The change of the Weierstrass structure under one row perturbation

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Abstract

In this work we study the change of the structure of a regular pencil when we perform small perturbations over some of its rows and the other rows remain unaltered. We provide necessary conditions when several rows are perturbed, and prove them to be sufficient to prescribe the homogenous invariant factors or the Weyr characteristic of the resulting pencil when one row is perturbed.

Keywords: matrix pencils, perturbation, Weierstrass structure, Weyr characteristic
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1 Introduction

The additive perturbation problem of a matrix can be stated as follows: given a matrix A, analyze the structure of $A + P$, where P is a perturbation matrix with certain properties. Different types of problems have been investigated, depending on different requirements over A and the perturbation P. Analogous problems can also be stated for a matrix pencil $A(s)$ and a perturbation pencil $P(s)$.

Results about perturbations of square matrices where the perturbation is a matrix of bounded rank can be found in [28, 30, 31, 33], among others. Changes of the Weierstrass or the Kronecker structure of regular or singular pencils, respectively, under pencil perturbations of bounded rank have also been obtained (see, for instance, [9, 10, 21, 2, 3, 18] and the references therein).

Other types of problems arise when the perturbation is required to be small. Thus, changes of the Jordan structure of a square matrix under small additive perturbations were studied in [11, 1]. Small additive perturbations have also been studied for pairs of matrices ([23]), and for pencils ([8, 22]). When small additive perturbations are performed only over

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one or several rows, changes in the similarity invariants of a matrix and changes in the feedback invariants of a pair of matrices have also been explored ([5, 7, 6, 16]).

Our target is to generalize the research of [5, 7, 6] to matrix pencils. It is natural to pose the following problem.

Problem 1.1 *Given a pencil* $A(s) = \begin{bmatrix} A_1(s) \\ A_2(s) \end{bmatrix} \in \mathbb{C}[s]^{(r+(m-r))\times n}$*, characterize the Kronecker structure of the pencils* $A'(s) = \begin{bmatrix} A'_1(s) \\ A_2(s) \end{bmatrix} \in \mathbb{C}[s]^{(r+(m-r))\times n}$ *obtained from* $A(s)$ *under small additive perturbations over* $A_1(s)$*.*

As mentioned in [5], the small perturbation problem of several rows is sort of a 'crossroad' of perturbation and completion problems. On one hand, the general small perturbation problem must be taken into account. On the other hand, when perturbing one or several rows of a square matrix (see [5, 7, 6]), the problem of characterizing the invariant factors of a square matrix with some prescribed rows plays an important role. This is the problem of completion of a rectangular matrix to a square one, and was solved in [32]. The problem of perturbing one row in a pair of matrices (see [16]) involves the problem of characterizing the feedback invariants of a pair of matrices with some prescribed rows. This problem was solved in [12].

For general pencils, the problem of characterizing the Kronecker structure of a matrix pencil with prescribed rows was solved in [13] (see also [14, 15]).

In this paper we study Problem 1.1 for regular pencils. We obtain necessary conditions when r rows of a regular pencil are perturbed, and solve the problem completely when $r = 1$, hence generalizing the results of [5]. To solve the problem we follow the ideas of [5], but we have to overcome the difficulties appearing due to the presence of infinite elementary divisors in the pencils.

The paper is organized as follows. We introduce some notation and basic definitions in Section 2. Section 3 is devoted to present previous results. This section is structured in two subsections. Subsection 3.1 contains results on perturbation of pencils, whilst results on completion problems are included in Subsection 3.2. Section 4 contains the main results of this work. In Theorems 4.1 and 4.2 we obtain necessary conditions that the Weierstrass invariants must satisfy when a regular pencil is perturbed on r rows. For $r = 1$, we prove that the necessary conditions obtained are sufficient for prescribing the homogeneous invariant factors (Theorem 4.9) or the Weyr characteristic (Theorem 4.13) of the perturbed pencil. Finally, Section 5 includes a summary of the results obtained in the paper and future work.

2 Notation and basic definitions

We start with the introduction of some properties of integers. We call *partition* of a positive integer n to a finite or infinite sequence of nonnegative integers $\mathbf{a} = (a_1, a_2, \dots)$ almost all zero, such that $a_1 \geq a_2 \geq \dots$ and $\sum_{i\geq n} a_i = n$. The number of components of $\mathbf{a}$ different from zero is the *length* of $\mathbf{a}$ (denoted $\ell(\mathbf{a})$). Notice that $\ell(\mathbf{a}) \leq n$. For $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$, $\mathbf{a}$ is *majorized by* $\mathbf{b}$ *in the Hardy-Littlewood-Pólya sense* ($\mathbf{a} \prec \mathbf{b}$) if $\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i$ for $1 \leq k \leq n-1$ and $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$. If $\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i$, $1 \leq k \leq n$, it is said that $\mathbf{a}$ is *weakly majorized* by $\mathbf{b}$ ($\mathbf{a} \prec\prec \mathbf{b}$) (see [27]).

The *conjugate partition* of $\mathbf{a}$, $\overline{\mathbf{a}} = (\bar{a}_1, \bar{a}_2, \ldots)$, is defined as $\bar{a}_k := \#\{i : a_i \geq k\},\ k \geq 1$. Given $\mathbf{a}$ and $\mathbf{b}$ two partitions, $\mathbf{a} \cup \mathbf{b}$ is the partition whose components are those of $\mathbf{a}$ and $\mathbf{b}$ arranged in decreasing order, and $\mathbf{a} + \mathbf{b}$ is the partition whose components are the sums of the corresponding components of $\mathbf{a}$ and $\mathbf{b}$. The following properties are satisfied: $\mathbf{a} \prec \mathbf{b} \Leftrightarrow \overline{\mathbf{b}} \prec \overline{\mathbf{a}}$ and $\overline{\mathbf{a} \cup \mathbf{b}} = \overline{\mathbf{a}} + \overline{\mathbf{b}}$.

In [17, Definition 2] a generalized majorization between three finite sequences of integers, $\mathbf{c} = (c_1, \ldots, c_m)$, $\mathbf{a} = (a_1, \ldots, a_s)$ and $\mathbf{d} = (d_1, \ldots, d_{m+s})$ is defined and it is denoted by $\mathbf{d} \prec' (\mathbf{c}, \mathbf{a})$. When $s = 0$, the generalided majorization reduces to $\mathbf{d} = \mathbf{c}$ and when $m = 0$, to $\mathbf{d} \prec \mathbf{a}$.

Through this paper, $\mathbb{C}$ denotes the field of complex numbers and $\mathbb{F}$ any arbitrary field. $\mathbb{F}[s]$ is the ring of polynomials in the indeterminate s with coefficients in $\mathbb{F}$ and $\mathbb{F}[s,t]$ the ring of polynomials in two variables s, t with coefficients in $\mathbb{F}$. We denote by $\mathbb{F}^{p\times q}$, $\mathbb{F}[s]^{p\times q}$ and $\mathbb{F}[s,t]^{p\times q}$ the vector spaces of $p \times q$ matrices with elements in $\mathbb{F}$, $\mathbb{F}[s]$, and $\mathbb{F}[s,t]$, respectively. $\mathrm{Gl}_p(\mathbb{F})$ will be the general linear group of invertible matrices in $\mathbb{F}^{p\times p}$.

Given a polynomial $\alpha(s) = \sum_{i=0}^{g} \alpha_i s^i \in \mathbb{F}[s]$, with $\deg(\alpha) = g$, and an integer $h \geq g$, we will denote by $\mathrm{rev}_h(\alpha)(t)$ the polynomial $\mathrm{rev}_h(\alpha)(t) = t^h \alpha(\frac{1}{t}) = t^{h-g} \sum_{i=0}^{g} \alpha_i t^{g-i} \in \mathbb{F}[t]$. We have $\deg(\mathrm{rev}_h(\alpha)) \leq h$. If $h = g$, we denote $\mathrm{rev}_g(\alpha) = \tilde{\alpha}(t)$. Then, $\tilde{\alpha}(0) \neq 0$.

The *companion matrix* of a monic polynomial $\alpha(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \in \mathbb{F}[s]$, will be

$$C = \begin{bmatrix} 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ -a_0 & -a_1 & \ldots & -a_{n-2} & -a_{n-1} \end{bmatrix} \in \mathbb{F}^{n\times n}.$$

Given a polynomial matrix $A(s) \in \mathbb{F}[s]^{m\times n}$, the *degree* of $A(s)$ ($\deg(A(s))$) is the maximum of the degrees of its entries, and the *normal rank* of $A(s)$ ($\mathrm{rank}(A(s))$) is the order of the largest non identically zero minor of $A(s)$, i.e., it is the rank of $A(s)$ considered as a matrix on the field of fractions of $\mathbb{F}[s]$. If $\deg(A(s)) = g$ and h is an integer $h \geq g$, then $\mathrm{rev}_h(A)(t) = t^h A(\frac{1}{t}) \in \mathbb{F}[t]^{m\times n}$.

A matrix $U(s) \in \mathbb{F}[s]^{n\times n}$ is *unimodular* if it is a unit in the ring $\mathbb{F}[s]^{n\times n}$, i.e., $0 \neq \det(U(s)) \in \mathbb{F}$. Two polynomial matrices $A(s), B(s) \in \mathbb{F}[s]^{m\times n}$ are *equivalent* ($A(s) \sim B(s)$) if there exist unimodular matrices $U(s) \in \mathbb{F}[s]^{m\times m}$, $V(s) \in \mathbb{F}[s]^{n\times n}$ such that $B(s) = U(s)A(s)V(s)$. If $A(s) \in \mathbb{F}[s]^{m\times n}$ and $\mathrm{rank}(A(s)) = \rho$, then (see for example [20, Ch. 6]) $A(s)$ is equivalent to a unique matrix of the form $S(s) = \begin{bmatrix} \mathrm{diag}(\alpha_1(s), \ldots, \alpha_\rho(s)) & 0 \\ 0 & 0 \end{bmatrix}$, where $\alpha_1(s), \ldots, \alpha_\rho(s)$ are monic polynomials and $\alpha_1(s) \mid \cdots \mid \alpha_\rho(s)$. The matrix $S(s)$ is the *Smith form* of $A(s)$ and the polynomials $\alpha_1(s), \ldots, \alpha_\rho(s)$ are the *invariant factors* of $A(s)$. We will take $\alpha_i(s) = 1$ for $i < 1$ and $\alpha_i(s) = 0$ for $i > \rho$. For $1 \leq k \leq \rho$, the monic greatest common divisor of the minors of $A(s)$ of order k is the *determinantal divisor of* $A(s)$ *of order* k, denoted by $D_k(s)$, and $D_k(s) = \alpha_1(s) \ldots \alpha_k(s)$. The invariant factors form a complete system of invariants for the equivalence of polynomial matrices, i.e., two polynomial matrices $A(s), B(s) \in \mathbb{F}[s]^{m\times n}$ are equivalent if and only if they have the same invariant factors.

A *matrix pencil* is a polynomial matrix $A(s) \in \mathbb{F}[s]^{m\times n}$ of degree at most one ($A(s) = A_0 + sA_1$). The pencil is *regular* if $m = n = \mathrm{rank}(A(s))$. Otherwise it is *singular*. If

$\operatorname{rank}(A(s)) = \min\{m, n\}$ the pencil is also called *quasi-regular*. The set of matrix pencils in $\mathbb{F}[s]^{m\times n}$ is denoted by $\mathcal{P}_{m\times n}(\mathbb{F})$.

Two matrix pencils $A(s) = A_0 + sA_1, B(s) = B_0 + sB_1 \in \mathcal{P}_{m\times n}(\mathbb{F})$ are *strictly equivalent* ($A(s) \overset{s.e.}{\sim} B(s)$) if there exist invertible matrices $P \in \mathrm{Gl}_m(\mathbb{F})$, $Q \in \mathrm{Gl}_n(\mathbb{F})$ such that $B(s) = PA(s)Q$.

