)\}$ and $\{(0.3473, -0.3473), (18.0176, -18.0176)\}$ appear in yellow. In this case, two more attracting orbits exist, with symmetric coordinates.

In general, it can be concluded that, the main performance of the members of M4 and M6 classes of iterative methods on this kind of polynomial systems is very stable. There are no attracting strange fixed points and the only attracting behavior different from the roots lay in a very small interval of r. These conclusions are numerically checked in the following numerical section.

6 Numerical results

In this section, some numerical problems are considered to demonstrate the convergence behavior and computational efficiency of the proposed methods. The proposed schemes (2) namely PM_4^1, PM_4^2, PM_4^3 for $a = \frac{1}{2}$, $a = \frac{5}{8}$, and $a = \frac{8}{10}$ respectively are considered and compared with existing fourth-order techniques namely HM_4, JM_4, MM_4^1, introduced by Sharma and Arora's



(a) $r = 18.659$ (b) $r = 18.659$, a detail

Fig. 6: Strange attractors of $M^6(x,r)$ for r in blue doubling-period cascade

method [34], Jarratt's method [35] and Narang's method [14], respectively. For $a = 1$, Jarratt's method is the special case of proposed scheme (2). The proposed scheme (25) for $a = \frac{5}{8}, r = 10$, for $a = \frac{1}{2}, r = \frac{1}{2}$, for $a = \frac{3}{2}, r = \frac{1}{4}$ and for $a = 1, r = \frac{39}{2}$ are denoted as PM_6^4, PM_6^5, PM_6^6 and PM_6^7 respectively and compared with existing schemes of sixth order namely, LM_6, MM_6^2, RM_6 proposed by Lotfi *et al.* [15], Narang's *et al.* [14] and Behl *et al.* [16], respectively. To verify the theoretical order of convergence, we have displayed the iterations k, $\|G(X^{(k)})\|$ and $||X^{(k+1)} - X^{(k)}||$, the approximation of the asymptotic error constant η and computational order of convergence (ρ) using the following formulas, respectively:

$$\rho \approx \frac{\ln \frac{\|X^{(k+1)}-X^{(k)}\|}{\|X^{(k)}-X^{(k-1)}\|}}{\ln \frac{\|X^{(k)}-X^{(k-1)}\|}{\|X^{(k-1)}-X^{(k-2)}\|}}, \qquad \text{for each } k = 2,\ 3, \dots, \tag{31}$$

where $X^{(k-2)}$, $X^{(k-1)}$, $X^{(k)}$, and $X^{(k+1)}$ are four consecutive approximations in the iterations process;

$$\eta = \lim_{k\to\infty} \frac{\|X^{(k+1)} - X^{(k)}\|}{\|X^{(k)} - X^{(k-1)}\|^{\rho}},\ where\ (\rho = 4\ or\ 6). \tag{32}$$

All numerical computations have been done on Mathematica 11 with variable precision arithmetics using a mantissa of 1000 digits to minimize rounding errors and in all tables, $b(\pm c)$ denotes $b \times 10^{\pm c}$.

Example 1 Consider the following small system of nonlinear equations

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= 1,\\ 2x_1^2 + x_2^2 - 4x_3 &= 0,\\ 3x_1^2 - 4x_2^2 + x_3^2 &= 0. \end{aligned}$$

