Applied Mathematics and Computation 412 (2022) 126549

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Combined matrices and conditioning

Rafael Bru[a], Maria T. Gassó[a,*], Máximo Santana[b]

[a] Institut de Matemàtica Multidisciplinar, Universitat Politècnica de València, Spain
[b] Universidad Autonóma de Santo Domingo, Dominican Republic

ARTICLE INFO

Article history:
Received 3 March 2021
Revised 3 July 2021
Accepted 23 July 2021
Available online 12 August 2021

MSC:
15A12
65F35

Keywords:
Combined matrix
Reduced combined matrix
Condition number

ABSTRACT

In this work, we study a lower bound of the condition number of a matrix by its combined matrix. In particular, we construct a special combined matrix in such a way that the sums of its columns are lower bounds of the condition number of the matrix. Cases for special matrices as unitary matrices are considered.

© 2021 The Authors. Published by Elsevier Inc.
This is an open access article under the CC BY-NC-ND license
(http://creativecommons.org/licenses/by-nc-nd/4.0/)

1. Introduction

We are going to work with a lower bound of the condition number of a square matrix M of size $n \times n$. In particular, we give the relationship between a special combined matrix of M and its condition number. The combined matrix of a nonsingular matrix M is defined as $\mathcal{C}(M) = M \circ M^{-T}$ where $\circ$ is the Hadamard product of matrices. It seems that the name of the combined matrix appears for the first time in the work by Fiedler [3]. However, those matrices are known as the relative gain array in Mathematical Control Theory as it is pointed out by Johnson and Shapiro [9], where some mathematical problems of those matrices are studied. Properties of combined matrices can be seen in Horn and Johnson [8] and some recent works are given in Alonso and Serrano [1] and Bru et al. [2].

It is worth saying that even though combined matrices can be defined for complex matrices, their applications are usually studied when they are real, for instance, to generate doubly stochastic matrices [6]. In this work we consider complex matrices since the concept of a combined matrix can be done in the complex field.

Wilkinson [13] studied the perturbation theory in problems involved in computing eigenvalues and eigenvectors. In Chapter 2 of [13], the spectral condition number of a matrix with respect to its eigenvalue problem is bounded below by the n condition numbers, namely $|s_i^{-1}|$, defined in Eq. (3) and considered in Definition 3. This result can be used for bounding below the condition number for inversion of an invertible matrix. The bounds called the n condition numbers of a matrix M with respect to its eigenvalue problem are the same bounds of the condition number when we consider the inversion problem of the matrix that diagonalize the original matrix. In the literature, we can find some bounds of the condition number of a matrix, see [12]. In that work, the authors give a lower bound and an upper bound of the condition number of M in

* Corresponding author.
E-mail addresses: rbru@mat.upv.es (R. Bru), mgasso@mat.upv.es (M.T. Gassó), msantana22@uasd.edu.do (M. Santana).

https://doi.org/10.1016/j.amc.2021.126549
0096-3003/© 2021 The Authors. Published by Elsevier Inc. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/)

terms of the trace of the matrices $M^\star M$, $(M^\star)^{-1}M^{-1}$, $M^\star MM^\star M$ and $(M^\star)^{-1}M^{-1}(M^\star)^{-1}M^{-1}$, where $M^\star$ denotes the complex conjugate transpose of M. An upper bound of the condition number in terms of the spectral norm is given in Piazza and Politi [10] and Rojo [11], and in terms of the Frobenius norm in Guggenheimer et al. [7].

In this work we study the above mentioned lower bounds $|s_i^{-1}|$ of the condition number for inversion of a square matrix M. In Section 2, we give the definitions and properties of combined matrices and the condition number for inversion. Then we construct a lower bound $s^{-1} = \max_i |s_i^{-1}|$ of that condition number by a special combined matrix of M (Theorem 4). In Section 3, some properties of the lower bound are given, concretely, for unitary matrices. Matrices for which the lower bound is tight are also studied. Left multiplication of the matrix M by unitary matrices preserves the lower bound and this property can be applied to the QR and polar factorizations of the invertible matrix M. Finally, Section 4 presents the conclusions.

2. Combined matrices and conditioning

First, we start with the two basic definitions, the condition number of a matrix and the combined matrix.