Given the pencil $A(s) = A_0 + sA_1 \in \mathcal{P}_{m\times n}(\mathbb{F})$ of $\operatorname{rank} A(s) = \rho$, a complete system of invariants for the strict equivalence is formed by a chain of homogeneous polynomials $\phi_1(s,t) \mid \cdots \mid \phi_\rho(s,t)$, $\phi_i(s,t) \in \mathbb{F}[s,t]$, $1 \le i \le \rho$, monic with respect to s, called the *homogeneous invariant factors* and two finite partitions of nonnegative integers, $c_1 \ge \cdots \ge c_{n-\rho}$ and $u_1 \ge \cdots \ge u_{m-\rho}$, called the *column and row minimal indices* of the pencil, respectively. In turn, the homogeneous invariant factors are determined by the invariant factors $\alpha_1(s) \mid \ldots \mid \alpha_\rho(s)$ and a chain of polynomials $t^{k_1} \mid \ldots \mid t^{k_\rho}$ in $\mathbb{F}[t]$, called the *infinite elementary divisors* (see [19, Ch. 2] or [20, Ch. 12]). In fact, we can write

$$\phi_i(s,t) = t^{k_i} t^{\deg(\alpha_i)} \alpha_i\left(\frac{s}{t}\right), \quad 1 \le i \le \rho. \tag{1}$$

Observe that $\alpha_i(s) = \phi_i(s,1)$, $1 \le i \le \rho$. If $\bar\alpha_1(t) \mid \ldots \mid \bar\alpha_\rho(t)$ are the invariant factors of the pencil $\operatorname{rev}_1(A)(t) = tA_0 + A_1 \in \mathbb{F}[t]^{m\times n}$, then for some $0 \ne l_i \in \mathbb{F}$,

$$l_i\bar\alpha_i(t) = \phi_i(1,t) = t^{k_i}\tilde\alpha_i(t), \quad 1 \le i \le \rho. \tag{2}$$

(Recall that $\tilde\alpha_i(t) = \operatorname{rev}_{g_i}(\alpha_i)$, where $g_i = \deg(\alpha_i)$). If $A(s) \in \mathcal{P}_{m\times n}(\mathbb{F})$ and $\operatorname{rank}(A(s)) = m$ ($\operatorname{rank}(A(s)) = n$), then $A(s)$ does not have row (column) minimal indices. As a consequence, the invariants for the strict equivalence of regular matrix pencils are reduced to the homogeneous invariant factors.

A canonical form for the strict equivalence of matrix pencils is the *Kronecker canonical form*. Let $A(s) \in \mathcal{P}_{m\times n}(\mathbb{F})$ be a pencil of $\operatorname{rank} A(s) = \rho$, with invariant factors $1 = \alpha_1(s) = \cdots = \alpha_{\rho-x}(s) \ne \alpha_{\rho-x+1}(s) \mid \cdots \mid \alpha_\rho(s)$, where $0 \le x \le \rho$ and $\deg(\alpha_{\rho-x+i}) = g_i > 0$, $1 \le i \le x$, infinite elementary divisors $1 = t^{k_1} = \cdots = t^{k_{\rho-y}} \ne t^{k_{\rho-y+1}} \mid \cdots \mid t^{k_\rho}$, where $0 \le y \le \rho$, column minimal indices $c_1 \ge \cdots \ge c_r > 0 = c_{r+1} = \cdots = c_{n-\rho}$, where $0 \le r \le n-\rho$, and row minimal indices $u_1 \ge \cdots \ge u_z > 0 = u_{z+1} = \cdots = u_{m-\rho}$, where $0 \le z \le m-\rho$. Then the Kronecker canonical form of $A(s)$ is

$$\begin{bmatrix} L(s) & 0 & 0 & 0 & 0 \\ 0 & R(s) & 0 & 0 & 0 \\ 0 & 0 & C(s) & 0 & 0 \\ 0 & 0 & 0 & N(s) & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathcal{P}_{m\times n}(\mathbb{F}),$$

where $L(s) = \operatorname{diag}(L_{c_1}(s), \ldots, L_{c_r}(s))$, $R(s) = \operatorname{diag}(R_{u_1}(s), \ldots, R_{u_z}(s))$, $C(s) = \operatorname{diag}(sI_{g_1} - C_1, \ldots, sI_{g_x} - C_x)$, $N(s) = \operatorname{diag}(N_{k_{\rho-y+1}}(s), \ldots, N_{k_\rho}(s))$, with C_i the companion matrix of $\alpha_{\rho-x+i}$, $1 \le i \le x$,

$$L_k(s) = \begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & s & -1 \end{bmatrix} \in \mathcal{P}_{k\times(k+1)}(\mathbb{F}),$$

$$R_k(s) = L_k^T(s) \in \mathcal{P}_{(k+1)\times k}(\mathbb{F}),$$

$$N_k(s) = \begin{bmatrix} 1 & -s & & \\ & \ddots & \ddots & \\ & & \ddots & -s \\ & & & 1 \end{bmatrix} \in \mathcal{P}_{k\times k}(\mathbb{F}),$$

understanding that the non specified components are zero. For details see [19, Ch. 2] or [20, Ch. 12] for infinite fields, and [29, Ch. 2] for arbitrary fields. From the Kronecker canonical form of the pencil $A(s)$ it is easy to see that $\sum_{i=1}^{\rho} \deg(\phi_i) + \sum_{i=1}^{n-\rho} c_i + \sum_{i=1}^{m-\rho} u_i = \rho$. If $A(s) \in \mathcal{P}_{p\times p}(\mathbb{F})$ is a regular pencil, then the Kronecker canonical form of $A(s)$ is reduced to $\operatorname{diag}(C(s), N(s))$ and is known as the *Weirstrass canonical form.*

Assume now that $A(s) = A_0 + sA_1 \in \mathcal{P}_{m\times n}(\mathbb{C})$ is a complex matrix pencil. Let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The *spectrum* of $A(s)$ is $\Lambda(A(s)) = \{\lambda \in \overline{\mathbb{C}} \ : \ \operatorname{rank}(A(\lambda)) < \operatorname{rank}(A(s))\}$, where we agree that $A(\infty) = A_1$. The elements $\lambda \in \Lambda(A(s))$ are the *eigenvalues* of $A(s)$.

We can factorize

$$\alpha_{\rho-i+1}(s) = \prod_{\lambda\in\Lambda(A(s))\setminus\{\infty\}} (s-\lambda)^{n_i(\lambda,A(s))}, \quad 1 \leq i \leq \rho,$$

$$\phi_{\rho-i+1}(s,t) = t^{n_i(\infty,A(s))} \prod_{\lambda\in\Lambda(A(s))\setminus\{\infty\}} (s-\lambda t)^{n_i(\lambda,A(s))}, \quad 1 \leq i \leq \rho.$$

The integers $n_1(\lambda, A(s)) \geq \cdots \geq n_\rho(\lambda, A(s))$ are the *partial multiplicities* of λ in $A(s)$, $\mathbf{n}(\lambda, A(s)) = (n_1(\lambda, A(s)), \ldots, n_\rho(\lambda, A(s)))$ is the *partition of* λ *in the Segre characteristic of* $A(s)$ and its conjugate partition $\mathbf{w}(\lambda, A(s)) = \overline{\mathbf{n}(\lambda, A(s))} = (w_1(\lambda, A(s)), \ldots, w_\rho(\lambda, A(s)))$ is *the partition of* λ *in the Weyr characteristic of* $A(s)$.

For $\lambda \in \overline{\mathbb{C}} \setminus \Lambda(A(s))$ we take $\mathbf{n}(\lambda, A(s)) = \mathbf{w}(\lambda, A(s)) = 0$. We agree that $n_i(\lambda, A(s)) = +\infty$ for $i < 1$ and $n_i(\lambda, A(s)) = 0$ for $i > \rho$, for $\lambda \in \overline{\mathbb{C}}$. We also agree that $\phi_i(s,t) = 1$ for $i < 1$ and $\phi_i(s,t) = 0$ for $i > \rho$. Observe that, in (1), $k_i = n_{\rho-i+1}(\infty, A(s))$, $1 \leq i \leq \rho$, and from (2) we conclude that $\mathbf{n}(\infty, A(s)) = \mathbf{n}(0, \operatorname{rev}_1(A)(t))$, thus $\mathbf{w}(\infty, A(s)) = \mathbf{w}(0, \operatorname{rev}_1(A)(t))$.

We will use the ℓ_1 norm in the vector space of polynomials of degree less than or equal to n. Given a polynomial $\alpha(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \in \mathbb{C}[s]$, $\|\alpha(s)\| = \sum_{i=1}^{n} |a_i|$. Notice that if $\alpha(s), \beta(s) \in \mathbb{C}[s]$ and h is an integer such that $\deg(\alpha) \leq h$, $\deg(\beta) \leq h$, then $\|\alpha(s) - \beta(s)\| = \|\operatorname{rev}_h(\alpha)(t) - \operatorname{rev}_h(\beta)(t)\|$.

Given a matrix $M(s) = (m_{i,j}(s)) \in \mathbb{C}[s]^{m\times n}$, $\|M(s)\| = \sum_{i=1}^{m} \sum_{j=1}^{n} \|m_{i,j}(s)\|$. If $M(s) \in \mathbb{C}[s]^{m\times n}, N(s) \in \mathbb{C}[s]^{n\times p}$, then $\|M(s)N(s)\| \leq \|M(s)\|\|N(s)\|$.

Given a real number $\eta > 0$ and $\lambda \in \mathbb{C}$, $B(\lambda, \eta) = \{z \in \mathbb{C} \ : \ |z - \lambda| < \eta\}$ denotes the open ball centered at λ and radius η. For $\lambda = \infty$, $B(\infty, \eta) = \{z \in \mathbb{C} \ : \ |z| > \eta^{-1}\} \cup \{\infty\}$.

For a given pencil $A(s) \in \mathcal{P}_{m\times n}(\mathbb{C})$, we define the *$\eta$-neighbourhood of the spectrum of* $A(s)$ as $\mathcal{V}_\eta(A(s)) = \bigcup_{\lambda\in\Lambda(A(s))} B(\lambda, \eta)$, whenever the balls $B(\lambda, \eta)$ are pairwise disjoint.

3 Previous results

In this section we present some preliminary results. We have grouped them in two subsections.

3.1 Perturbation results

First of all, we show that in Problem 1.1 we can assume that $A_2(s)$ is in Kronecker canonical form. The proof follows the scheme of that of [5, Lemma 3.2].

Lemma 3.1 *Let* $A(s) = \begin{bmatrix} A_1(s) \\ A_2(s) \end{bmatrix} \in \mathcal{P}_{(r+(m-r))\times n}(\mathbb{C})$ *and* $B(s) \in \mathcal{P}_{m\times n}(\mathbb{C})$ *be matrix pencils.*

Let $\bar{A}_2(s) = PA_2(s)Q$ and $\bar{A}_1(s) = A_1(s)Q$ with $P \in \mathrm{Gl}_{m-r}(\mathbb{C})$, $Q \in \mathrm{Gl}_n(\mathbb{C})$. Let $\bar{A}(s) = \begin{bmatrix} \bar{A}_1(s) \\ \bar{A}_2(s) \end{bmatrix}$. The following propositions are equivalent:

(i) For every $\epsilon > 0$, there exists a pencil $A'(s) = \begin{bmatrix} A'_1(s) \\ A_2(s) \end{bmatrix} \in \mathcal{P}_{(r+(m-r))\times n}(\mathbb{C})$ such that $\|A'(s) - A(s)\| < \epsilon$ and $A'(s) \overset{s.e.}{\sim} B(s)$.

(ii) For every $\epsilon' > 0$, there exists a pencil $\bar{A}'(s) = \begin{bmatrix} \bar{A}'_1(s) \\ \bar{A}_2(s) \end{bmatrix} \in \mathcal{P}_{(r+(m-r))\times n}(\mathbb{C})$ such that $\|\bar{A}'(s) - \bar{A}(s)\| < \epsilon'$ and $\bar{A}'(s) \overset{s.e.}{\sim} B(s)$.

Proof. We have $\mathrm{diag}(I_r, P)A(s)Q = \bar{A}(s)$.