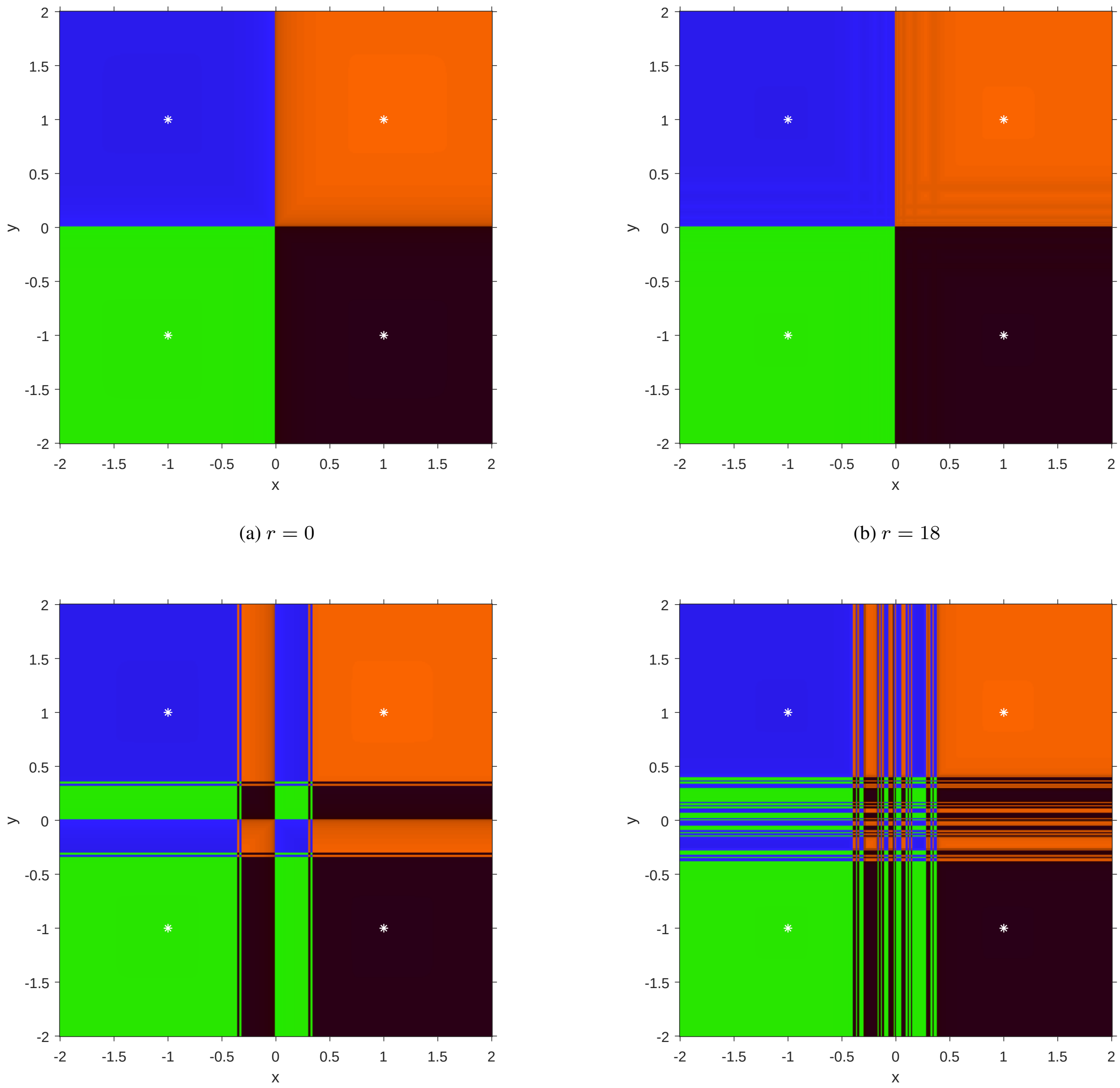
The convergence of this system towards the solution $X^* = (0.6982886\dots, 0.6285243 \dots, 0.3425642\dots)^T$ is tested and shown in Table 1.

Example 2 One of the most famous physical problem of radiative transfer theory which is the Chandrasekhar integral equation (see [36, pages 251-252]) is considered for applicability and effectiveness of proposed schemes compared to existing methods, as follows:

$$F(P,c) = 0,\ \ P : [0,1] \to \mathbb{R}$$

with parameter c and operation F as

$$F(P,c)(y) = P(y) - \Big(1 - \frac{c}{2}\int_0^1 \frac{yP(y)}{y+v}dv\Big)^{-1}.$$



(a) $r = 0$ (b) $r = 18$

(c) $r = -2$ (d) $r = 20$

Fig. 7: Stable dynamical planes of M6 on $p(x)$

We transform the Chandrasekhar equation in a nonlinear system by means of a Gauss-Legendre quadrature formula, $\int_0^1 f(t)dt \simeq \sum_{j=1}^{8} w_j f(t_j)$, where t_j denotes the nodes and w_j the weights of the Gauss-Legendre quadrature formula.

Denoting the approximations of $P(y(t_i))$ by $y_i (i = 1, 2, ..., 8)$, one gets the system of nonlinear equations

