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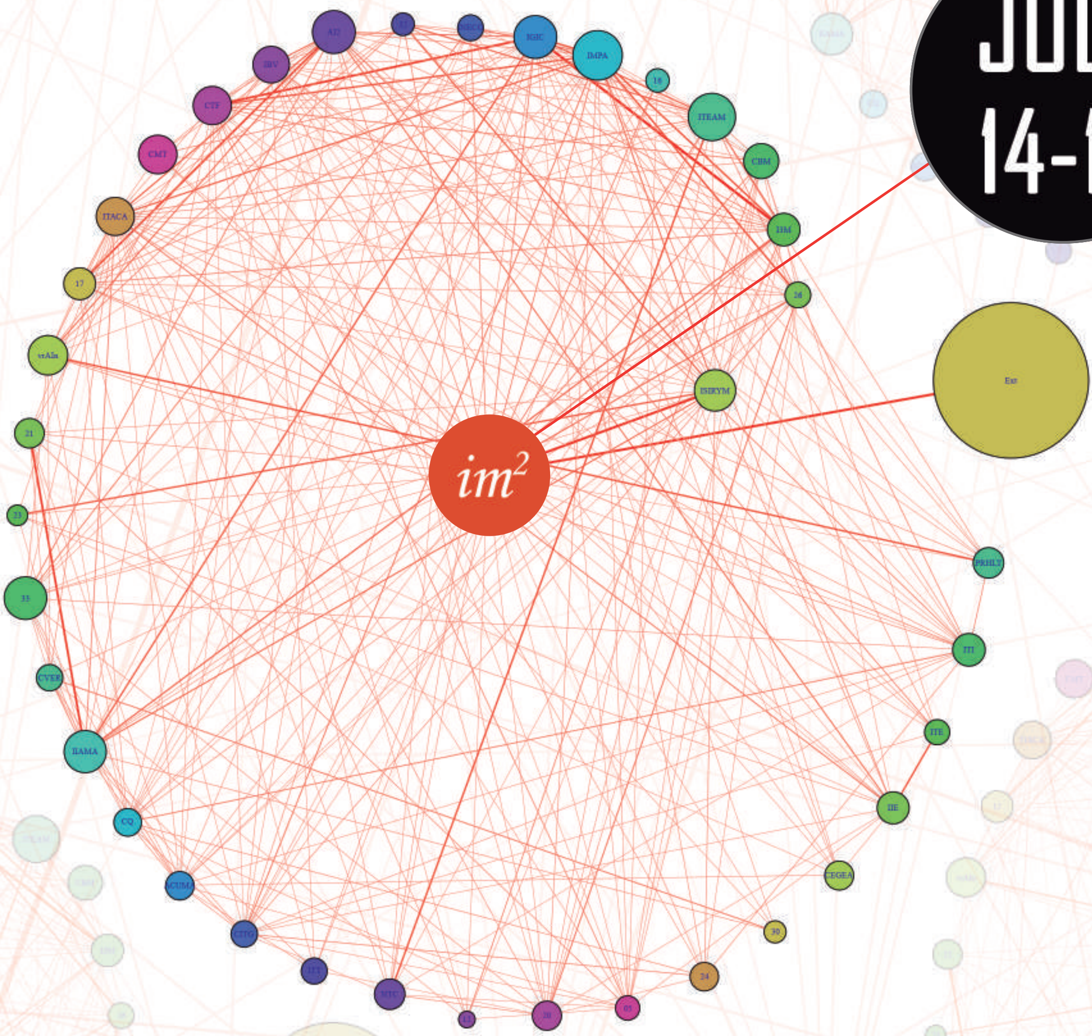

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An Optimal Eighth Order Derivative-Free Scheme for Multiple Roots of Non-linear Equations

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1 Introduction

Many practical problems are nonlinear in nature, therefore, the problem of solving a nonlinear equation is considered to be one of the significant domain. In addition, construction of higher order optimal iterative methods for multiple roots having prior knowledge of multiplicity ($m > 1$) has remained one of the most important and challenging tasks in computational mathematics. Due to advancement in computer technology, various higher order as well as optimal schemes have been proposed for computing the multiple roots of the nonlinear equations. Most of these require derivative evaluation of the involved function [1, 2, 8–10] while very few are derivative-free [4, 5, 7] and only one is of optimal eighth order [6] so far.