$(i) \Rightarrow (ii)$: Let $\epsilon' > 0$. From (i) we know that given $\epsilon > 0$ there exists a pencil $A'(s) = \begin{bmatrix} A'_1(s) \\ A_2(s) \end{bmatrix} \in \mathcal{P}_{(r+(m-r))\times n}(\mathbb{C})$ such that $A'(s) \overset{s.e.}{\sim} B(s)$ and $\|A'(s) - A(s)\| < \epsilon$. Let $\bar{A}'(s) = \mathrm{diag}(I_r, P)A'(s)Q$. Then, $\bar{A}'(s) \overset{s.e.}{\sim} A'(s) \overset{s.e.}{\sim} B(s)$ and

$$\|\bar{A}'(s) - \bar{A}(s)\| = \|\mathrm{diag}(I_r, P)(A'(s) - A(s))Q\| \leq \|\mathrm{diag}(I_r, P)\|\|A'(s) - A(s)\|\|Q\|$$

$$< \epsilon\|\mathrm{diag}(I_r, P)\|\|Q\|.$$

Taking $\epsilon = \frac{\epsilon'}{\|\mathrm{diag}(I_r,P)\|\|Q\|}$, the result follows.

$(ii) \Rightarrow (i)$: The proof is analogous.

The following lemma can also be found in [5, Lemma 2.1].

Lemma 3.2 *[4, Theorem VI.1.2] Let $\alpha(s) \in \mathbb{C}[s]$ be a polynomial of degree g, $\alpha(s) = \sum_{i=0}^{g} a_i s^i = a_g(s - \mu_1)\dots(s - \mu_g)$.*

(a) For every $\epsilon > 0$, there exists $\delta > 0$ such that if $\alpha'(s)$ is a polynomial of degree at most g and $\|\alpha'(s) - \alpha(s)\| < \delta$, then the roots of $\alpha'(s)$ are in $\bigcup_{i=1}^{g} B(\mu_i, \epsilon)$.

(b) Reciprocally, given $\epsilon > 0$ there exists $\delta > 0$ such that if $\mu'_i \in B(\mu_i, \delta)$, $1 \leq i \leq g$, and $\alpha'(s) = a_g(s - \mu'_1)\dots(s - \mu'_g)$, then $\|\alpha'(s) - \alpha(s)\| < \epsilon$.

In the next theorem, necessary conditions are given for perturbations of quasi-regular pencils.

Theorem 3.3 *[8, Ch. 2, Theorem 2.6], [22, Theorem 4.2, particular case] Let $A(s) \in \mathcal{P}_{m\times n}(\mathbb{C})$ be a pencil such that $\mathrm{rank}(A(s)) = \min\{m, n\}$. Let the partition $\mathbf{r}$ (the partition $\mathbf{s}$) be the conjugate partition of that of the column (row) minimal indices of $A(s)$.*

Let $\mathcal{V}_\eta(A(s))$ be an η-neighbourhood of the spectrum of $A(s)$. There exists $\delta > 0$ such that if $\|A'(s) - A(s)\| < \delta$, then $\mathrm{rank}\, A'(s) = \mathrm{rank}\, A(s)$ and

(i) If the partition $\mathbf{r}'$ (the partition $\mathbf{s}'$) is the conjugate partition of that of the column (row) minimal indices of $A'(s)$, then $\mathbf{r} \prec\prec \mathbf{r}'$ ($\mathbf{s} \prec\prec \mathbf{s}'$).

(ii)

$$\Lambda(A'(s)) \subseteq \mathcal{V}_\eta(A(s)), \tag{3}$$

(iii) $\bigcup_{\mu \in B(\lambda,\eta)} \mathbf{w}(\mu, A'(s)) \prec\prec \mathbf{w}(\lambda, A(s))$*, for every* $\lambda \in \Lambda(A(s))$.

Remark 3.4 *If* $\operatorname{rank}(A(s)) = m = n$*, i.e., if* $A(s)$ *is regular, then* $A'(s)$ *is also regular, condition (i) disappears and condition (iii) becomes*

$$\bigcup_{\mu \in B(\lambda,\eta)} \mathbf{w}(\mu, A'(s)) \prec \mathbf{w}(\lambda, A(s)), \text{ for every } \lambda \in \Lambda(A(s)). \tag{4}$$

The following results on perturbation of matrix pencils are stated for more general pencils in the corresponding references; we present here the particular cases for regular pencils.

Theorem 3.5 *[22, Theorem 5.1, particular case] Let* $A(s) \in \mathcal{P}_{n\times n}(\mathbb{C})$ *be a regular pencil, and let* $\mathcal{V}_\eta(A(s))$ *be an* η*-neighbourhood of the spectrum of* $A(s)$*. For every* $\lambda \in \Lambda(A(s))$*, let* t_λ *be a given integer* $t_\lambda \geq 0$ *and let* $\mathbf{w}^{(\lambda,j)} = (w_1^{(\lambda,j)}, \dots)$ *be given partitions,* $1 \leq j \leq t_\lambda$*. For every* $\epsilon > 0$*, there exists a pencil* $A'(s) \in \mathcal{P}_{n\times n}(\mathbb{C})$ *such that* $\|A'(s) - A(s)\| < \epsilon$*, its spectrum satisfies (3), and*

$$\begin{array}{l} A'(s) \text{ has just } t_\lambda \text{ eigenvalues } \mu_{\lambda,1}, \dots, \mu_{\lambda,t_\lambda} \text{ in } B(\lambda,\eta), \text{ with} \\ \mathbf{w}(\mu_{\lambda,j}, A'(s)) = \mathbf{w}^{(\lambda,j)},\ 1 \leq j \leq t_\lambda, \text{ for every } \lambda \in \Lambda(A(s)), \end{array} \tag{5}$$

if and only if

$$\bigcup_{j=1}^{t_\lambda} \mathbf{w}^{(\lambda,j)} \prec \mathbf{w}(\lambda, A(s)), \text{ for every } \lambda \in \Lambda(A(s)). \tag{6}$$

Theorem 3.6 *[8, Ch. 2, Theorem 3.1, particular case] Let* $A(s), B(s) \in \mathcal{P}_{n\times n}(\mathbb{C})$ *be regular pencils. For every* $\epsilon > 0$*, there exists a pencil* $A'(s) \in \mathcal{P}_{n\times n}(\mathbb{C})$ *such that* $\|A'(s) - A(s)\| < \epsilon$ *and* $A'(s) \overset{s.e.}{\sim} B(s)$ *if and only if*

$$\mathbf{w}(\lambda, B(s)) \prec \mathbf{w}(\lambda, A(s)), \text{ for every } \lambda \in \overline{\mathbb{C}}. \tag{7}$$

Corollary 3.7 *Let* $A(s), B(s) \in \mathcal{P}_{n\times n}(\mathbb{C})$ *be regular pencils with homogeneous invariant factors* $\psi_1(s,t) \mid \cdots \mid \psi_n(s,t)$ *and* $\psi'_1(s,t) \mid \cdots \mid \psi'_n(s,t)$*, respectively. For every* $\epsilon > 0$*, there exists a pencil* $A'(s) \in \mathcal{P}_{n\times n}(\mathbb{C})$ *such that* $\|A'(s) - A(s)\| < \epsilon$ *and* $A'(s) \overset{s.e.}{\sim} B(s)$ *if and only if*

$$\prod_{j=1}^{i} \psi'_j(s,t) \mid \prod_{j=1}^{i} \psi_j(s,t), \quad 1 \leq i \leq n. \tag{8}$$

Proof. Taking into account that $\deg(\prod_{j=1}^n \psi_j(s,t)) = \deg(\prod_{j=1}^n \psi'_j(s,t)) = n$, from (8), for $i = n$ we obtain

$$\prod_{j=1}^{n} \psi'_j(s,t) = \prod_{j=1}^{n} \psi_j(s,t). \tag{9}$$

By Theorem 3.6, we must prove that (8) and (9) are equivalent to (7). It is easy to see that (8) and (9) are equivalent to

$$\prod_{j=1}^{i} \psi_{n-j+1}(s,t) \mid \prod_{j=1}^{i} \psi'_{n-j+1}(s,t),\ 1 \leq i \leq n, \prod_{j=1}^{n} \psi_{n-j+1}(s,t) = \prod_{j=1}^{n} \psi'_{n-j+1}(s,t). \tag{10}$$

As

$$\psi_{n-i+1}(s,t) = t^{n_i(\infty,A(s))} \prod_{\lambda\in\Lambda(A(s))\setminus\{\infty\}} (s-\lambda t)^{n_i(\lambda,A(s))},$$

$$\psi'_{n-i+1}(s) = t^{n_i(\infty,B(s))} \prod_{\lambda\in\Lambda(B(s))\setminus\{\infty\}} (s-\lambda t)^{n_i(\lambda,B(s))}, \quad 1\leq i\leq n,$$

condition (10) is equivalent to

$$(\mathbf{n}(\lambda, A(s)) \prec (\mathbf{n}(\lambda, B(s)), \text{ for every } \lambda\in\overline{\mathbb{C}}, \tag{11}$$

which is equivalent to (7).

Remark 3.8 *Following [6] we denote condition (8) by*

$$(\psi'_1(s,t),\ldots,\psi'_n(s,t)) \prec\prec (\psi_1(s,t),\ldots,\psi_n(s,t)),$$

and conditions (8) and (9) by

$$(\psi'_1(s,t),\ldots,\psi'_n(s,t)) \prec (\psi_1(s,t),\ldots,\psi_n(s,t)). \tag{12}$$

3.2 Completion results

The following theorem contains a solution of a polynomial matrix completion problem.

Theorem 3.9 *[26, 31] Let $A(s)\in\mathbb{F}[s]^{(m-r)\times(n-s)}$, $B(s)\in\mathbb{F}[s]^{m\times n}$ be polynomial matrices matrices with invariant factors $\alpha_1(s)\mid\cdots\mid\alpha_\rho(s)$ and $\beta_1(s)\mid\cdots\mid\beta_{\bar\rho}(s)$, respectively, where $\rho=\operatorname{rank}(A(s))$ and $\bar\rho=\operatorname{rank}(B(s))$. Then, there exist $X(s)\in\mathbb{F}[s]^{r\times(n-s)}$, $Y(s)\in\mathbb{F}[s]^{r\times s}$, $Z(s)\in\mathbb{F}[s]^{(m-r)\times s}$ such that $B(s)\sim\begin{bmatrix}X(s) & Y(s)\\ A(s) & Z(s)\end{bmatrix}$ if and only if*

$$\beta_i(s)\mid\alpha_i(s)\mid\beta_{i+r+s}(s), \quad 1\leq i\leq\rho.$$

As mentioned, the problem of row completion of matrix pencils was solved in [13, 14]. We state here the version of [15] for pencils without row minimal indices.

Theorem 3.10 *[15, Theorem 2, particular case] Let $A_2(s)\in\mathcal{P}_{(n-r)\times(n+m)}(\mathbb{F})$ be a matrix pencil of $\operatorname{rank}(A_2(s))=n-r$, with homogeneous invariant factors $\phi_1(s,t)\mid\cdots\mid\phi_{n-r}(s,t)$ and column minimal indices $c_1\geq\cdots\geq c_{r+m}$. Let $A(s)\in\mathcal{P}_{n\times(n+m)}(\mathbb{F})$ be a matrix pencil of $\operatorname{rank}(A(s))=n$ with homogeneous invariant factors $\psi_1(s,t)\mid\cdots\mid\psi_n(s,t)$ and column minimal indices $d_1\geq\cdots\geq d_m$. Let $\mathbf{c}=(c_1,\ldots,c_{r+m})$, $\mathbf{d}=(d_1,\ldots,d_m)$ and $\mathbf{a}=(a_1,\ldots,a_r)$, where*

$$a_i=\deg(\tau_{r-i+1}(s,t))-\deg(\tau_{r-i}(s,t))-1, \quad 1\leq i\leq r, \text{ with}$$

$\tau_j(s,t)=\prod_{i=1}^{n-r+j}\operatorname{lcm}(\phi_{i-j}(s,t),\psi_i(s,t))$, $0\leq j\leq r$.