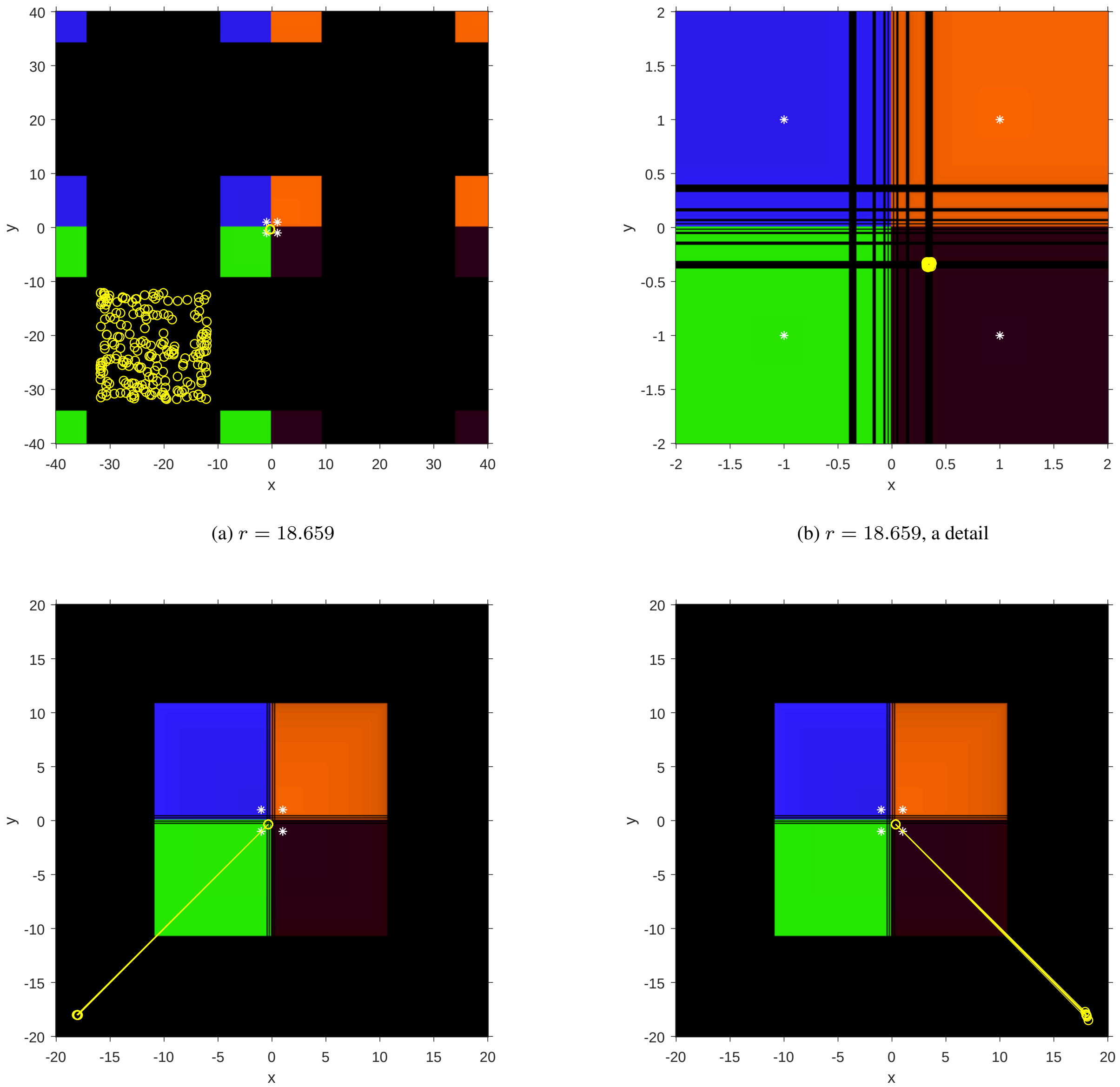
$$y_i - \left(1 - \frac{c}{2} \sum_{j=1}^{8} a_{ij} y_j\right)^{-1} = 0, \;\; i = 1, 2, ..., 8,$$

where

$$a_{ij} = \frac{w_j \, t_i}{t_i + t_j}, \; and \; c = 0.5.$$

Nodes t_j and weights w_j are known and given in the following table for $t = 8$.

The performance of the methods to estimate the solution:
$X^* = (1.021720\ldots, 1.073186\ldots, 1.125725\ldots, 1.169753\ldots, 1.203072\ldots,$
$1.226491\ldots, 1.241525\ldots, 1.249449\ldots)^T$ is checked in Table 3.



(a) $r = 18.659$

(b) $r = 18.659$, a detail

(c) $r = 18.5$

(d) $r = 18.5$

Fig. 8: Unstable dynamical planes of M4 on $p(x)$

Example 3 Consider the Frank-Kamenetskii Problem [37] described by the following differential equation

$$xy'' + y' + xe^y = 0,\ y'(0) = y(1) = 0. \tag{33}$$

To convert the above boundary value problem (33) into nonlinear system of size 50×50 with step size $h = \frac{1}{51}$, the finite difference discretization is used. For second derivative central difference has been used which is as follows:

$$y_i'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2},\ i = 1,\ 2,\ \ldots,\ 50.$$

The solution $(0.4742933\ldots, 0.4738815\ldots, 0.4730579\ldots, 0.4718232\ldots, 0.4701778\ldots,$
$0.4681227\ldots, 0.4656589\ldots, 0.4627878\ldots, 0.4595107\ldots, 0.4558294\ldots, 0.4517458\ldots,$
$0.4472620\ldots, 0.4423803\ldots, 0.4371031\ldots, 0.4314332\ldots, 0.4253735\ldots, 0.4189268\ldots,$
$0.4120966\ldots, 0.4048861\ldots, 0.3972989\ldots, 0.3893387\ldots, 0.3810094\ldots, 0.3723149\ldots,$
$0.3632593\ldots, 0.3538470\ldots, 0.3440823\ldots, 0.3339696\ldots, 0.3235137\ldots, 0.3127193\ldots,$

Table 1: Convergence behavior of different methods using initial guess $X^{(0)} = (1, 1, 1)^T$ for Example 1.