Motivated by the exploration going on in this area and with a requirement to achieve more optimal derivative-free schemes, we present an eighth-order optimal derivative-free method to find repeated zeros of the nonlinear equation when $m \geq 1$. This proposed family of the method has four functional evaluations and is based on the first-order divided differences and weight functions. There are two weight function involved in this family of methods, one is univariate and the other is multivariate. We compare our methods with two of the recent derivative free methods of seventh [5] and eighth order [6] using standard test problems and modelling applications.

2 Construction of Optimal Eighth Order Method

We propose a family of methods of eight-order for finding repeated roots with multiplicity $m \geqslant 1$.

$$
\begin{aligned}
s_k &= t_k + \gamma f\left(t_k\right), \text{ where } \gamma \in \mathbb{R} - \{0\}, \\
y_k &= t_k - m\frac{f(t_k)}{f[t_k, s_k]},
\end{aligned}
$$

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$$z_k = y_k - m u_k H(u_k) \frac{f(t_k)}{f[t_k, s_k]},$$
$$t_{k+1} = z_k - m v_k P(u_k, v_k, w_k) \frac{f(t_k)}{f[t_k, s_k]}, \tag{1}$$

where $u_k = (\frac{f(y_k)}{f(t_k)})^{\frac{1}{m}}$, $v_k = (\frac{f(z_k)}{f(t_k)})^{\frac{1}{m}}$ and $w_k = (\frac{f(z_k)}{f(y_k)})^{\frac{1}{m}}$. Let $H : \mathbb{C} \to \mathbb{C}$ and $P : \mathbb{C}^3 \to \mathbb{C}$ be analytic in the neighborhood of 0 and $(0, 0, 0)$. The investigation on the convergence analysis of the proposed family (1) and the conditions on weight functions $H(u_k)$ and $P(u_k, v_k, w_k)$ are apparent from the following result.

Theorem 26. *Let $t = \alpha$ be a repeated root with multiplicity $m \geq 1$ of the function $f(t)$. In addition, we suppose that $f : \mathbb{C} \to \mathbb{C}$ be analytic in a region enclosing repeated zero of $f(t)$ with known multiplicity m. Suppose that the initial guess t_0 be sufficiently close to the multiple zero α. Then, the proposed method defined by equation* (1) *has eighth-order of convergence when the conditions given below are satisfied:*

$$H(0) = 1, \quad H^{'}(0) = 2, \quad H^{''}(0) = -2 \textit{ and } H'''(0) = 36,$$
$$P_{000} = 1, \quad P_{100} = 2, \quad P_{001} = 1, \quad P_{101} = 4 - P_{010},$$
$$|P_{110}| < \infty, \quad |P_{002}| < \infty$$
$$\textit{where,} \quad P_{ijl} = \frac{\partial^{i+j+l}}{\partial u^i \partial v^j \partial w^l} P\left(u, v, w\right)|_{(0,0,0)}. \tag{2}$$

Then, the proposed scheme has the following error equation:

$$\begin{aligned} e_{k+1} = &-\frac{1}{24m^7}(b_1((11+m)b_1^2 - 2mb_2)(-24(1+m)^2 b_1^3 + \\ &(3P_{002}(11+m)^2 + 2(-665 - 84m + 5m^2 + 6P_{110}(11+m)))b_1^4 \\ &- 12m(P_{002}(11+m) + 2(-10 + P_{110} + 4m))b_1^2 b_2 + 12(-2 + P_{002})m^2 b_2^2 \\ &+ 120m^2 b_1 b_3))e_k^8 + O\left(e_k^9\right), \end{aligned} \tag{3}$$

$$b_j = \frac{m!}{(m+j)!} \frac{f^{(m+j)}\left(\alpha\right)}{f^m\left(\alpha\right)}, \; j \in \mathbb{N}$$

From Theorem 1, we can obtain several new multiple root finding two-point methods by using different cases for $H(u_k)$ and $P(u_k, v_k, w_k)$ in the proposed scheme (1). Some particular cases of the proposed scheme are given as follows:

FZ1: We take $H(u_k) = 1 + 2u_k - u_k^2 + 6u_k^3$ and $P(u_k, v_k, w_k) = 1 + 2u_k + 4v_k + w_k$ in (1).