There exists a matrix pencil $A_1(s)\in\mathcal{P}_{r\times(n+m)}(\mathbb{F})$ such that $\begin{bmatrix}A_1(s)\\ A_2(s)\end{bmatrix}\overset{s.e.}{\sim}A(s)$ if and only if

$$\psi_i(s,t)\mid\phi_i(s,t)\mid\psi_{i+r}(s,t), \quad 1\leq i\leq n-r, \tag{13}$$

and

$$\mathbf{c}\prec'(\mathbf{d},\mathbf{a}). \tag{14}$$

Remark 3.11 *If $m = 0$, then $A(s)$ is regular. As it has no column minimal indices, condition (14) becomes*

$$\mathbf{c} \prec \mathbf{a}.$$

4 Main results

In this section we study Problem 1.1 when $m = n$ and the pencil $A(s) = \begin{bmatrix} A_1(s) \\ A_2(s) \end{bmatrix} \in \mathcal{P}_{(r+(n-r))\times n}(\mathbb{C})$ is regular, hence $A_2(s) \in \mathcal{P}_{(n-r)\times n}(\mathbb{C})$ is quasi-regular with $\operatorname{rank}(A_2(s)) = n - r$. We denote the homogeneous invariant factors and column minimal indices of $A_2(s)$ by $\phi_1(s,t) \mid \cdots \mid \phi_{n-r}(s,t)$ and $c_1 \geq \cdots \geq c_r$, respectively, and the homogeneous invariant factors of $A(s)$ are $\psi_1(s,t) \mid \cdots \mid \psi_n(s,t)$.

In the following results we give some necessary conditions a pencil must satisfy when obtained after perturbation of $A_1(s)$. The next theorem is straightforward from Theorems 3.3, 3.10 and Remarks 3.4 and 3.11.

Theorem 4.1 *Let $\mathcal{V}_\eta(A(s))$ be an η-neighbourhood of the spectrum of $A(s)$. There exists $\epsilon > 0$ such that if $A'(s) = \begin{bmatrix} A'_1(s) \\ A_2(s) \end{bmatrix} \in \mathcal{P}_{(r+(n-r))\times n}(\mathbb{C})$ has $\psi'_1(s,t) \mid \cdots \mid \psi'_n(s,t)$ as homogeneous invariant factors and $\|A'(s) - A(s)\| < \epsilon$, then the spectrum of $A'(s)$ satisfies (3), (4), and*

$$\psi'_i(s,t) \mid \phi_i(s,t) \mid \psi'_{i+r}(s,t), \quad 1 \leq i \leq n-r, \tag{15}$$

and

$$\mathbf{c} \prec \mathbf{a}', \tag{16}$$

where $\mathbf{c} = (c_1, \ldots, c_r)$ and $\mathbf{a}' = (a'_1, \ldots, a'_r)$, with

$$a'_i = \deg(\tau'_{r-i+1}(s,t)) - \deg(\tau'_{r-i}(s,t)) - 1, \quad 1 \leq i \leq r,$$

and $\tau'_j(s,t) = \prod_{i=1}^{n-r+j} \operatorname{lcm}(\phi_{i-j}(s,t), \psi'_i(s,t))$, $0 \leq j \leq r$.

From Corollary 3.7 (see Remark 3.8), Theorem 3.10 and Remark 3.11, we obtain

Theorem 4.2 *Let $\psi'_1(s,t) \mid \cdots \mid \psi'_n(s,t)$ be homogeneous polynomials, monic with respect to s. If for every $\epsilon > 0$, there exists a pencil $A'(s) = \begin{bmatrix} A'_1(s) \\ A_2(s) \end{bmatrix} \in \mathcal{P}_{(r+(n-r))\times n}(\mathbb{C})$ with $\psi'_1(s,t) \mid \cdots \mid \psi'_n(s,t)$ as homogeneous invariant factors and $\|A'(s) - A(s)\| < \epsilon$, then (15), (16) and (12) hold.*

In the rest of the section we will assume that $r = 1$, i.e., $A(s) = \begin{bmatrix} a(s) \\ A_2(s) \end{bmatrix} \in \mathcal{P}_{(1+(n-1))\times n}(\mathbb{C})$, and $\psi'_1(s,t) \mid \cdots \mid \psi'_n(s,t)$ will be prescribed homogeneous polynomials, monic with respect to s.

Since $\operatorname{rank} A_2(s) = n - 1$, the pencil $A_2(s)$ has only one column minimal index, $c_1 \geq 0$. Let us assume that $A_2(s)$ has $p \geq 0$ nontrivial invariant factors $1 \neq \alpha_1(s) \mid \cdots \mid \alpha_p(s)$,

$\alpha_i(s) = s^{g_i} + \sum_{j=0}^{g_i-1} a_{i,j}s^j$, $g_i > 0$, $1 \leq i \leq p$, and $q \geq 0$ nontrivial infinite elementary divisors $t^{\mu_1} \mid \cdots \mid t^{\mu_q}$, $\mu_i > 0$, $1 \leq i \leq q$. Notice that if $\hat{\alpha}_1(s) \mid \cdots \mid \hat{\alpha}_{n-1}(s)$ are the invariant factors of $A_2(s)$, then $\hat{\alpha}_i(s) = 1$ for $1 \leq i \leq n-1-p$ and $\hat{\alpha}_{n-1-p+i}(s) = \alpha_i(s)$ for $1 \leq i \leq p$. Analogously, if $t^{\hat{\mu}_1} \mid \cdots \mid t^{\hat{\mu}_{n-1}}$ are the infinite elementary divisors of $A_2(s)$, then $\hat{\mu}_i = 0$ for $1 \leq i \leq n-1-q$ and $\hat{\mu}_{n-1-q+i} = \mu_i$ for $1 \leq i \leq q$. The next identities are satisfied

$$c_1 + \sum_{i=1}^{n-1} \deg(\phi_i) = c_1 + \sum_{i=1}^{p} g_i + \sum_{i=1}^{q} \mu_i = n-1.$$

In the case that $\sum_{i=1}^{n} \deg(\psi_i') = n$ and (15) holds for $r = 1$, i.e.,

$$\psi_i'(s,t) \mid \phi_i(s,t) \mid \psi_{i+1}'(s,t), \quad 1 \leq i \leq n-1, \tag{17}$$

we have

$$\tau_1'(s,t) = \prod_{i=1}^{n} \operatorname{lcm}(\phi_{i-1}(s,t), \psi_i'(s,t)) = \prod_{i=1}^{n} \psi_i'(s,t),$$

and

$$\tau_0'(s,t) = \prod_{i=1}^{n-1} \operatorname{lcm}(\phi_i(s,t), \psi_i'(s,t)) = \prod_{i=1}^{n-1} \phi_i(s,t),$$

hence

$$a_1' = \deg(\tau_1') - \deg(\tau_0') - 1 = n - (n-1-c_1) - 1 = c_1.$$

Therefore, for $r = 1$, (12) and (17) imply (16).

Our aim is to prove that conditions (12) and (17) are sufficient to guarantee that in every neighbourhood of $A(s)$ there exists a pencil $A'(s) \in \mathcal{P}_{n\times n}(\mathbb{C})$ with $\psi_1'(s,t) \mid \cdots \mid \psi_n'(s,t)$ as homogeneous invariant factors.

By Lemma 3.1, we can assume that $A_2(s)$ is in Kronecker canonical form. There are two cases to consider depending on the value of c_1.

$$\begin{array}{ll} c_1 > 0, & A_2(s) = \operatorname{diag}(L_{c_1}(s), C(s), N(s)) \in \mathcal{P}_{(n-1)\times n}(\mathbb{C}), \\ c_1 = 0, & A_2(s) = \begin{bmatrix} O & \operatorname{diag}(C(s), N(s)) \end{bmatrix} \in \mathcal{P}_{(n-1)\times n}(\mathbb{C}), \end{array} \tag{18}$$

where $C(s) = \operatorname{diag}(sI_{g_1} - C_1, \ldots, sI_{g_p} - C_p)$, $N(s) = \operatorname{diag}(N_{\mu_1}(s), \ldots, N_{\mu_q}(s))$, with C_i the companion matrix of $\alpha_i(s)$, $1 \leq i \leq p$, and $L_{c_1}(s)$ and $N_{\mu_i}(s)$, $1 \leq i \leq q$, are defined in (3).

The proof of the following lemma is analogous to that of Lemma 3.4 of[5].

Lemma 4.3 *Let* $A(s) = A_0 + sA_1 = \begin{bmatrix} a(s) \\ A_2(s) \end{bmatrix} \in \mathcal{P}_{(1+(n-1))\times n}(\mathbb{C})$ *be a regular pencil, with* $A_2(s)$ *as (18). We partition* $a(s)$ *according to the blocks of* $A_2(s)$,

$$a(s) = \begin{bmatrix} a^L(s) & a^C(s) & a^N(s) \end{bmatrix} =$$

$$\Big[\; a_1^L(s) \; \ldots \; a_{c_1+1}^L(s) \;\Big\|\; a_{1,1}^C(s) \; \ldots \; a_{1,g_1}^C(s) \;\Big|\; \ldots \;\Big|\; a_{p,1}^C(s) \; \ldots \; a_{p,g_p}^C(s) \;\Big\|$$

$$\Big\|\; a_{1,1}^N(s) \; \ldots \; a_{1,\mu_1}^N(s) \;\Big|\; \ldots \;\Big|\; a_{q,1}^N(s) \; \ldots \; a_{q,\mu_q}^N(s) \;\Big].$$

Then, there exist unimodular matrices $U(s), V(s) \in \mathbb{C}[s]^{n\times n}$ and $\bar{U}(t), \bar{V}(t) \in \mathbb{C}[t]^{n\times n}$ such that

$$U(s)A(s)V(s) = \operatorname{diag}(I_{n-1-p}, M(s)), \quad \bar{U}(t)\operatorname{rev}_1(A)(t)\bar{V}(t) = \operatorname{diag}(I_{n-1-p-q}, \bar{M}(t)), \tag{19}$$

with

$$M(s) = \begin{bmatrix} \gamma^L(s) & \gamma_1^C(s) & \dots & \gamma_p^C(s) \\ 0 & \alpha_1(s) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_p(s) \end{bmatrix},$$

$$\bar{M}(t) = \begin{bmatrix} \operatorname{rev}_{c_1+1}(\gamma^L)(t) & \operatorname{rev}_{g_1}(\gamma_1^C)(t) & \dots & \operatorname{rev}_{g_p}(\gamma_p^C)(t) & \operatorname{rev}_{\mu_1}(\gamma_1^N)(t) & \dots & \operatorname{rev}_{\mu_p}(\gamma_q^N)(t) \\ 0 & \tilde{\alpha}_1(t) & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \tilde{\alpha}_p(t) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & t^{\mu_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & t^{\mu_q} \end{bmatrix}, \tag{20}$$

where

$$\gamma^L(s) = \sum_{j=1}^{c_1+1} s^{j-1}a_j^L(s),\ \gamma_i^C(s) = \sum_{j=1}^{g_i} s^{j-1}a_{i,j}^C(s), 1 \le i \le p,\ \gamma_i^N(s) = \sum_{j=1}^{\mu_i} s^{\mu_i-j}a_{i,j}^N(s), 1 \le i \le q. \tag{21}$$

Remark 4.4 *1. In Lemma 4.3,*

$$\deg(\gamma^L) \le c_1+1, \quad \deg(\gamma_i^C) \le g_i,\ 1 \le i \le p, \quad \deg(\gamma_i^N) \le \mu_i,\ 1 \le i \le q. \tag{22}$$

2. *The pencil $A(s)$ (the pencil $\operatorname{rev}_1(A)(t)$) has the same nontrivial invariant factors as the polynomial matrix $M(s)$ (the polynomial matrix $\bar{M}(t)$).*

3. *By Theorem 3.10, the pencil $A(s)$ has at most $p+1$ nontrivial invariant factors $\beta_1(s) \mid \cdots \mid \beta_{p+1}(s)$ and $q+1$ nontrivial infinite elementary divisors $t^{\eta_1} \mid \cdots \mid t^{\eta_{q+1}}$ satisfying*

$$\beta_i(s) \mid \alpha_i(s) \mid \beta_{i+1}(s), \quad 1 \le i \le p, \qquad \eta_i \le \mu_i \le \eta_{i+1}, \quad 1 \le i \le q. \tag{23}$$

We have $\alpha_1(s)\dots\alpha_p(s)\gamma^L(s) = \kappa\beta_1(s)\dots\beta_{p+1}(s)$ for some $\kappa \in \mathbb{C}$, hence

$$\begin{aligned}\eta_1 + \cdots + \eta_{q+1} &= n - \sum_{i=1}^{p+1}\deg(\beta_i) = n - \sum_{i=1}^{p}\deg(\alpha_i) - \deg(\gamma^L) \\ &= \mu_1 + \cdots + \mu_p + c_1 + 1 - \deg(\gamma^L).\end{aligned}$$

Next lemma provides a characterization of the invariant factors of matrices of the form (20).