$Cases$	k	$\|\|G(X^{(k+1)})\|\|$	$\|\|X^{(k+1)} - X^{(k)}\|\|$	ρ	$\frac{\|X^{(k+1)}-X^{(k)}\|}{\|X^{(k)}-X^{(k-1)}\|^{\rho}}$	η
HM_4	1	9.2(−2)	3.4(−2)			
	2	3.3(−6)	8.4(−7)		6.56632768(−1)	1.20819612
	3	3.0(−24)	6.0(−25)	3.9425	1.20819612	
JM_4	1	5.0(−2)	1.6(−2)			
	2	5.1(−8)	1.1(−8)		3.25047966(−1)	3.67096134(−1)
	3	2.9(−32)	5.6(−33)	4.0214	3.67096134(−1)	
MM_4^1	1	5.3(−2)	1.7(−2)			
	2	1.1(−9)	3.1(−10)		3.31909055(−3)	3.48171436(−2)
	3	1.4(−39)	3.1(−40)	3.9536	3.48171436(−2)	
PM_4^1	1	2.3(−2)	3.8(−3)			
	2	7.1(−11)	1.6(−11)		8.22152464(−2)	3.21217484(−1)
	3	1.2(−43)	2.3(−44)	4.0045	3.21217484(−1)	
PM_4^2	1	3.3(−2)	8.3(−3)			
	2	7.1(−10)	1.4(−10)		2.98786575(−2)	4.56924693(−2)
	3	9.2(−41)	1.8(−41)	3.9961	4.56924693(−2)	
PM_4^3	1	4.5(−2)	1.4(−2)			
	2	2.2(−8)	4.8(−9)		1.29912566(−1)	2.31165655(−1)
	3	6.5(−34)	1.3(−34)	4.0153	2.31165655(−1)	
LM_6	1	2.3(−2)	6.9(−3)			
	2	7.5(−13)	1.6(−13)		1.48465013	3.38894420
	3	2.8(−76)	5.4(−77)	6.0165	3.38894420	
MM_6^2	1	1.2(−2)	3.6(−3)			
	2	4.4(−16)	1.0(−16)		4.34999343(−2)	1.60951775(−1)
	3	9.4(−97)	1.8(−97)	6.0065	1.60951775(−1)	
RM_6	1	1.6(−2)	5.0(−3)			
	2	4.6(−14)	1.1(−14)		6.43957889(−1)	1.93917293
	3	1.4(−83)	2.7(−84)	6.0153	1.93917293	
PM_6^4	1	3.2(−3)	5.9(−4)			
	2	6.1(−20)	1.2(−20)		2.68196800	2.77107704
	3	3.8(−120)	7.3(−121)	5.9995	2.77107704	
PM_6^5	1	3.1(−3)	5.3(−4)			
	2	5.7(−20)	1.1(−20)		4.81238189	1.62837380
	3	1.3(−119)	2.5(−120)	6.0015	1.62837380	
PM_6^6	1	4.0(−3)	8.4(−4)			
	2	1.6(−18)	3.1(−19)		8.56644577(−1)	1.04568928
	3	5.0(−111)	9.6(−112)	5.9998	1.04568928	
PM_6^7	1	7.7(−3)	2.6(−3)			
	2	5.3(−15)	1.0(−15)		3.16181092	9.11341844
	3	4.9(−89)	9.3(−90)	6.0034	9.11341844	

Table 2: Nodes and weights of Gauss-Legendre quadrature formula

j	t_j	w_j
1	0.01985507175123188415821957...	0.05061426814518812957626567...
2	0.10166676129318663020422303...	0.11119051722668723527217800...
3	0.23723379504183550709113047...	0.15685332293894364366898110...
4	0.40828267875217509753026193...	0.18134189168918099148257522...
5	0.59171732124782490246973807...	0.18134189168918099148257522...
6	0.76276620495816449290886952...	0.15685332293894364366898110...
7	0.89833323870681336979577696...	0.11119051722668723527217800...
8	0.98014492824876811584178043...	0.05061426814518812957626567...

$0.3015910\ldots, 0.2901340\ldots, 0.2783530\ldots, 0.2662533\ldots, 0.2538399\ldots, 0.2411180\ldots,$ $0.2280928\ldots, 0.2147698\ldots, 0.2011542\ldots, 0.1872513\ldots, 0.1730667\ldots, 0.1586057\ldots,$ $0.1438739\ldots, 0.1288766\ldots, 0.1136194\ldots, 0.09810770\ldots, 0.08234704\ldots, 0.06634287\ldots,$ $0.05010065\ldots, 0.03362582\ldots, 0.01692381\ldots)^T$ of this problem is tested and shown in Table 4.