FZ2: Also as another special case let $H\left(u_k\right) = \frac{1-9u_k^2}{1-2u_k-4u_k^2}$ and $P(u_k, v_k, w_k) = 1 + 2u_k + w_k + 4u_k w_k$ in (1).

It is noteworthy that the selection of specific values of parameter γ can be made under the point of view of an improvement of the stability and a widening of the set of converging initial estimations. We are also analyzing these aspects.

3 Numerical Results

We investigate the performance and convergence behavior of our proposed eighth order methods namely denoted by FZ1 and FZ2, respectively, by carrying out some test functions involving standard nonlinear functions and some applied examples. We compare the methods with the recent derivative free methods of seventh order (see [5], Case I(a), Case I(b), Case II(c)) denoted by SH1, SH2 and eighth order (see [6], M-1, M-4) denoted as SH3, SH4. We take the value of $\gamma = 0.001$.

For numerical tests, all computations have been performed in computer algebra software Maple 16 using 300 significant digits of precision. Tables show per step numerical errors of approximating real root $|t_k - t_{k-1}|$ of first three iterations and the absolute residual error of the test function at the third iteration and the computational order of convergence (see [3]). The numerical errors are shown with 5 sf.

$$COC \approx \frac{\ln |f(t_{k+2})/f(t_{k+1})|}{\ln |f(t_{k+1})/f(t_k)|}, \quad k = 1, 2, ...$$

We have taken into consideration the following test problems.

Example 1. *Consider the nonlinear function given by:*

$$f_1(t) = \left(\cos \frac{\pi t}{2} + t^2 - \pi\right)^5.$$

This function has multiple zero at $\alpha = 2.034724896$ *with multiplicity* $m = 5$ *and we take initial guess* $t_0 = 2.35$. *The numerical results are presented in the Table 1.*

Table 1: Comparison of multiple root finding methods for $f_1(t)$

$Methods$	$\|t_1 - t_0\|$	$\|t_2 - t_1\|$	$\|t_3 - t_2\|$	$\|f_1(t)\|$	COC
$SH1$	0.31518	9.0774×10^{-5}	1.3313×10^{-28}	3.4471×10^{-971}	6.99
$SH2$	0.31518	9.2540×10^{-5}	1.5655×10^{-28}	1.1472×10^{-968}	6.99
$SH3$	0.31512	1.4983×10^{-4}	3.5252×10^{-30}	4.9603×10^{-1170}	7.99
$SH4$	0.31504	2.3131×10^{-4}	2.4969×10^{-28}	2.5964×10^{-1094}	7.99
$FZ1$	0.31522	5.1854×10^{-5}	2.6901×10^{-34}	6.9572×10^{-1337}	7.99
$FZ2$	0.31522	6.3806×10^{-5}	1.5845×10^{-33}	7.8605×10^{-1306}	7.99

Example 2. *Consider the nonlinear function with multiple root* $\alpha = 1.365230013$ *having multiplicity* $m = 6$ *as follows:*

$$f_2(t) = \left(t^3 + 4t^2 - 10\right)^6.$$

We choose the initial guess $t_0 = 1.33$. *The numerical results are shown in Table 2.*

Table 2: Comparison of multiple root finding methods for $f_2(t)$

$Methods$	$\|t_1 - t_0\|$	$\|t_2 - t_1\|$	$\|t_3 - t_2\|$	$\|f_2(t)\|$	COC
$SH1$	0.048173	0.012943	6.3782×10^{-14}	7.9072×10^{-547}	6.98
$SH2$	0.048173	0.012943	6.5758×10^{-14}	3.3895×10^{-546}	6.98
$SH3$	0.041758	6.5287×10^{-3}	2.4152×10^{-17}	4.1759×10^{-786}	8.00
$SH4$	0.039425	4.1955×10^{-3}	1.3974×10^{-18}	1.9635×10^{-843}	7.99
$FZ1$	0.041775	6.5451×10^{-3}	9.7966×10^{-18}	1.4011×10^{-807}	8.00
$FZ2$	0.041775	6.5451×10^{-3}	1.0799×10^{-17}	3.1622×10^{-805}	8.00