Lemma 4.5 *Let $\sigma_1(s) \mid \cdots \mid \sigma_p(s)$ be monic polynomials and*

$$M(s) = \begin{bmatrix} \gamma(s) \\ M_2(s) \end{bmatrix} = \begin{bmatrix} \gamma_0(s) & \gamma_1(s) & \dots & \gamma_p(s) \\ 0 & \sigma_1(s) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_p(s) \end{bmatrix} \in \mathbb{C}[s]^{(1+p)\times(1+p)},$$

a polynomial matrix with invariant factors $\tau_1(s) \mid \cdots \mid \tau_{p+1}(s)$. Then,

$$\gamma_0(s)\sigma_1(s)\dots\sigma_p(s) = \kappa\tau_1(s)\dots\tau_{p+1}(s),$$

for some $\kappa \in \mathbb{C}$, and for $1 \leq k \leq p$, $\tau_1(s)\dots\tau_k(s)$ is the monic greatest common divisor of the polynomials in the following list.

1. $\sigma_1(s)\dots\sigma_k(s)$,
2. $\gamma_0(s)\sigma_1(s)\dots\sigma_{k-1}(s)$,
3. $\gamma_i(s)\sigma_1(s)\dots\sigma_{i-1}(s)\sigma_{i+1}(s)\dots\sigma_k(s), \quad 1 \leq i \leq k.$
4. $\gamma_i(s)\sigma_1(s)\dots\sigma_{k-1}(s), \quad k+1 \leq i \leq p,$

where we take $\sigma_a(s)\dots\sigma_b(s) = 1$ for $a > b$.

Proof. The proof can be obtained taking into account that

$$\det(M(s)) = \kappa\tau_1(s)\dots\tau_{p+1}(s),$$

for some $\kappa \in \mathbb{C}$, and that, for $1 \leq k \leq p$, the minors of order k of $M(s)$ are multiples of one polynomial of the list (see the proof of [5, Theorem 3.5]).

Lemma 4.6 *Given a pencil $a(s) = \begin{bmatrix} a_1(s) & \dots & a_g(s) \end{bmatrix} \in \mathcal{P}_{1\times g}(\mathbb{C})$, let $\gamma(s) = \sum_{j=1}^{g} s^{j-1}a_j(s)$ and let $\gamma'(s) \in \mathbb{C}[s]$ be a polynomial such that $\deg(\gamma') \leq g$. Then, there exists a pencil $a'(s) = \begin{bmatrix} a'_1(s) & \dots & a'_g(s) \end{bmatrix} \in \mathcal{P}_{1\times g}(\mathbb{C})$ such that $\gamma'(s) = \sum_{j=1}^{g} s^{j-1}a'_j(s)$ and $\|a'(s) - a(s)\| = \|\gamma'(s) - \gamma(s)\|$.*

Proof. Let $a_j(s) = sa_{j,1} + a_{j,0}$, $1 \leq j \leq g$, $\gamma(s) = \sum_{j=0}^{g} s^j\gamma_j$, $\gamma'(s) = \sum_{j=0}^{g} s^j\gamma'_j$ and $e_j = \gamma'_j - \gamma_j$, $0 \leq j \leq g$. Then $\gamma_0 = a_{1,0}$, $\gamma_g = a_{g,1}$, $\gamma_j = a_{j,1} + a_{j+1,0}$, $1 \leq j \leq g-1$ and $\|\gamma'(s) - \gamma(s)\| = \sum_{j=0}^{g}|e_j|$.

Define $a'_j(s) = sa'_{j,1} + a'_{j,0}$, $1 \leq j \leq g$, with $a'_{1,0} = a_{1,0} + e_0$, $a'_{g,1} = a_{g,1} + e_g$, and, for $1 \leq j \leq g-1$, $a'_{j,1} = a_{j,1} + e_j$, $a'_{j+1,0} = a_{j+1,0}$ (or $a'_{j,1} = a_{j,1}$, $a'_{j+1,0} = a_{j+1,0} + e_j$). Then $a'(s) = \begin{bmatrix} a'_1(s) & \dots & a'_g(s) \end{bmatrix}$ satisfies the desired conditions.

Lemma 4.7 *Let*

$$\bar{M}(t) = \begin{bmatrix} \bar{a}(t) & \bar{b}(t) & \bar{c}(t) \\ O & \bar{M}_1(t) & 0 \\ O & O & \bar{M}_2(t) \end{bmatrix} \in \mathbb{C}[t]^{(1+p+q)\times(1+p+q)},$$

and

$$\bar{N}(t) = \begin{bmatrix} \bar{a}(t) & \bar{c}(t) \\ O & \bar{M}_2(t) \end{bmatrix} \in \mathbb{C}[t]^{(1+q)\times(1+q)},$$

be regular polynomial matrices such that $\bar{M}_1(0) = I_p$. Then

$$w(0, \bar{M}(t)) = w(0, \bar{N}(t)).$$

Proof. By [24, Theorem 2],

$$\sum_{i=1}^{k} w_i(0, \bar{M}(t)) = (1+p+q)k - \operatorname{rank} \mathcal{R}_k,$$

$$\sum_{i=1}^{k} w_i(0, \bar{N}(t)) = (1+q)k - \operatorname{rank} \mathcal{S}_k, \quad k \geq 1,$$

with

$$\mathcal{R}_k = \begin{bmatrix} \bar{M}(0) & 0 & \dots & 0 & 0 \\ \frac{1}{1!}\bar{M}^{(1)}(0) & \bar{M}(0) & \dots & 0 & 0 \\ \frac{1}{2!}\bar{M}^{(2)}(0) & \frac{1}{1!}\bar{M}^{(1)}(0) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{(k-1)!}\bar{M}^{(k-1)}(0) & \frac{1}{(k-2)!}\bar{M}^{(k-2)}(0) & \dots & \frac{1}{1!}\bar{M}^{(1)}(0) & \bar{M}(0) \end{bmatrix},$$

$$\mathcal{S}_k = \begin{bmatrix} \bar{N}(0) & 0 & \dots & 0 & 0 \\ \frac{1}{1!}\bar{N}^{(1)}(0) & \bar{N}(0) & \dots & 0 & 0 \\ \frac{1}{2!}\bar{N}^{(2)}(0) & \frac{1}{1!}\bar{N}^{(1)}(0) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{(k-1)!}\bar{N}^{(k-1)}(0) & \frac{1}{(k-2)!}\bar{N}^{(k-2)}(0) & \dots & \frac{1}{1!}\bar{N}^{(1)}(0) & \bar{N}(0) \end{bmatrix},$$

where the superscript $^{(j)}$ indicates the jth-derivative of the matrix with respect to t. Taking into account that

$$\bar{M}(0) = \begin{bmatrix} \bar{a}(0) & \bar{b}(0) & \bar{c}(0) \\ O & I_p & 0 \\ O & O & \bar{M}_2(0) \end{bmatrix},$$

performing elementary operations it is easy to see that $\operatorname{rank} \mathcal{R}_k = pk + \operatorname{rank} \mathcal{S}_k$, from where $\sum_{i=1}^{k} w_i(0, \bar{M}(t)) = \sum_{i=1}^{k} w_i(0, \bar{N}(t))$, for $k \geq 1$, and the lemma follows.

Lemma 4.8 *Let* $\sigma_1(s) \mid \cdots \mid \sigma_p(s)$, $\tau_1(s) \mid \cdots \mid \tau_{p+1}(s)$, $\tau'_1(s) \mid \cdots \mid \tau'_{p+1}(s)$ *be monic polynomials such that*

$$\tau'_i(s) \mid \sigma_i(s) \mid \tau'_{i+1}(s), \quad 1 \leq i \leq p, \tag{24}$$

and

$$(\tau'_1(s), \dots, \tau'_{p+1}(s)) \prec (\tau_1(s), \dots, \tau_{p+1}(s)). \tag{25}$$

Let $\gamma_0(s), \gamma_1(s), \dots, \gamma_p(s) \in \mathbb{C}[s]$ *be polynomials such that* $\deg(\gamma_i) \leq \deg(\sigma_i)$, $1 \leq i \leq p$, *and let*

$$M(s) = \begin{bmatrix} \gamma_0(s) & \gamma_1(s) & \dots & \gamma_p(s) \\ 0 & \sigma_1(s) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_p(s) \end{bmatrix} \in \mathbb{C}[s]^{(1+p)\times(1+p)},$$

be a matrix with invariant factors $\tau_1(s) \mid \cdots \mid \tau_{p+1}(s)$*. Then, for every* $\epsilon > 0$ *there exists*

$$M'(s) = \begin{bmatrix} \gamma_0(s) & \gamma'_1(s) & \dots & \gamma'_p(s) \\ 0 & \sigma_1(s) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_p(s) \end{bmatrix} \in \mathbb{C}[s]^{(1+p)\times(1+p)},$$

with invariant factors $\tau'_1(s) \mid \cdots \mid \tau'_{p+1}(s)$*, such that* $\deg(\gamma'_i) \leq \deg(\sigma_i)$*,* $1 \leq i \leq p$*, and* $\|M'(s) - M(s)\| < \epsilon$*.*

Proof. The invariant factors of the polynomial matrix

$$\begin{bmatrix} 0 & \sigma_1(s) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_p(s) \end{bmatrix} \in \mathbb{C}[s]^{p\times(1+p)},$$

are $\sigma_1(s) \mid \cdots \mid \sigma_p(s)$. By Theorem 3.9,

$$\tau_i(s) \mid \sigma_i(s) \mid \tau_{i+1}(s), \quad 1 \leq i \leq p.$$

Now the proof follows the steps of that of Theorem 3.5 in [5].

In the next theorem we solve the announced problem of prescription of the homogeneous invariant factors.

Theorem 4.9 *Let* $\psi'_1(s,t) \mid \cdots \mid \psi'_n(s,t)$ *be homogeneous polynomials, monic with respect to s. For every* $\epsilon > 0$*, there exists a pencil* $A'(s) = \begin{bmatrix} a'(s) \\ A_2(s) \end{bmatrix} \in \mathcal{P}_{(1+(n-1))\times n}(\mathbb{C})$ *with* $\psi'_1(s,t) \mid \cdots \mid \psi'_n(s,t)$ *as homogeneous invariant factors such that* $\|A'(s) - A(s)\| < \epsilon$ *if and only if (12) and (17) hold.*

Proof. The necessity of the conditions follows from Theorem 4.2. Let us prove the sufficiency.

Since $A_2(s)$ has p nontrivial invariant factors $\alpha_1(s) \mid \cdots \mid \alpha_p$ and q nontrivial infinite elementary divisors $t^{\mu_1} \mid \cdots \mid t^{\mu_q}$, by Theorem 3.10, the pencil $A(s)$ has at most $p+1$ nontrivial invariant factors $\beta_1(s) \mid \cdots \mid \beta_{p+1}(s)$ and $q+1$ nontrivial infinite elementary divisors $s^{\eta_1} \mid \cdots \mid s^{\eta_{q+1}}$ satisfying

$$\beta_i(s) \mid \alpha_i(s) \mid \beta_{i+1}(s), \quad 1 \leq i \leq p, \quad \text{and} \quad \eta_i \leq \mu_i \leq \eta_{i+1}, \quad 1 \leq i \leq q.$$

Denoting

$$\hat{\beta}'_i(s) = \psi'_i(s,1) \quad \text{and} \quad \psi'_i(s,t) = t^{\hat{\eta}'_i} t^{\deg(\hat{\beta}'_i)} \hat{\beta}'_i(\frac{s}{t}), \quad 1 \leq i \leq n,$$

from (17)

$$\hat{\beta}'_i(s) = 1,\ 1 \leq i \leq n-p-1, \quad \text{and} \quad \hat{\eta}'_i = 0,\ 1 \leq i \leq n-q-1,$$

and, taking $\beta'_i(s) = \hat{\beta}'_{n-p-1+i}(s)$, $1 \leq i \leq p+1$, $\eta'_i = \hat{\eta}'_{n-q-1+i}$, $1 \leq i \leq q+1$, from (17) and (12) we obtain

$$\beta'_i(s) \mid \alpha_i(s) \mid \beta'_{i+1}(s), \quad 1 \leq i \leq p, \text{ and } (\beta'_1(s), \dots, \beta'_{p+1}(s)) \prec (\beta_1(s), \dots, \beta_{p+1}(s)). \quad (26)$$

$$\eta'_i \leq \mu_i \leq \eta'_{i+1}, \quad 1 \leq i \leq q, \quad \text{and} \quad (t^{\eta'_1}, \dots, t^{\eta'_{q+1}}) \prec (t^{\eta_1}, \dots, t^{\eta_{q+1}}). \tag{27}$$

We partition $a(s)$ as in Lemma 4.3. According to this lemma, $A(s)$ and $\mathrm{rev}_1(A)(t)$ are equivalent to $\mathrm{diag}(I_{n-1-p}, M(s))$ and $\mathrm{diag}(I_{n-1-p-q}, \bar{M}(t))$, respectively, where $M(s)$ and $\bar{M}(t)$ are defined in (20) and (21).