Table 3: Convergence of different methods using initial value $X^{(0)} = (1, 1, \ldots, 1)^T$ for Example 2.

$Cases$	k	$\|\|G(X^{(k+1)})\|\|$	$\|\|X^{(k+1)} - X^{(k)}\|\|$	ρ	$\frac{\|X^{(k+1)}-X^{(k)}\|}{\|X^{(k)}-X^{(k-1)}\|^{\rho}}$	η
	1	$7.9(-7)$	$8.9(-7)$			
HM_4	2	$2.1(-30)$	$2.4(-30)$		3.71092028	$3.19792695(-6)$
	3	$9.0(-125)$	$1.0(-124)$	4.0027	$3.19792695(-6)$	
	1	$6.3(-7)$	$7.0(-7)$			
JM_4	2	$5.4(-31)$	$6.0(-31)$		$2.48977678(-6)$	$2.07422265(-6)$
	3	$2.4(-127)$	$2.7(-127)$	4.0556	$2.07422265(-6)$	
	1	$5.8(-7)$	$6.5(-7)$			
MM_4^1	2	$3.4(-31)$	$3.7(-31)$		$2.14640690(-6)$	$1.75506541(-6)$
	3	$3.1(-128)$	$3.4(-128)$	4.0550	$1.75506541(-6)$	
	1	$5.4(-7)$	$6.0(-7)$			
PM_4^1	2	$2.3(-31)$	$2.5(-31)$		$1.90277319(-6)$	$1.54003293(-6)$
	3	$5.6(-129)$	$6.2(-129)$	4.0547	$1.54003293(-6)$	
	1	$5.8(-7)$	$6.5(-7)$			
PM_4^2	2	$3.4(-31)$	$3.8(-31)$		$2.15684152(-6)$	$1.77057582(-6)$
	3	$3.2(-128)$	$3.5(-128)$	4.0551	$1.77057582(-6)$	
	1	$6.7(-7)$	$6.9(-7)$			
PM_4^3	2	$4.8(-31)$	$5.3(-31)$		$2.3944553(-6)$	$1.98712419(-6)$
	3	$1.4(-127)$	$1.5(-127)$	4.0550	$1.98712419(-6)$	
	1	$8.8(-11)$	$9.7(-11)$			
LM_6	2	$9.3(-62)$	$9.4(-62)$		$1.14365077(-1)$	$2.09430142(-47)$
	3	$1.3(-319)$	$1.4(-319)$	5.2138	$2.09430142(-47)$	
	1	$1.2(-10)$	$1.3(-10)$			
MM_6^2	2	$6.3(-69)$	$7.1(-69)$		$1.38796360(-9)$	$1.19766180(-9)$
	3	$1.4(-418)$	$1.5(-418)$	6.0382	$1.19766180(-9)$	
	1	$1.3(-10)$	$1.5(-10)$			
RM_6	2	$1.5(-68)$	$1.7(-68)$		$1.52873178(-9)$	$1.31247377(-9)$
	3	$2.9(-416)$	$3.3(-416)$	6.0385	$1.31247377(-9)$	
	1	$3.3(-11)$	$3.9(-11)$			
PM_6^4	2	$7.4(-72)$	$8.7(-72)$		$2.58565512(-9)$	$1.89554607(-9)$
	3	$6.8(-436)$	$8.0(-436)$	6.0369	$1.89554607(-9)$	
	1	$1.0(-10)$	$1.1(-10)$			
PM_6^5	2	$2.1(-69)$	$2.4(-69)$		$1.15228376(-9)$	$9.90919774(-10)$
	3	$1.8(-421)$	$2.0(-421)$	6.0380	$9.90919774(-10)$	
	1	$1.1(-10)$	$1.2(-10)$			
PM_6^6	2	$2.8(-69)$	$3.2(-69)$		$1.21461310(-9)$	$1.04639321(-9)$
	3	$9.2(-421)$	$1.0(-420)$	6.0380	$1.04639321(-9)$	
	1	$1.9(-10)$	$2.2(-10)$			
PM_6^7	2	$4.9(-67)$	$5.7(-67)$		$5.21111400(-9)$	$4.50911963(-9)$
	3	$1.3(-406)$	$1.5(-406)$	6.0392	$4.50911963(-9)$	

Example 4 In this example, we want to check the effectiveness of our methods on a large system of nonlinear equations as compared to the existing methods. Therefore, we consider the following (100×100) system of polynomial equations.

$$\begin{cases} x_i^2 \sin(x_{i+1}) - 1 = 0, \ 1 \le i \le 99, \\ x_i^2 \sin(x_1) - 1 = 0, \qquad i = 100. \end{cases}$$

The required solution of this system $X^* = (-1.114157\ldots, -1.114157\ldots, \ldots, -1.114157\ldots)^T$ is tested and depicted in the Table 5.