Definition 1. Let M be an invertible matrix of size $n \times n$. The condition number of M is

$$\kappa(M) = ||M||||M^{-1}||,$$

where $||\cdot||$ denotes a matrix norm.

This condition number refers to the solution of linear systems $Mx = c$, where M is the coefficient matrix. It measures the sensibility of the solution x with respect to some perturbations of the entries of M or c. Also, it refers to the inversion problem of M. Usually, the condition number $\kappa(M)$ is given with the spectral norm of M, that is,

$$\kappa(M) = ||M||_2||M^{-1}||_2,$$

where $||M||_2 = \sigma_1$. Here, σ_1 denotes the maximum singular value of M. Recall that if M is invertible, then $1 \le \kappa(M) < \infty$. If M is not invertible, we have $\kappa(M) = \infty$.

As we said in the Introduction we are going to bound $\kappa(M)$ by the combined matrix of M. Let us remind its definition.

Definition 2. Let M be an invertible matrix. The matrix

$$\mathcal{C}(M) = M \circ M^{-T}$$

is called the **combined matrix** of M, where $\circ$ is the Hadamard product and M^{-T} stands for the inverse transpose of M.

Remark 1. Some properties of combined matrices that we are going to use are:

(i) Let D be an invertible diagonal matrix. Then $MD \circ M^{-T} = M \circ M^{-T}D$ and $DM \circ M^{-T} = M \circ DM^{-T}$ (see [8]).
(ii) $\mathcal{C}(MD) = \mathcal{C}(DM) = \mathcal{C}(M)$ for any nonsingular diagonal matrix D.
(iii) $\mathcal{C}(M^T) = \mathcal{C}(M^{-1})$.
(iv) the sum of the entries of any column and any row of a combined matrix is 1. Note that if the combined matrix is nonnegative then it is double stochastic (see [5]).

Throw the paper we use the following factorizations.

Let $M = [m_{ij}]$ be an $n \times n$ nonsingular matrix and denote the ith column by m_i. Consider the factorization

$$M = AF, \tag{1}$$

where the matrix $A = [a_1, a_2, \dots, a_n]$ with $a_i = \dfrac{m_i}{||m_i||_2}$ and the diagonal matrix $F = diag(||m_1||_2, ||m_2||_2, \dots, ||m_n||_2)$.

Let $P = [p_{ij}]$ be the inverse of M and denote by p^i the ith row vector of P. Consider the factorization

$$P = M^{-1} = GB. \tag{2}$$

Denoting the rows of a matrix with super indices, we have

$$B = \begin{bmatrix} b^1 \\ b^2 \\ \vdots \\ b^n \end{bmatrix},$$

with $b^i = \dfrac{p^i}{||p^i||_2}$ and the diagonal matrix $G = diag\big(||p^1||_2, ||p^2||_2, \dots, ||p^n||_2\big)$.

Since $PM = I$, we have that $p^j m_i = 0$, if $j \neq i$, that is, the column vectors of M are orthogonal to the row vectors of P and so, $b^j a_i = 0$, for all $j \neq i$. Then for a given i, $i = 1, 2, \dots, n$, the vector b^i is orthogonal to the hyperplane spanned by the vectors $\{a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_n\}$.

The coefficient

$$s_i = b^i a_i, \tag{3}$$

$i = 1, 2, \dots, n$, can be seen as the cosine of the angle between these two vectors and it is the coefficient of the projection of a_i onto b^i when both vectors are in a real vector space. This cosine is the sine of the angle between the column vector a_i and the above hyperplane because both angles are complementary. The coefficient (3) will be called *projection* throughout this work.

Remark 2. If s_i is one, the vector a_i is orthogonal to the hyperplane spanned by the vectors $\{a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_n\}$. However, if the projection s_i is small, then the vector a_i is close enough to the hyperplane indicating that the ith column of A is almost dependent on the other columns of the matrix A and then M is almost singular.

The absolute value of the inverse of the projections s_i, $i = 1, 2, \dots, n$, are called the n condition numbers with respect to its eigenvalue problem in Wilkinson [13]. We have the following definition.

Definition 3. Consider the factorizations $M = AF$ and $P = GB$ of M and its inverse P, given in Eqs. (1) and (2), respectively. The **minimal projection** of M is the constant

$$s = \min_{1 \le i \le n} \{|s_i|\},$$

where s_i is given by the Eq. (3). The minimal projection of the matrix M will be denoted by $s(M)$.