Let $\epsilon > 0$. By Lemma 4.8, from (26) there exists

$$M'(s) = \begin{bmatrix} \gamma^L(s) & \bar{\gamma}_1^C(s) & \dots & \bar{\gamma}_p^C(s) \\ 0 & \alpha_1(s) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_p(s) \end{bmatrix} \in \mathbb{C}[s]^{(p+1)\times(p+1)},$$

with invariant factors $\beta'_1(s) \mid \dots \beta'_{p+1}(s)$, such that $\deg(\bar{\gamma}_i^C) \leq g_i$, $1 \leq i \leq p$, and $\|M'(s) - M(s)\| < \frac{\epsilon}{2}$. Moreover, by Lemma 4.6, there exists a pencil

$$\bar{a}^C(s) = \left[\begin{array}{ccc|c|ccc} \bar{a}_{1,1}^C(s) & \dots & \bar{a}_{1,g_1}^C(s) & \dots & \bar{a}_{p,1}^C(s) & \dots & \bar{a}_{p,g_p}^C(s) \end{array}\right] \in \mathcal{P}_{1\times\sum_{i=1}^p g_i}(\mathbb{C}),$$

such that $\|\bar{a}^C(s) - a^C(s)\| < \frac{\epsilon}{2}$ and

$$\bar{\gamma}_i^C(s) = \sum_{j=1}^{g_i} s^{j-1}\bar{a}_{i,j}^C(s), \quad 1 \leq i \leq p. \tag{28}$$

Let $\bar{N}(t) = \begin{bmatrix} \mathrm{rev}_{c_1+1}(\gamma^L)(t) & \mathrm{rev}_{\mu_1}(\gamma_1^N)(t) & \dots & \mathrm{rev}_{\mu_q}(\gamma_q^N)(t) \\ 0 & t^{\mu_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t^{\mu_q} \end{bmatrix}$, and let $\delta_1(t) \mid \dots \mid \delta_{q+1}(t)$ be its invariant factors. Let $g_0 = \deg(\gamma^L)$. By Remark 4.4 item 3, $\kappa\delta_1(t)\dots\delta_{q+1}(t) = t^{\mu_1+\dots+\mu_q+c_1+1-g_0}\tilde{\gamma}^L(t) = t^{\eta_1+\dots+\eta_{q+1}}\tilde{\gamma}^L(t)$ for some $\kappa \in \mathbb{C}$. By Lemma 4.7, $w(0, \bar{N}(t)) = w(0, \bar{M}(t))$. As $\mathrm{diag}(I_{n-1-p-q}, \bar{M}(t))$ and $\mathrm{rev}_1(A)(t)$ are equivalent, we have $w(0, \bar{M}(t)) = w(0, \mathrm{rev}_1(A)(t)) = w(\infty, A(s)) = \overline{(\eta_1, \dots, \eta_{q+1})}$, and by Theorem 3.9, $\delta_i(t) \mid t^{\mu_i}$, $1 \leq i \leq q$. Therefore, $\delta_i(t) = t^{\eta_i}$, $1 \leq i \leq q$ and $\kappa\delta_{q+1}(t) = t^{\eta_{q+1}}\tilde{\gamma}^L(t)$.

We define $\delta'_i(t) = t^{\eta'_i}$, $1 \leq i \leq q$ and $\delta'_{q+1}(t) = \frac{1}{\kappa}t^{\eta'_{q+1}}\tilde{\gamma}^L(t)$. Then, from (27),

$$\delta'_i(t) \mid t^{\mu_i} \mid \delta'_{i+1}(t), \quad 1 \leq i \leq q, \qquad (\delta'_1(t), \dots, \delta'_{q+1}(t)) \prec (\delta_1(t), \dots, \delta_{q+1}(t)). \tag{29}$$

By Lemma 4.8, from (29), there exists

$$\bar{N}'(t) = \begin{bmatrix} \mathrm{rev}_{c_1+1}(\gamma^L)(t) & \gamma'_1(t) & \dots & \gamma'_q(t) \\ 0 & t^{\mu_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t^{\mu_q} \end{bmatrix} \in \mathbb{C}[t]^{(q+1)\times(q+1)},$$

with invariant factors $\delta'_1(t) \mid \dots \delta'_{q+1}(s)$, such that $\deg({\gamma'}_i) \leq \mu_i$, $1 \leq i \leq q$, and $\|\bar{N}(t) - \bar{N}'(t)\| < \frac{\epsilon}{2}$. Observe that $w(0, \bar{N}'(t)) = \overline{(\eta'_1, \dots, \eta'_{q+1})}$.

Let $\bar{\gamma}_1^N(s), \dots, \bar{\gamma}_q^N(s) \in \mathbb{C}[s]$ be polynomials such that $\mathrm{rev}_{\mu_i}(\bar{\gamma}_i^N)(t) = \gamma'_i(t)$. Recall that $\|\bar{\gamma}_i^N(s) - \gamma_i^N(s)\| = \|\mathrm{rev}_{\mu_i}(\bar{\gamma}_i^N)(t) - \mathrm{rev}_{\mu_i}(\gamma_i^N)(t)\|$. By Lemma 4.6, there exists a pencil

$$\bar{a}^N(s) = \left[\begin{array}{ccc|c|ccc} \bar{a}_{1,1}^N(s) & \dots & \bar{a}_{1,\mu_1}^N(s) & \dots & \bar{a}_{q,1}^N(s) & \dots & \bar{a}_{q,\mu_q}^N(s) \end{array}\right] \in \mathcal{P}_{1\times\sum_{i=1}^q \mu_i}(\mathbb{C}),$$

such that $\|\bar{a}^N(s) - a^N(s)\| < \frac{\epsilon}{2}$ and

$$\bar{\gamma}_i^N(s) = \sum_{j=1}^{\mu_i} s^{\mu_i - j} \bar{a}_{i,j}^N(s),\ 1 \leq i \leq q. \tag{30}$$

Let

$$\bar{M}'(t) = \begin{bmatrix} \operatorname{rev}_{c_1+1}(\bar{\gamma}^L)(t) & \operatorname{rev}_{g_1}(\bar{\gamma}_1^C)(t) & \dots & \operatorname{rev}_{g_p}(\bar{\gamma}_p^C)(t) & \operatorname{rev}_{\mu_1}(\bar{\gamma}_1^N)(t) & \dots & \operatorname{rev}_{\mu_p}(\bar{\gamma}_q^N)(t) \\ 0 & \tilde{\alpha}_1(t) & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \tilde{\alpha}_p(t) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & t^{\mu_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & t^{\mu_q} \end{bmatrix}.$$

By Lemma 4.7, $w(0, \bar{M}'(t)) = w(0, \bar{N}'(t)) = \overline{(\eta_1', \dots, \eta_{q+1}')}$.

Let $a'(s) = \begin{bmatrix} a^L(s) & \bar{a}^C(s) & \bar{a}^N(s) \end{bmatrix}$ and $A'(s) = \begin{bmatrix} a'(s) \\ A_2(s) \end{bmatrix}$. Then $\|A'(s) - A(s)\| < \epsilon$. By Lemma 4.3, the pencils $A'(s)$ and $\operatorname{rev}_1(A')(t)$ are equivalent to $\operatorname{diag}(I_{n-1-p}, M'(s))$ and $\operatorname{diag}(I_{n-1-p-q}, \bar{M}'(t))$, respectively. Therefore, the invariant factors of $A'(s)$ are

$$\hat{\beta}_i'(s) = 1,\ 1 \leq i \leq n-p-1, \quad \hat{\beta}_{n-p-1+i}'(s) = \beta_i'(s),\ 1 \leq i \leq p+1.$$

Moreover,

$$w(\infty, A'(s)) = w(0, \bar{M}'(t)) = \overline{(\eta_1', \dots, \eta_{q+1}')},$$

therefore, the homogeneous invariant factors of $A'(s)$ are $\psi_1'(s,t) \mid \cdots \mid \psi_n'(s,t)$.

In order to prescribe the Weyr characteristic of $A'(s)$ we will use some auxiliary lemmas. First of all, we state Lemma 4.5 in terms of the partial multiplicities of the eigenvalues of $M(s)$.

Lemma 4.10 *Let $M(s) = \begin{bmatrix} \gamma(s) \\ M_2(s) \end{bmatrix} \in \mathbb{C}[s]^{(1+p)\times(1+p)}$ be the matrix in Lemma 4.5, let $\lambda \in \mathbb{C}$ and write*

$$\gamma_i(s) = (s-\lambda)^{x_i} \hat{\gamma}_i(s), \quad \hat{\gamma}_i(\lambda) \neq 0, \quad x_i \geq 0, \quad 0 \leq i \leq p.$$

Then, $\sum_{i=1}^{p+1} n_i(\lambda, M(s)) = x_0 + \sum_{i=1}^{p} n_i(\lambda, M_2(s))$ and for $2 \leq \ell \leq p+1$, $\sum_{i=\ell}^{p+1} n_i(\lambda, M(s))$ is the minimun of the integers in the following list.

1. $\sum_{i=\ell-1}^{p} n_i(\lambda, M_2(s))$,
2. $x_0 + \sum_{i=\ell}^{p} n_i(\lambda, M_2(s))$,
3. $x_j + \sum_{i=p-j+2}^{p} n_i(\lambda, M_2(s)) + \sum_{i=\ell-1}^{p-j} n_i(\lambda, M_2(s)), \quad 1 \leq j \leq p-\ell+2$,
4. $x_j + \sum_{i=\ell}^{p} n_i(\lambda, M_2(s)), \quad p-\ell+3 \leq j \leq p$,

where we take $\sum_{i=a}^{b} n_i(\lambda, M_2(s)) = 0$ for $a > b$.

Observe that Theorem 3.9 implies, for $\lambda \in \mathbb{C}$,

$$n_{i+1}(\lambda, M(s)) \leq n_i(\lambda, M_2(s)) \leq n_i(\lambda, M(s)), \quad 1 \leq i \leq p,$$

hence $x_0 \geq n_1(\lambda, M(s)) - n_1(\lambda, M_2(s))$.

Lemma 4.11 *Under the notation of Lemmas 4.5 and 4.10, let z be an integer, $0 \leq z \leq n_1(\lambda, M(s)) - n_1(\lambda, M_2(s))$, $\gamma_0'(s) = (s-\lambda)^{x_0 - z}\hat{\gamma}_0'(s)$, with $\hat{\gamma}_0'(\lambda) \neq 0$, and $M'(s) = \begin{bmatrix} \gamma'(s) \\ M_2(s) \end{bmatrix} = \begin{bmatrix} \gamma_0'(s) & \gamma_1(s) & \dots & \gamma_p(s) \\ 0 & \sigma_1(s) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_p(s) \end{bmatrix}$. Then*

$$n_1(\lambda, M'(s)) = n_1(\lambda, M(s)) - z \text{ and } n_k(\lambda, M'(s)) = n_k(\lambda, M(s)),\ 2 \leq k \leq p+1. \tag{31}$$

Proof. Notice that for $M'(s)$, the values of the expressions in items 1, 3 and 4 in Lemma 4.10 coincide with those of $M(s)$, whereas the expression in item 2 turns into $x_0 - z + \sum_{i=\ell}^{p} n_i(\lambda, M_2(s))$ for $M'(s)$.