Example 5 Consider the Bratu Problem [38] that has large variety of application areas such as the fuel ignition model of thermal combustion, radioactive heat transfer, thermal reaction, the Chandrasekhar model of the expansion of the universe, chemical reactor theory and nanotechnology. The problem is defined as:

$$y'' + C_1 e^y = 0, \ y(0) = y(1) = 0. \tag{34}$$

The finite difference discretization is used convert this boundary value problem into nonlinear system of size 40×40 with C_1=1 and step size $h = \frac{1}{41}$. For second derivative central difference has been used which is as follows:

$$y_i'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}, \ i = 1, \ 2, \ \ldots, \ 40.$$

Table 4: Convergence behavior of different methods using initial value $X^{(0)} = (\frac{1}{10}, \frac{1}{10}, \ldots, \frac{1}{10})^T$ for Example 3.

$Cases$	k	$\|\|G(X^{(k+1)})\|\|$	$\|\|X^{(k+1)} - X^{(k)}\|\|$	ρ	$\frac{\|X^{(k+1)}-X^{(k)}\|}{\|X^{(k)}-X^{(k-1)}\|^{\rho}}$	η
HM_4	1	$1.5(-6)$	$2.6(-3)$			
	2	$1.6(-17)$	$2.9(-14)$		$6.23789565(-4)$	$6.23762642(-4)$
	3	$2.6(-61)$	$4.5(-58)$	4.000	$6.23762642(-4)$	
JM_4	1	$7.3(-7)$	$1.3(-3)$			
	2	$3.9(-19)$	$7.1(-16)$		$2.34263687(-4)$	$2.34236756(-4)$
	3	$3.2(-68)$	$5.8(-65)$	4.2229	$2.34236756(-4)$	
MM_4^1	1	$6.2(-7)$	$1.1(-3)$			
	2	$9.9(-20)$	$1.9(-16)$		$1.19116526(-4)$	$1.19364071(-4)$
	3	$7.6(-71)$	$1.5(-67)$	4.2122	$1.19364071(-4)$	
PM_4^1	1	$2.9(-7)$	$5.4(-4)$			
	2	$1.9(-21)$	$3.5(-18)$		$4.06900764(-5)$	$4.29296835(-5)$
	3	$3.5(-78)$	$6.5(-75)$	4.1842	$4.29296835(-5)$	
PM_4^2	1	$4.8(-7)$	$8.9(-4)$			
	2	$4.1(-20)$	$7.8(-17)$		$1.25036152(-4)$	$1.25171462(-4)$
	3	$2.4(-72)$	$4.6(-69)$	4.2079	$1.25171462(-4)$	
PM_4^3	1	$6.6(-7)$	$1.2(-3)$			
	2	$2.3(-19)$	$4.2(-16)$		$2.03161618(-4)$	$2.03144555(-4)$
	3	$3.4(-69)$	$6.2(-66)$	4.2193	$2.03144555(-4)$	
LM_6	1	$5.9(-9)$	$1.0(-5)$			
	2	$3.8(-36)$	$5.2(-34)$		$5.12639506(-4)$	$1.30020237(-26)$
	3	$1.7(-177)$	$2.7(-174)$	5.0727	$1.30020237(-26)$	
MM_6^2	1	$1.8(-8)$	$3.2(-5)$			
	2	$1.0(-36)$	$1.8(-33)$		$1.82615946(-6)$	$1.82531370(-6)$
	3	$3.5(-206)$	$6.2(-203)$	6.1550	$1.82531370(-6)$	
RM_6	1	$2.7(-8)$	$4.7(-5)$			
	2	$3.3(-35)$	$5.8(-32)$		$5.12016539(-6)$	$5.11568289(-6)$
	3	$1.2(-196)$	$2.0(-193)$	6.1630	$5.11568289(-6)$	
PM_6^4	1	$1.1(-8)$	$1.7(-5)$			
	2	$4.1(-38)$	$7.0(-35)$		$2.76740190(-6)$	$2.76658573(-6)$
	3	$1.9(-214)$	$3.2(-211)$	6.1484	$2.76658573(-6)$	
PM_6^5	1	$8.0(-9)$	$1.4(-5)$			
	2	$2.2(-39)$	$4.0(-36)$		$5.30775163(-7)$	$5.32026749(-7)$
	3	$1.2(-222)$	$2.2(-219)$	6.1429	$5.32026749(-7)$	
PM_6^6	1	$9.6(-9)$	$1.7(-5)$			
	2	$7.3(-39)$	$1.3(-35)$		$5.67371161(-7)$	$5.68749186(-7)$
	3	$1.5(-219)$	$2.7(-216)$	6.1450	$5.68749186(-7)$	
PM_6^7	1	$5.1(-8)$	$8.5(-5)$			
	2	$3.0(-33)$	$5.1(-30)$		$1.36031353(-5)$	$1.35828720(-5)$
	3	$1.4(-184)$	$2.4(-181)$	6.1741	$1.35828720(-5)$	