The following result shows how a small value of s gives rise to an ill–conditioned matrix. The result is in (Wilkinson [13], Chapter 2, Section 31, equation 31.7) where the proof is given for left and right eigenvectors.

Theorem 1. *Let M be an invertible square matrix. Then*

$$1 \le \frac{1}{s(M)} \le \kappa(M).$$

Proof. We use the above notation. We have

$$\kappa(M) = ||M||_2 ||M^{-1}||_2 \ge ||Me_i||_2 ||e^i M^{-1}||_2,$$

where e_i is the ith unit vector, i.e., $e_i = (0, \dots, 1, \dots, 0)$ and so $Me_i = m_i$ and $e^i M^{-1} = p^i$. Then

$$\frac{1}{|s_i|} = \frac{1}{|b^i a_i|} = \frac{||p^i||_2 ||m_i||_2}{|p^i m_i|} = ||p^i||_2 ||m_i||_2 = ||e^i M^{-1}||_2 ||Me_i||_2,$$

since $PM = I$. Hence

$$\frac{1}{|s_i|} \le \kappa(M),$$

for any $i = 1, 2, \dots, n$. Then $\frac{1}{s(M)} \le \kappa(M)$. Further $1 \le \frac{1}{s(M)}$ by the Cauchy–Schwarz inequality. Hence the proof follows. □

Therefore, if the minimal projection is small the condition number bounded below by the inverse of $s(M)$ could be really large. Thus, we can say that $\kappa(M) = \infty$ when the matrix M is singular.

Now, let us see how we can compute the minimal projection of an invertible matrix by its combined matrix (see Fiedler and Markham [4], Horn and Johnson [8]).

Definition 4. Let M be an invertible matrix and let A be the matrix considered in the Eq. (1). The matrix

$$\mathcal{R}(M) = A \circ B^T$$

is said to be the **reduced combined matrix** of M, where B is given in (2).

The reduced combined matrix of M^T and M^{-1} has the same relationship as the combined matrix of both matrices (point (iii) of Remark 1), as we see in the following result.

Theorem 2. *Let M be an invertible matrix. Then*

$$\mathcal{R}(M^T) = \mathcal{R}(M^{-1}),$$

where $\mathcal{R}(M^T)$ and $\mathcal{R}(M^{-1})$ are the reduced combined matrices of M^T and M^{-1}, respectively.

Proof. Suppose that $M^{-1} = ST$ is the factorization as in (1) and $\left(M^{-1}\right)^{-1} = VW$ the factorization as in (2). Then $\mathcal{R}(M^{-1}) = S \circ W^T$ according to the above definition. In addition

$$M^T = \left(M^{-T}\right)^{-1} = (VW)^T = W^T V,$$

and

$$(M^T)^{-1} = (ST)^T = TS^T$$

since T and V are diagonal. Then

$$\mathcal{R}(M^T) = W^T \circ (S^T)^T = S \circ W^T = \mathcal{R}(M^{-1}),$$

which is the same relation that the combined matrix. □

The relationship between the combined matrix and the reduced combined matrix is the following.

Theorem 3. *Let M be an invertible matrix. Let $\mathcal{C}(M)$ and $\mathcal{R}(M)$ be the combined and the reduced combined matrix of M, respectively. Then*

$$\mathcal{C}(M) = \mathcal{R}(M)FG, \tag{4}$$

where F and G are the matrices of the factorizations (1) and (2), respectively.

Proof. From Definition 2 and factorizations (1) and (2) we have

$$\mathcal{C}(M) = M \circ \left(M^{-1}\right)^T = AF \circ (GB)^T = AF \circ B^T G.$$

Then by Remark 1, we have

$$\mathcal{C}(M) = AF \circ B^T G = \left(A \circ B^T\right) GF = \mathcal{R}(M)FG.$$

since matrices F and G are nonsingular and diagonal. □

The column sum property of Remark 1 does not hold when we work with the reduced combined matrix $\mathcal{R}(M) = [r_{ij}]$. We are interested in that sum to obtain the lower bound of the condition number.