From Lemma 4.10, we obtain that

$$\sum_{i=1}^{p+1} n_i(\lambda, M'(s)) = x_0 - z + \sum_{i=1}^{p} n_i(\lambda, M_2(s)) = \sum_{i=1}^{p+1} n_i(\lambda, M(s)) - z,$$

and

$$\sum_{i=k}^{p+1} n_i(\lambda, M'(s)) \leq \sum_{i=k}^{p+1} n_i(\lambda, M(s)),\ 2 \leq k \leq p+1.$$

Assume that for some $k \in \{2, \dots, p+1\}$, $\sum_{i=k}^{p+1} n_i(\lambda, M'(s)) < \sum_{i=k}^{p+1} n_i(\lambda, M(s))$. Then

$$\sum_{i=k}^{p+1} n_i(\lambda, M'(s)) = x_0 - z + \sum_{i=k}^{p} n_i(\lambda, M_2(s))$$

$$= \sum_{i=1}^{p+1} n_i(\lambda, M(s)) - \sum_{i=1}^{p} n_i(\lambda, M_2(s)) - z + \sum_{i=k}^{p} n_i(\lambda, M_2(s))$$

$$= \sum_{i=1}^{k-1} (n_i(\lambda, M(s)) - n_i(\lambda, M_2(s)) - z + \sum_{i=k}^{p+1} n_i(\lambda, M(s)).$$

Bearing in mind that $n_i(\lambda, M(s)) \geq n_i(\lambda, M_2(s))$, $1 \leq i \leq p$ and $n_1(\lambda, M(s)) - n_1(\lambda, M_2(s)) \geq z$, we obtain that $\sum_{i=k}^{p+1} n_i(\lambda, M'(s)) \geq \sum_{i=k}^{p+1} n_i(\lambda, M(s))$, which is a contradiction. Therefore, for $2 \leq k \leq p+1$,

$$\sum_{i=k}^{p+1} n_i(\lambda, M'(s)) = \sum_{i=k}^{p+1} n_i(\lambda, M(s)),$$

hence (31) is satisfied.

The proof of the following lemma can be found in [25, Lemma 3.2].

Lemma 4.12 *Let* $(a_1, \dots)$ *and* $(b_1, \dots)$ *be partitions of nonnegative integers. Let* $\mathbf{p} = (p_1, \dots) = \overline{(a_1, \dots)}$ *and* $\mathbf{q} = (q_1, \dots) = \overline{(b_1, \dots)}$ *be the conjugate partitions. Let* $k \geq 0$ *be an integer. Then,* $a_i \geq b_{i+k}$*,* $i \geq 1$*, if and only if* $p_i \geq q_i - k$*,* $i \geq 1$*.*

In the next theorem, given a matrix pencil $A(s)$, we provide conditions that some prescribed partitions must satisfy in order to be the Weyr characteristic of a pencil obtained from $A(s)$ by a small perturbation of one row. Recall that the pencil $A(s)$ is split as $A(s) = \begin{bmatrix} a(s) \\ A_2(s) \end{bmatrix} \in \mathcal{P}_{(1+(n-1))\times n}(\mathbb{C})$.

Theorem 4.13 *Let* $\mathcal{V}_\eta(A(s))$ *be an* η*-neighbourhood of the spectrum of* $A(s)$*. For each* $\lambda \in \Lambda(A(s))$ *let* t_λ *be a given integer* $t_\lambda \geq 1$ *and let* $\mathbf{w}^{(\lambda,j)} = (w_1^{(\lambda,j)}, \dots)$ *be given partitions,* $1 \leq j \leq t_\lambda$*, such that* $\mathbf{w}^{(\lambda,j)} \neq (0)$*,* $2 \leq j \leq t_\lambda$*.*

For every $\epsilon > 0$*, there exists a pencil* $A'(s) = \begin{bmatrix} a'(s) \\ A_2(s) \end{bmatrix} \in \mathcal{P}_{(1+(n-1))\times n}(\mathbb{C})$ *such that* $\|A'(s) - A(s)\| < \epsilon$*, the spectrum of* $A'(s)$ *satisfies condition (3), and*

$$\mathbf{w}(\lambda, A'(s)) = \mathbf{w}^{(\lambda,1)},$$

$A'(s)$ has $t_\lambda - 1$ eigenvalues $\mu_{\lambda,2}, \dots, \mu_{\lambda,t_\lambda}$ in $B(\lambda, \eta)$, different from λ, with

$$\mathbf{w}(\mu_{\lambda,j}, A'(s)) = \mathbf{w}^{(\lambda,j)}, \ 2 \leq j \leq t_\lambda, \tag{32}$$

if and only if condition (6) and

$$\left.\begin{array}{c} 0 \leq w_i^{(\lambda,1)} - w_i(\lambda, A_2(s)) \leq 1, \quad i \geq 1, \\ 0 \leq w_i^{(\lambda,j)} \leq 1, \quad i \geq 1, \quad 2 \leq j \leq t_\lambda, \end{array}\right\} \tag{33}$$

are satisfied.

Proof. The proof is inspired by that of [5, Theorem 3.8]. Assume that for every $\epsilon > 0$, there exists a pencil $A'(s) = \begin{bmatrix} a'(s) \\ A_2(s) \end{bmatrix} \in \mathcal{P}_{(1+(n-1))\times n}(\mathbb{C})$ satisfying (3) and (32) and such that $\|A'(s) - A(s)\| < \epsilon$. Then, from Theorem 3.5, condition (6) holds, and from Theorem 4.9 we obtain (17), which is equivalent to

$$n_{i+1}(\mu, A'(s)) \leq n_i(\mu, A_2(s)) \leq n_i(\mu, A'(s)), \quad i \geq 1, \quad \text{for each } \mu \in \overline{\mathbb{C}}. \tag{34}$$

For $\lambda \in \Lambda(A(s))$ and $2 \leq j \leq t_\lambda$, $\mu_{\lambda,j} \notin \Lambda(A(s))$, hence $\mathbf{n}(\mu_{\lambda,j}, A(s)) = \mathbf{n}(\mu_{\lambda,j}, A_2(s)) = (0)$. Then, by Lemma 4.12, (34) implies (33).

Conversely, assume that (6) and (33) hold, and let $\epsilon > 0$. Recall that $A_2(s)$ is in the Kronecker canonical form given in (18). For each $\lambda \in \Lambda(A(s))$ and $1 \leq j \leq t_\lambda$, let $\mathbf{n}^{(\lambda,j)} = \overline{\mathbf{w}^{(\lambda,j)}} = (n_1^{(\lambda,j)}, \dots)$. Then, from (33) and Lemma 4.12 we obtain

$$n_{i+1}^{(\lambda,1)} \leq n_i(\lambda, A_2(s)) \leq n_i^{(\lambda,1)}, \quad i \geq 1, \tag{35}$$

and

$$n_2^{(\lambda,j)} = 0, \quad 2 \leq j \leq t_\lambda. \tag{36}$$

For $\lambda \in \Lambda(A(s))$ we denote $\mathbf{p}^\lambda = \sum_{j=1}^{t_\lambda} \mathbf{n}^{(\lambda,j)} = (p_1^\lambda, \dots)$ and define

$$\xi_{n-i+1}(s,t) = t^{p_i^\infty} \prod_{\lambda \in \Lambda(A(s)) \setminus \{\infty\}} (s - \lambda t)^{p_i^\lambda}, \quad 1 \le i \le n, \tag{37}$$

where, if $\infty \notin \Lambda(A(s)$, we take $\mathbf{p}^\infty = (0)$. Then, (6) and (35) are equivalent to

$$(\xi_1(s,t), \dots, \xi_n(s,t)) \prec (\psi_1(s,t), \dots, \psi_n(s,t))$$

and

$$\xi_1(s,t) \mid \phi_i(s,t) \mid \xi_{i+1}(s,t), \; 1 \le i \le n-1,$$

respectively. By Theorem 4.9, there exists a pencil $B(s) = \begin{bmatrix} b(s) \\ A_2(s) \end{bmatrix} \in \mathcal{P}_{(1+(n-1))\times n}(\mathbb{C})$ with $\xi_1(s,t) \mid \cdots \mid \xi_n(s,t)$ as homogeneous invariant factors and such that $\|B(s) - A(s)\| < \frac{\epsilon}{2}$. Observe that $\mathbf{n}(\lambda, B(s)) = \mathbf{p}^\lambda$ for each $\lambda \in \Lambda(A(s))$.

We partition $b(s)$ according to the blocks of $A_2(s)$,

$$b(s) = \begin{bmatrix} b^L(s) & b^C(s) & b^N(s) \end{bmatrix} =$$

$$\Big[\; b_1^L(s) \;\; \dots \;\; b_{c_1+1}^L(s) \;\Big\|\; b_{1,1}^C(s) \;\; \dots \;\; b_{1,g_1}^C(s) \;\Big|\; \dots \;\Big|\; b_{p,1}^C(s) \;\; \dots \;\; b_{p,g_p}^C(s) \;\Big\|$$

$$\Big\|\; b_{1,1}^N(s) \;\; \dots \;\; b_{1,\mu_1}^N(s) \;\Big|\; \dots \;\Big|\; b_{q,1}^N(s) \;\; \dots \;\; b_{q,\mu_q}^N(s) \;\Big].$$

By Lemma 4.3, the pencils $B(s)$ and $\mathrm{rev}_1(B)(t)$ are equivalent to $\mathrm{diag}(I_{n-1-p}, N(s))$ and $\mathrm{diag}(I_{n-1-p-q}, \bar{N}(t))$, respectively, where

$$N(s) = \begin{bmatrix} \theta(s) \\ M_2(s) \end{bmatrix} = \begin{bmatrix} \theta^L(s) & \theta_1^C(s) & \dots & \theta_p^C(s) \\ 0 & \alpha_1(s) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_p(s) \end{bmatrix},$$

$$\bar{N}(t) = \begin{bmatrix} \mathrm{rev}_{c_1+1}(\theta^L)(t) & \mathrm{rev}_{g_1}(\theta_1^C)(t) & \dots & \mathrm{rev}_{g_p}(\theta_p^C)(t) & \mathrm{rev}_{\mu_1}(\theta_1^N)(t) & \dots & \mathrm{rev}_{\mu_p}(\theta_q^N)(t) \\ 0 & \tilde{\alpha}_1(t) & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \tilde{\alpha}_p(t) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & t^{\mu_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & t^{\mu_q} \end{bmatrix},$$

with

$$\theta^L(s) = \sum_{j=1}^{c_1+1} s^{j-1} b_j^L(s), \; \theta_i^C(s) = \sum_{j=1}^{g_i} s^{j-1} b_{i,j}^C(s), \; 1 \le i \le p, \; \theta_i^N(s) = \sum_{j=1}^{\mu_i} s^{\mu_i - j} b_{i,j}^N(s), \; 1 \le i \le q.$$

Then, $N(s)$ has the same nontrivial invariant factors as $B(s)$, hence $\alpha_1(s) \dots \alpha_p(s) \theta^L(s) = \kappa \psi_1(s,1) \dots \psi_n(s,1) = \kappa \xi_1(s,1) \dots \xi_n(s,1)$ for some $\kappa \in \mathbb{C}$. Therefore

$$\theta^L(s) = \kappa \prod_{\lambda \in \Lambda(A(s)) \setminus \{\infty\}} (s - \lambda)^{x_{0,\lambda}},$$

where, for $\lambda \in \Lambda(A(s)$, $x_{0,\lambda} = \sum_{i=1}^{n} n_i(\lambda, B(s)) - \sum_{i=1}^{n-1} n_i(\lambda, A_2(s)) = \sum_{i=1}^{n} n_i(\lambda, A(s)) - \sum_{i=1}^{n-1} n_i(\lambda, A_2(s))$. Observe that

$$\deg(\theta^L) = \deg(\psi_1(s,1)\dots\psi_n(s,1)) - \deg(\phi_1(s,1)\dots\phi_{n-1}(s,1)) =$$

$$c_1 + 1 + \sum_{i=1}^{n-1} n_i(\infty, A_2(s)) - \sum_{i=1}^{n} n_i(\infty, A(s)) = c_1 + 1 - x_{0,\infty},$$

thus

$$\mathrm{rev}_{c_1+1}(\theta^L)(t) = t^{c_1+1-\deg(\theta^L)}\tilde{\theta}^L(t) = t^{x_{0,\infty}}\tilde{\theta}^L(t).$$

For each $\lambda \in \Lambda(A(s))$ and $2 \le j \le t_\lambda$, let $\mu_{\lambda,j} \in \mathbb{C}$ with $\mu_{\lambda,j} \neq \lambda$, and $z_\lambda = \sum_{j=2}^{t_\lambda} n_1^{(\lambda,j)}$. We have

$$n_1(\lambda, N(s)) - n_1(\lambda, M_2(s)) = n_1(\lambda, B(s)) - n_1(\lambda, A_2(s)) = p_1^\lambda - n_1(\lambda, A_2(s))$$

$$= n_1^{(\lambda,1)} - n_1(\lambda, A_2(s)) + z_\lambda.$$

By (35), we obtain $z_\lambda \le n_1(\lambda, N(s)) - n_1(\lambda, M_2(s)) \le x_{0,\lambda}$.