The required solution of this system $X^* = (0.055685\ldots, 0.109484\ldots, 0.161292\ldots,$ $0.211002\ldots, 0.258509\ldots, 0.303705\ldots, 0.346483\ldots, 0.386737\ldots, 0.424363\ldots,$ $0.459262\ldots, 0.491336\ldots, 0.520492\ldots, 0.546646\ldots, 0.569716\ldots, 0.589632\ldots,$ $0.606329\ldots, 0.619754\ldots, 0.629862\ldots, 0.636620\ldots, 0.640005\ldots, 0.640005\ldots,$ $0.636620\ldots, 0.629862\ldots, 0.619754\ldots, 0.606329\ldots, 0.589632\ldots, 0.569716\ldots,$ $0.546646\ldots, 0.520492\ldots, 0.491336\ldots, 0.459262\ldots, 0.424363\ldots, 0.386737\ldots,$ $0.346483\ldots, 0.303705\ldots, 0.258509\ldots, 0.211002\ldots, 0.161292\ldots, 0.109484\ldots,$ $0.0556856\ldots)^T$ is tested and shown in Table 6.

7 Conclusions

In this work, we have developed new families of fourth and sixth-order iterative methods for solving systems of nonlinear equations numerically. As these classes depend on parameters, a stability analysis has been performed, by using tools from multidimensional discrete dynamics, in order to select those values of the parameters with better properties. Then some specific elements of both families are chosen. In order to check their effectiveness, the proposed schemes are applied on some large-scale systems arising from various academic problems. Further, the numerical results show that the proposed techniques perform better than the existing methods of same order in terms of residual error, difference between two consecutive approximations and asymptotic error constant.

Table 5: Convergence behavior of different methods using initial value $X^{(0)} = (-1, -1, \ldots, -1)^T$ for Example 4.

$Cases$	k	$\|\|G(X^{(k+1)})\|\|$	$\|\|X^{(k+1)} - X^{(k)}\|\|$	ρ	$\frac{\|X^{(k+1)}-X^{(k)}\|}{\|X^{(k)}-X^{(k-1)}\|^{\rho}}$	η
	1	3.1(−5)	2.3(−5)			
HM_4	2	6.8(−24)	4.9(−24)		1.91814312(−5)	1.91807387(−5)
	3	1.6(−98)	1.1(−98)	4.000	1.91807387(−5)	
	1	3.0(−5)	2.2(−5)			
JM_4	2	6.1(−24)	4.4(−24)		1.88736669(−5)	1.88730286(−5)
	3	9.6(−99)	6.9(−99)	4.1695	1.88730286(−5)	
	1	3.0(−5)	2.2(−5)			
MM_4^1	2	5.9(−24)	4.2(−24)		1.87824776(−5)	1.87818553(−5)
	3	8.3(−99)	5.9(−99)	4.1694	1.87818553(−5)	
	1	3.0(−5)	2.2(−5)			
PM_4^1	2	5.7(−24)	4.1(−24)		1.87197857(−5)	1.87191736(−5)
	3	7.5(−99)	5.4(−99)	4.1693	1.87191736(−5)	
	1	3.0(−5)	2.2(−5)			
PM_4^2	2	5.9(−24)	4.2(−24)		1.87871086(−5)	1.87864852(−5)
	3	8.4(−99)	6.0(−99)	4.1694	1.87864852(−5)	
	1	3.0(−5)	2.2(−5)			
PM_4^3	2	6.0(−24)	4.3(−24)		1.88490459(−5)	1.88484118(−5)
	3	9.3(−99)	6.7(−99)	4.1695	1.88484118(−5)	
	1	2.0(−7)	1.4(−7)			
LM_6	2	8.7(−49)	6.3(−49)		7.96435784(−8)	7.96435977(−8)
	3	6.7(−297)	4.8(−297)	6.1239	7.96435977(−8)	
	1	1.5(−7)	1.1(−7)			
MM_6^2	2	1.7(−49)	1.2(−49)		7.38016606(−8)	7.38016476(−8)
	3	3.1(−301)	2.2(−301)	6.1221	7.38016476(−8)	
	1	1.5(−7)	1.1(−7)			
RM_6	2	1.8(−49)	1.3(−49)		7.42388427(−8)	7.42388292(−8)
	3	5.1(−301)	3.7(−301)	6.1222	7.42388292(−8)	
	1	1.4(−7)	9.9(−8)			
PM_6^4	2	8.8(−50)	6.3(−50)		6.77788219(−8)	6.77788132(−8)
	3	5.9(−303)	4.2(−303)	6.1214	6.77788132(−8)	
	1	1.5(−7)	1.1(−7)			
PM_6^5	2	1.6(−49)	1.1(−49)		7.32543890(−8)	7.32543764(−8)
	3	2.1(−301)	1.5(−301)	6.1220	7.32543764(−8)	
	1	1.5(−7)	1.1(−7)			
PM_6^6	2	1.6(−49)	1.1(−49)		7.34048736(−8)	7.34048610(−8)
	3	2.3(−301)	1.7(−301)	6.1220	7.34048610(−8)	
	1	5.1(−8)	8.5(−6)			
PM_6^7	2	3.0(−33)	5.1(−30)		1.36031353(−5)	1.35828720(−5)
	3	1.4(−184)	2.4(−181)	6.1741	1.35828720(−5)	