Theorem 4. *Let M be an invertible matrix of order n and let A be the matrix considered in Eq. (1). Then the projections s_j are the sum of the entries of the columns of the reduced combined matrix $\mathcal{R}(M) = A \circ B^T$, that is,*

$$s_j = \sum_{i=1}^{n} r_{ij}, \quad 1 \le j \le n.$$

where B is given in the Eq. (2).

Proof. We use the notation of Section 1. Then the projections are

$$s_j = b^j a_j = \sum_{i=1}^{n} b_{ji} a_{ij}.$$

Since the (i, j) entry of $\mathcal{R}(M)$ is $r_{ij} = a_{ij} b_{ji}$, we conclude

$$s_j = \sum_i r_{ij}, \quad 1 \le j \le n,$$

that is, the projections are the sum of the entries of the columns of the reduced combined matrix. □

Thus, we can obtain the minimal projection s of M, $s(M)$, by the reduced combined matrix of M.

Example 1. Consider the invertible matrix

$$M = \begin{bmatrix} 1.3202 & -3.4438 & 4.4723 \\ 2.6406 & -3.4438 & 0 \\ 1.3202 & 0 & 4.4723 \end{bmatrix}$$

where $||m_1||_2 = 3.2341,\ ||m_2||_2 = 4.8703,\ ||m_3||_2 = 6.3249$. We have the factorization $M = AF$, where

$$A = \begin{bmatrix} 0.4082 & -0.7071 & 0.7071 \\ 0.8165 & -0.7071 & 0 \\ 0.4082 & 0 & 0.7071 \end{bmatrix}$$

and $F = diag(3.2341, 4.8703, 6.3249)$.

The inverse of M is

$$P = \begin{bmatrix} -0.3787 & 0.3787 & 0.3787 \\ -0.2904 & 0.0000 & 0.2904 \\ 0.1118 & -0.1118 & 0.1118 \end{bmatrix}.$$

The factorization $P = GB$ gives rise to the matrix

$$B = \begin{bmatrix} -0.5774 & 0.5774 & 0.5774 \\ -0.7071 & 0.0000 & 0.7071 \\ 0.5773 & -0.5773 & 0.5774 \end{bmatrix}$$

where B has normalized rows and

$$G = diag(||p_1||_2 = 0.6559, ||p_2||_2 = 0.4107, ||p_3||_2 = 0.1936).$$

Now, we compute the reduced combined matrix

$$\mathcal{R}(M) = A \circ B^T = \begin{bmatrix} -0.2357 & 0.5000 & 0.4082 \\ 0.4714 & 0.0000 & 0 \\ 0.2357 & 0 & 0.4083 \end{bmatrix}.$$

The column sums of the matrix $\mathcal{R}(M)$ are $s_1 = 0.4714, s_2 = 0.5000, s_3 = 0.8165$. The minimal projection is $s = 0.4714$ and the lower bound of $\kappa(M) = 5.5832$ is $\frac{1}{s(M)} = 2.1213$.

3. Properties of $s(M)$

The first property we can say about the minimal projection is a direct consequence of Theorem 2.

Theorem 5. *Let M be an invertible matrix. Then, the minimal projection of M^T is equal to the minimal projection of M^{-1}, that is, $s(M^T) = s(M^{-1})$.*

Proof. It is a direct consequence of Theorem 2 join with Theorem 4. □

To compute the minimal projection we have to normalize the columns of the matrix M and then the rows of the matrix $P = M^{-1}$. Then we have the following result.

Lemma 1. *Let M be a unitary matrix of order n. Then its combined matrix is equal to its reduced combined matrix, that is, $\mathcal{C}(M) = \mathcal{R}(M)$. Moreover, the projections $s_i = 1$, for all $i = 1, 2, \dots, n$, and then the minimal projection $s(M)$ is 1.*

Proof. Since M and M^{-1} are unitary they have normalized columns, the factorization (1) is $M = MI$, and the factorization (2) of its inverse is $M^{-1} = IM^{-1}$. Moreover, since $M^{-1} = \overline{M}^T$ the combined matrix is

$$\mathcal{C}(M) = M \circ \overline{M}.$$

Note that, in our notation of Eqs. (1) and (2) $M = A$ and $M^T = B$. Then the reduced combined matrix

$$\mathcal{R}(M) = A \circ B^T = M \circ \left(\overline{M}^T\right)^T = M \circ \overline{M} = \mathcal{C}(M).$$

Then the column sums of $\mathcal{R}(M)$ are equal to those of $\mathcal{C}(M)$ which are 1 by Remark 1. Therefore, it is clear that $s_i = 1$, for all $i = 1, 2, \dots, n$, and hence the minimal projection $s(M)$ is one. □

The converse is not true as the following example shows.