Let

$$\bar{\theta}_1^L(s) = \prod_{\lambda \in \Lambda(A(s)) \setminus \{\infty\}} \left((s-\lambda)^{x_{0,\lambda} - z_\lambda} \prod_{j=2}^{t_\lambda} (s - \mu_{\lambda,j})^{n_1^{(\lambda,j)}} \right), \quad \bar{\theta}^L(s) = \prod_{j=2}^{t_\infty} \left(1 - (\frac{1}{\mu_{\infty,j}})s \right)^{n_1^{(\infty,j)}} \bar{\theta}_1^L(s), \quad (38)$$

and

$$N'(s) = \begin{bmatrix} \theta'(s) \\ M_2(s) \end{bmatrix} = \begin{bmatrix} \bar{\theta}^L(s) & \theta_1^C(s) & \dots & \theta_p^C(s) \\ 0 & \alpha_1(s) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_p(s) \end{bmatrix},$$

$$\bar{N}'(t) = \begin{bmatrix} \mathrm{rev}_{c_1+1}(\bar{\theta}^L)(t) & \mathrm{rev}_{g_1}(\theta_1^C)(t) & \dots & \mathrm{rev}_{g_p}(\theta_p^C)(t) & \mathrm{rev}_{\mu_1}(\theta_1^N)(t) & \dots & \mathrm{rev}_{\mu_p}(\theta_q^N)(t) \\ 0 & \tilde{\alpha}_1(t) & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \tilde{\alpha}_p(t) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & t^{\mu_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & t^{\mu_q} \end{bmatrix}.$$

Observe that for each $\lambda \in \Lambda(A(s))$ and $2 \le j \le t_\lambda$,

$$n_1(\mu_{\lambda,j}, N'(s)) = n_1^{(\lambda,j)}, \quad n_k(\mu_{\lambda,j}, N'(s)) = 0 = n_k^{(\lambda,j)}, \quad 2 \le k \le p+1,$$

i.e., $\mathbf{w}(\mu_{\lambda,j}, N'(s)) = \mathbf{w}^{(\lambda,j)}$, $2 \le j \le t_\lambda$. Moreover, $\deg(\bar{\theta}^L) = \deg(\theta^L) + z_\infty$, hence

$$\mathrm{rev}_{c_1+1}(\bar{\theta}^L)(t) = t^{c_1+1-\deg(\bar{\theta}^L)}\tilde{\bar{\theta}}^L(t) = t^{x_{0,\infty} - z_\infty}\tilde{\bar{\theta}}^L(t). \quad (39)$$

As $\deg(\bar{\theta}^L) = c_1 + 1 - (x_{0,\infty} - z_\infty) \le c_1 + 1$, by Lemma 4.11, from (38) and (39), for $\lambda \in \Lambda(A(s))$

$$n_1(\lambda, N'(s)) = n_1(\lambda, N(s)) - z_\lambda = n_1^{(\lambda,1)}, \text{ and}$$

$$n_k(\lambda, N'(s)) = n_k(\lambda, N(s)) = n_k^{(\lambda,1)}, \ 2 \le k \le p+1,$$

i.e., $\mathbf{w}(\lambda, N'(s)) = \mathbf{w}^{(\lambda,1)}$. By Lemma 4.6, there exists a pencil $\bar{b}^L(s) \in \mathcal{P}_{1\times(c_1+1)}$ such that $\|\bar{b}^L(s) - b^L(s)\| = \|\bar{\theta}^L(s) - \theta^L(s)\|$ and if $B'(s) = \begin{bmatrix} b'(s) \\ A_2(s) \end{bmatrix} \in \mathcal{P}_{(1+(n-1))\times n}(\mathbb{C})$ with $b'(s) = \begin{bmatrix} \bar{b}^L(s) & b^C(s) & b^N(s) \end{bmatrix}$, then $B'(s)$ and $\text{rev}_1(B')(t))$ are equivalent to $\text{diag}(I_{n-1-p}, N'(s))$ and $\text{diag}(I_{n-1-p-q}, N'(t))$, respectively. Hence, for each $\lambda \in \Lambda(A(s))$, $\mathbf{w}(\lambda, B'(s)) = \mathbf{w}(\lambda, N'(s)) = \mathbf{w}^{(\lambda,1)}$ and $\mathbf{w}(\mu_{\lambda,j}, B'(s)) = \mathbf{w}(\mu_{\lambda,j}, N'(s)) = \mathbf{w}^{(\lambda,j)}$, $2 \le j \le t_\lambda$.

By Lemma 3.2, for $\lambda \in \Lambda(A(s) \setminus \{\infty\}$, the values of $\mu_{\lambda,j}$ can be chosen in $B(\lambda, \eta)$, $2 \le j \le t_\lambda$, in such a way that $\|\bar{\theta}_1^L(s) - \theta^L(s)\| < \frac{\epsilon}{4}$, and it is easy to see that we can choose $\mu_{\infty,j} \in B(\infty, \eta)$ such that $\|\bar{\theta}^L(s) - \bar{\theta}_1^L(s)\| < \frac{\epsilon}{4}$, $2 \le j \le t_\infty$. Therefore, $\|B'(s) - B(s)\| = \|\bar{b}^L(s) - b^L(s)\| = \|\bar{\theta}^L(s) - \theta^L(s)\| < \frac{\epsilon}{2}$, and as a consequence we obtain $\|B'(s) - A(s)\| < \epsilon$.

5 Conclusions and future research

The effect of small perturbations of a regular pencil when some of its rows remain unchanged is investigated. It is a twofold problem. On one hand, it involves characteristics of general small perturbation problems. On the other hand, it is closely related to matrix pencil completion problems. We have obtained necessary conditions to be satisfied by the Weierstrass invariants of a pencil which is a one-row small perturbation of another regular pencil (see Theorems 4.1 and 4.2).

Moreover, when perturbing a single row, we also prove the sufficiency of the necessary conditions obtained (see Theorems 4.9 and 4.13). Our results generalize to pencils previous results on the problem obtained for square matrices. To achieve them, we had to tackle the difficulties appearing due to the presence of infinite elementary divisors in the pencils.

Our next step is to extend the sufficiency part of this work to regular pencils when more than one row is perturbed. The research can also be extended to singular pencils.

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References

[1] A.S. Markus and E.È. Parilis. The change of the jordan structure of a matrix under small perturbations. *Linear Algebra and its Applications 51* (1983), 139–152.

[2] Baragaña, I., Dodig, M., Roca, A., and Stošić, M. Bounded rank perturbations of regular pencils over arbitrary fields. *Linear Algebra Appl 601* (2020), 180–188.

[3] Baragaña, I., and Roca, A. Rank-one perturbations of matrix pencils. *Linear Algebra Appl 606* (2020), 170–191.

[4] Bathia, R. *Matrix Analysis.* Springer, New York, 1997.

[5] Beitia, M., de Hoyos, I., and Zaballa, I. The change of the Jordan structure under one row perturbations. *Linear Algebra and its Applications 401* (2005), 119–134.

[6] Beitia, M., de Hoyos, I., and Zaballa, I. The change of similarity invariants under row perturbations. *Linear Algebra and its Applications 409* (2008), 1302–1333.

[7] Beitia, M., de Hoyos, I., and Zaballa, I. The change of similarity invariants under row perturbations: Generic cases. *Linear Algebra and its Applications 409* (2008), 482–496.

[8] de Hoyos, I. *Perturbación de matrices rectangulares y haces de matrices.* PhD thesis, Universidad del País Vasco, Euskal Heriko Unibertsitatea, Bilbao, 1990.

[9] De Terán, F., and Dopico, F. Low rank perturbation of Kronecker structures without full rank. *SIAM Journal on Matrix Analysis and Applications 29*, 2 (2007), 496–529.

[10] De Terán, F., Dopico, F., and Moro, J. Low rank perturbation of Weierstrass structure. *SIAM Journal on Matrix Analysis and Applications 30*, 2 (2008), 538–547.

[11] den Boer, H., and Thijsse, G. Semi-stability of sums of partial multiplicities under additive perturbation. *Integral Equations and Operator Theory 3*, 1 (1980), 23–42.

[12] Dodig, M. Feedback invariants of matrices with prescribed rows. *Linear Algebra and its Applications 405* (2005), 121–154.

[13] Dodig, M. Matrix pencils completion problems. *Linear Algebra and its Applications 428* (2008), 259–304.

[14] Dodig, M. Explicit solution of the row completion problem for matrix pencils. *Linear Algebra and its Applications 432* (2010), 1299–1309.

[15] Dodig, M. Completion up to a matrix pencil with column minimal indices as the only nontrivial Kronecker invariants. *Linear Algebra and its Applications 438* (2013), 3155–3173.

[16] Dodig, M., and Stošić, M. The change of feedback invariants under one row perturbation. *Linear Algebra and its Applications 422* (2007), 582–603.

[17] Dodig, M., and Stošić, M. On convexity of polynomial paths and generalized majorizations. *Electronic Journal of Combinatorics 17*, 1 (2010), 61.

[18] Dodig, M., and Stošić, M. Rank one perturbations of matrix pencils. *SIAM Journal on Matrix Analysis and Applications 41*, 4 (2020), 1889–1911.

[19] Friedland, S. *Matrices: algebra, analysis and applications.* World Scientific, Singapore, 2016.

[20] Gantmacher, F. *Matrix Theory, Vols I, II.* Chelsea, New York, 1974.

[21] Gernandt, H., and Trunk, C. Eigenvalue placement for regular matrix pencils with rank one perturbations. *SIAM Journal on Matrix Analysis and Applications 38*, 1 (2017), 134–154.

[22] Gracia, J., and de Hoyos, I. Safety neighbourhoods for the kronecker canonical form. preprint, `http://www.vc.ehu.es/campus/centros/farmacia/deptos-f/depme/gracia1.htm`, 1998.

[23] Gracia, J., de Hoyos, I., and Zaballa, I. Perturbation of linear control systems. *Linear Algebra and its Applications 121* (1989), 353–383.

[24] Kalogeropoulos, G., Psarrakos, P., and Karcanias, N. On the computation of the Jordan canonical form of regular matrix polynomials. *Linear Algebra and its Applications 385* (2004), 117–130.

[25] Lippert, R., and Strang, G. The Jordan forms of AB and BA. *Electronic Journal of the International Linear Algebra Society 18* (2009), 281–288.

[26] Marques de Sà, E. Imbedding conditions for λ-matrices. *Linear Algebra and its Applications 24* (1979), 33–50.

[27] Marshall, W., and Olkin, I. *Inequalities: Theory of majorization and its applications.* Academic, London, 1979.

[28] Mehl, C., Mehrmann, V., Ran, A., and Rodman, L. Eigenvalue perturbation theory of classes of structured matrices under generic structured rank one perturbations. *Linear Algebra and its Applications 435* (2011), 687–716.

[29] Roca, A. *Asignación de Invariantes en Sistemas de Control.* PhD thesis, Universitat Politècnica València, 2003.

[30] Silva, F. The Rank of the Difference of Matrices with Prescribed Similarity Classes. *Linear and Multilinear Algebra 24* (1988), 51–58.

[31] Thompson, R. Interlacing inequalities for invariant factors. *Linear Algebra and its Applications 24* (1979), 1–31.

[32] Zaballa, I. Matrices with prescribed rows and invariant factors. *Linear Algebra and its Applications 87* (1987), 113–146.

[33] Zaballa, I. Pole assignment and additive perturbations of fixed rank. *SIAM Journal on Matrix Analysis and Applications 12*, 1 (1991), 16–23.