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Table 6: Convergence results of different methods at initial value $X^{(0)} = (sin(\pi h), \ldots, sin(40\pi h))^T$ for Example 5.

$Cases$	k	$\|\|G(X^{(k+1)})\|\|$	$\|\|X^{(k+1)} - X^{(k)}\|\|$	ρ	$\frac{\|\|X^{(k+1)}-X^{(k)}\|\|}{\|\|X^{(k)}-X^{(k-1)}\|\|^{\rho}}$	η
HM_4	1	$1.6(-3)$	$5.3(-1)$			
	2	$1.0(-6)$	$3.5(-4)$		$4.30224576(-3)$	$7.38471573(-3)$
	3	$3.1(-19)$	$1.1(-16)$	3.9264	$7.38471573(-3)$	
JM_4	1	$1.4(-4)$	$4.9(-2)$			
	2	$3.4(-11)$	$1.2(-8)$		$2.08601907(-3)$	$2.15606045(-3)$
	3	$1.2(-37)$	$4.2(-35)$	3.9978	$2.15606045(-3)$	
MM_4^1	1	$1.8(-3)$	$6.9(-1)$			
	2	$3.0(-6)$	$1.1(-3)$		$4.58499896(-3)$	$5.95449773(-4)$
	3	$2.3(-18)$	$7.7(-16)$	4.3150	$5.95449773(-4)$	
PM_4^1	1	$1.1(-4)$	$4.0(-2)$			
	2	$3.5(-12)$	$1.3(-9)$		$5.08552301(-4)$	$4.64468796(-4)$
	3	$3.2(-42)$	$1.2(-39)$	4.0053	$4.64468796(-4)$	
PM_4^2	1	$1.1(-5)$	$3.9(-3)$			
	2	$4.5(-16)$	$1.5(-13)$		$6.61035020(-4)$	$6.85480490(-4)$
	3	$1.0(-57)$	$3.4(-55)$	3.9985	$6.85480490(-4)$	
PM_4^3	1	$9.4(-5)$	$3.2(-2)$			
	2	$5.3(-12)$	$1.8(-9)$		$1.704924209(-3)$	$1.73755622(-3)$
	3	$5.3(-41)$	$1.8(-38)$	3.9989	$1.73755622(-3)$	
LM_6	1	$3.1(-4)$	$1.1(-1)$			
	2	$8.0(-13)$	$2.8(-10)$		$1.87422921(-4)$	54.2818430
	3	$1.5(-57)$	$2.7(-56)$	5.7883	54.2818430	
MM_6^2	1	$1.7(-3)$	$6.5(-1)$			
	2	$1.1(-7)$	$3.7(-5)$		$4.94903487(-4)$	$4.75329756(-5)$
	3	$3.4(-34)$	$1.2(-31)$	7.4585	$4.75329756(-5)$	
RM_6	1	$2.7(-4)$	$9.1(-2)$			
	2	$4.0(-13)$	$1.4(-10)$		$2.47237315(-4)$	$2.82988099(-4)$
	3	$5.9(-66)$	$2.1(-63)$	6.4927	$2.82988099(-4)$	
PM_6^4	1	$6.6(-3)$	$2.3(-2)$			
	2	$3.7(-17)$	$1.3(-14)$		$8.79261373(-5)$	$9.05968935(-5)$
	3	$1.2(-90)$	$4.1(-88)$	6.3491	$9.05968935(-5)$	
PM_6^5	1	$6.7(-5)$	$2.3(-2)$			
	2	$1.5(-17)$	$5.3(-15)$		$3.40642112(-5)$	$3.19132910(-5)$
	3	$2.0(-93)$	$6.8(-91)$	6.3408	$3.19132910(-5)$	
PM_6^6	1	$2.6(-3)$	$9.2(-2)$			
	2	$8.6(-14)$	$3.0(-11)$		$4.91936579(-5)$	$3.43933916(-5)$
	3	$7.3(-71)$	$2.5(-68)$	6.4793	$3.43933916(-5)$	
PM_6^7	1	$2.6(-4)$	$9.2(-2)$			
	2	$8.6(-14)$	$3.0(-11)$		$4.91936579(-5)$	$3.43933916(-5)$
	3	$7.3(-71)$	$2.5(-68)$	6.4793	$3.43933916(-5)$	

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