Example 2. Suppose the combined and the reduced combined matrix are the identity matrix. Take any nonsingular diagonal matrix, for instance, the matrix

$$M = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Its combined and reduced combined matrix are the identity matrix, that is, $\mathcal{C}(M) = \mathcal{R}(M) = I$. Note that the factorization (1) is $M = IF$, where $F = M$ and the factorization (2) is $M^{-1} = P = GI$, where $G = M^{-1}$. However, the matrix M is not unitary since $||m_1||_2 = 3,\ ||m_2||_2 = 2$ and $||m_3||_2 = 2$.

The previous example gives some insight to establish a result that is like the reciprocal of Lemma 1.

Lemma 2. *Suppose the combined and the reduced combined matrix of a matrix M are equal. Then M has its columns orthogonal.*

Proof. Since $\mathcal{R}(M) = \mathcal{C}(M)$ the column sums of the reduced combined matrix are 1 by Remark 1. Then the projections $s_i = 1$ by Theorem 4. Therefore, each column vector of M is orthogonal to the hyperplane spanned by the remaining column vectors (see Remark 2). Hence, M has orthogonal columns. □

Then, we have the following theorem.

Theorem 6. *Let M be an invertible matrix. Then $\mathcal{R}(M) = \mathcal{C}(M)$ if and only if $M = UD$, for some unitary matrix U and nonsingular diagonal matrix D.*

Proof. Suppose $M = UD$, where U is a unitary matrix and D is a nonsingular diagonal matrix. By Remark 1 we have $\mathcal{C}(M) = \mathcal{C}(UD) = \mathcal{C}(U)$. Then $\mathcal{C}(M) = \mathcal{R}(U)$ by Lemma 1.

It remains to prove that $\mathcal{R}(U) = \mathcal{R}(M)$. Consider $U = UI$ and $U^{-1} = IU^{-1}$ the factorizations (1) and (2), respectively. Then $UD = U(DI)$ and $(UD)^{-1} = (ID^{-1})U^{-1}$ are the factorizations (1) and (2) of $M = UD$, respectively. Then $\mathcal{R}(UD) = U \circ U^{-T} = \mathcal{R}(U)$. Therefore $\mathcal{R}(M) = \mathcal{R}(U)$ and hence $\mathcal{R}(M) = \mathcal{C}(M)$.

Conversely, let $M = [m_{ij}]$ and consider $\mathcal{R}(M) = \mathcal{C}(M)$. Then M has its columns orthogonal by Lemma 2. Let $D = diag(||m_1||_2, ||m_2||_2, \ldots, ||m_n||_2)$. Then it is clear that M can be written as

$$M = UD,$$

where U is unitary.
□

The following result shows that the condition number attains the lower bound s^{-1} when we work with unitary matrices.

Theorem 7. *The lower bound s^{-1} of the condition number is sharp for unitary matrices, that is, the equality is attained when the matrix is unitary.*

Proof. The minimal projection $s = 1$ by Lemma 1. Since unitary matrices have the property that $\kappa(M) = 1$, then it is clear that $s^{-1} = \kappa(M)$ showing that the lower bound s^{-1} is sharp when the matrix is unitary. □

Let us see that the lower bound s^{-1} is invariant under left unitary transformations.

Given the matrix M, we consider the projections s_i defined in (3) and the factorizations (1) and (2). Now let us consider the unitary matrix U and the matrix

$$UM = [Um_1 \ Um_2 \ \ldots \ Um_n].$$

The factorization of UM as in (1) is

$$UM = \tilde{A}\tilde{F},$$

where the diagonal matrix

$$\tilde{F} = diag(||Um_1||, ||Um_2||, \ \ldots, ||Um_n||) = diag(||m_1||, ||m_2||, \ \ldots, ||m_n||),$$

since U is unitary. Then $UM = \tilde{A}F$. Using this equation and Eq. (1) we have

$$\tilde{A} = UA.$$

Moreover,

$$(UM)^{-1} = M^{-1}U^{-1} = \begin{bmatrix} p^1U^{-1} \\ p^2U^{-1} \\ \vdots \\ p^nU^{-1} \end{bmatrix}.$$

In the same form, we have the factorization of $(UM)^{-1}$ as in (2)

$$(UM)^{-1} = \tilde{G}\tilde{B},$$

where

$$\tilde{G} = diag(||p^1U^{-1}||, ||p^2U^{-1}||, \ \ldots, ||p^nU^{-1}||) = diag(||p^1||, ||p^2||, \ \ldots, ||p^n||),$$

since U^{-1} is unitary as well. Then $(UM)^{-1} = G\tilde{B}$. Again with this equation and equation (2) we obtain

$$\tilde{B} = BU^{-1}.$$

This discussion is based on Chapter 2, Section 32 of [13]. Then we have the following result.

Theorem 8. *Let M be an invertible matrix. Then the lower bounds $s^{-1}(UM)$ and $s^{-1}(M)$ are equal for every unitary matrix U.*

Proof. From the above discussion, the reduced combined matrix of UM is

$$\mathcal{R}(UM) = \tilde{A} \circ \tilde{B}^T = (UA) \circ (BU^{-1})^T.$$

Denoting by $\tilde{s}_i$ the sum of the entries of the ith column of $\mathcal{R}(UM)$ (see Theorem 4) we have, $\tilde{s}_i = \left((BU^{-1})^T e_i\right)^T ((UA)e_i)$. Then

$$\left((BU^{-1})^T e_i\right)^T ((UA)e_i) = e_i^T (BU^{-1})(UA)e_i = (e_i^T B)(Ae_i) = (B^T e_i)^T (Ae_i),$$

which is the sum of the entries of the ith column of the reduced combined matrix $\mathcal{R}(M) = A \circ B^T$ of M, that is, s_i. Since $\tilde{s}_i = s_i$, for all $i = 1, 2, \ldots, n$. Then $s^{-1}(UM) = s^{-1}(M)$. □

The result of Theorem 8 can be applied to the following factorizations. Consider the QR factorization of M: $M = QR$, where Q is unitary and R is the upper triangular factor. Then $s^{-1}(M) = s^{-1}(R)$ by Theorem 8. Moreover, consider now the polar factorization of the invertible matrix M: $M = UH$, where U is unitary and H is the Hermitian positive definite factor. Then $s^{-1}(M) = s^{-1}(H)$ by Theorem 8.

However, right multiplication by unitary matrices does not preserve the lower bound s^{-1}.

Example 3. Let

$$M = \begin{bmatrix} 1 & 0 & -2 \\ -1 & 3 & 2 \\ 0 & 1 & -1 \end{bmatrix}.$$

Its reduced combined matrix is

$$\mathcal{R}(M) = \begin{bmatrix} 0.4385 & 0 & -0.2010 \\ -0.1754 & 0.6708 & 0.2010 \\ 0 & 0 & 0.3015 \end{bmatrix}.$$

Then $s_1 = 0.2631$, $s_2 = 0.6708$ and $s_3 = 0.3015$. The lower bound is $s^{-1}(M) = 3, 8006$.

Consider the unitary matrix

$$U = \begin{bmatrix} 0.7071 & 0.4082 & 0.5774 \\ 0 & -0.8165 & 0.5774 \\ 0.7071 & -0.4082 & -0.5774 \end{bmatrix}.$$

The computation of the reduced combined matrix of MU gives rise to

$$\mathcal{R}(MU) = \begin{bmatrix} -0.3086 & 0.1681 & 0.6749 \\ 0.1543 & 0.2522 & 0.0000 \\ 0.4629 & 0.0841 & -0.2700 \end{bmatrix}.$$

Then $\tilde{s}_1 = 0.3086$, $\tilde{s}_2 = 0.5043$ and $\tilde{s}_3 = 0.4049$. The lower bound is $s^{-1}(MU) = 3.2404$, which is different from that of M. The condition number of both matrices is $\kappa(M) = \kappa(MU) = 11.6684$.

Finally, consider $(MU)^{-1} = U^{-1}M^{-1}$, where U is unitary. We have $\mathcal{R}((MU)^{-1}) = \mathcal{R}(M^{-1})$ by Theorem 8. Therefore, $\mathcal{R}((MU)^{-1}) = \mathcal{R}(M^T)$ by Theorem 2. Hence, the lower bound of the inverse of the right multiplication of the matrix M by a unitary matrix U equals the lower bound of M^T.

4. Conclusions

A special combined matrix of the invertible matrix M has been defined to compute a lower bound $s^{-1}(M)$ of the condition number for inversion of an invertible matrix. That combined matrix is called reduced combined $\mathcal{R}(M)$ and some properties of it are given together with the relationship between the combined matrix and the reduced combined.

Then, a lower bound $s^{-1}(M)$ of the condition number of the M is considered. That lower bound is constructed by the reduced combined matrix, concretely with the sum of its columns. Moreover, properties of the lower bound are given, concretely, $s^{-1}(M) = 1$ for unitary matrices in which case the lower bound is tight. In addition, the equality $s(M^T) = s(M^{-1})$ has been obtained. Left multiplication of the matrix M by unitary matrices preserves the lower bound, that is, $s^{-1}(UM) = s^{-1}(M)$, where U is unitary. This property can be applied to the QR and polar factorizations of the invertible matrix M.

Acknowledgments

We thank the anonymous referees for their valuable reports that have improved this work. This work was partially supported by Spanish grants MTM2017–85669–P and MTM2017–90682-REDT from “Ministerio de Economa y Competitividad”, and by the Dominican Republic FONDOCYT grant number 2016–2017–057.

References

[1] P. Alonso, M.L. Serrano, Combined matrices of almost strictly sign regular matrices, J. Comput. Appl. Math. 354 (2019) 144–151, doi:10.1016/j.cam.2018.09.029.
[2] R. Bru, M.T. Gasso, M. Santana, I. Gimenez, Nonnegative combined matrices, J. Appl. Math. 2014 (2014) 5, doi:10.1155/2014/182354. ID 182354
[3] M. Fiedler, Notes on hilbert and cauchy matrices, Linear Algebra Appl. 432 (2010) 351–356, doi:10.1016/j.laa.2009.08.014.
[4] M. Fiedler, T.L. Markham, Combined matrices in special classes of matrices, Linear Algebra Appl. 435 (2011) 1945–1955, doi:10.1016/j.laa.2011.03.054.
[5] R. Fuster, M.T. Gasso, I. Gimenez, CMMSE algorithms for constructing doubly stochastic matrices with the relative gain array (combined matrix) $A \circ A^{-T}$, J. Math. Chem. 57 (2019) 1700–1709, doi:10.1007/s10910-019-01032-1.
[6] M.T. Gasso, I. Gil, I. Gimenez, M. Santana, E. Segura, Diagonal entries of the combined matrix of sign regular matrices of order three, Proyecciones (Antofagasta) 40 (1) (2021) 255–271, doi:10.22199/0717-6279-2021-01-0016.

[7] H.W. Guggenheimer, A.S. Edelman, C.R. Johnson, A simple estimate of the condition number of a linear system, College Math. J. 26 (1995) 2–5, doi:10.1080/07468342.1995.11973657.
[8] R.A. Horn, C.R. Johnson, Topics in Matrix Analysis, Cambridge Univ. Press, 1991.
[9] C. Johnson, H. Shapiro, Mathematical aspects of the relative gain array, SIAM J. Algebraic Discrete Methods 7 (4) (1986) 627–644, doi:10.1137/0607069.
[10] G. Piazza, T. Politi, An upper bound for the condition number of a matrix in spectral norm, J. Comput. Appl. Math. 143 (2002) 141–144, doi:10.1016/S0377-0427(02)00396-5.
[11] O. Rojo, Further bounds for the smallest singular value and the spectral condition number, Comput. Math. Appl. 38 (1999) 215–228, doi:10.1016/S0898-1221(99)00252-7.
[12] J. Lam, Z. Li, Y. Wei, J.-e. Feng, K.W. Chung, Estimates of the spectral condition number, Linear Multilinear Algebra 59 (3) (2011) 249–260, doi:10.1080/03081080903369419.
[13] J.H. Wilkinson, The Algebraic Eigenvalue Problem, Clarendon Press, Oxford, 1965.