Example 3. *Let us take another nonlinear function given by:*

$$f_3(t) = \left(t^{\frac{1}{2}} - \frac{1}{t} - 1\right)^7.$$

The above function has one multiple root $\alpha = 2.147899036$ *with multiplicity* $m = 7$. *We choose the initial guess* $t_0 = 1.30$ *then, the numerical results are shown in Table 3.*

Table 3: Comparison of multiple root finding methods for $f_3(t)$

$Methods$	$\lvert t_1 - t_0\rvert$	$\lvert t_2 - t_1\rvert$	$\lvert t_3 - t_2\rvert$	$\lvert f_3(t)\rvert$	COC
$SH1$	0.84388	4.0157×10^{-3}	2.2190×10^{-19}	1.0710×10^{-929}	6.99
$SH2$	0.84384	4.0566×10^{-3}	2.4840×10^{-19}	3.6146×10^{-927}	6.99
$SH3$	0.84274	5.1515×10^{-3}	1.8224×10^{-20}	6.1847×10^{-1118}	7.99
$SH4$	0.83965	8.2398×10^{-3}	1.8360×10^{-18}	3.8481×10^{-1003}	7.99
$FZ1$	0.84580	2.0942×10^{-3}	4.1687×10^{-24}	2.0506×10^{-1325}	7.99
$FZ2$	0.84522	2.6780×10^{-3}	3.5857×10^{-23}	1.6149×10^{-1272}	7.99

Example 4. *Beam Positioning Problem:*

Consder an r meter long beam is leaning against the edge of the cubical box with the sides of a meter length each such that one of its end touches the wall and the other touches the floor. Let y be the distance in meters along the beam from the floor to the edge of the box and let x be the distance in meter from the bottom of the box to the bottom of the beam. To find x for the given value of $r = 4$ and considering this case four times, we get

$$f_4(t) = (t^4 + 2t^3 - 14t^2 + 2t + 1)^4$$

The multiple root is $\alpha = 0.3622$ with multiplicity $m = 4$. Taking the initial guess $t_0 = 0.5$ gives the numerical results that are shown in Table 4**.**

Table 4: Comparison of multiple root finding methods for $f_4(t)$

$Methods$	$\lvert t_1 - t_0\rvert$	$\lvert t_2 - t_1\rvert$	$\lvert t_3 - t_2\rvert$	$\lvert f_7(t)\rvert$	COC
$SH1$	0.13772	7.7986×10^{-5}	2.7338×10^{-26}	2.6596×10^{-700}	6.99
$SH2$	0.13772	7.8767×10^{-5}	3.0090×10^{-26}	4.3340×10^{-699}	6.99
$SH3$	0.13771	8.3337×10^{-5}	5.5524×10^{-29}	5.7643×10^{-884}	7.99
$SH4$	0.13766	1.3381×10^{-4}	5.3097×10^{-27}	3.0490×10^{-819}	7.99
$FZ1$	0.13776	3.8085×10^{-5}	4.0232×10^{-32}	4.0156×10^{-986}	7.99
$FZ2$	0.13775	4.4573×10^{-5}	1.5751×10^{-31}	5.7141×10^{-967}	7.99

Table 5: Comparison of multiple root finding methods for $f_5(t)$

$Methods$	$\|t_1 - t_0\|$	$\|t_2 - t_1\|$	$\|t_3 - t_2\|$	$\|f_8(t)\|$	COC
$SH1$	0.04987	6.2000×10^{-4}	7.8458×10^{-14}	3.7028×10^{-665}	6.98
$SH2$	0.04937	6.2058×10^{-4}	8.0778×10^{-14}	2.3076×10^{-664}	6.98
$SH3$	0.04934	6.5713×10^{-4}	1.1775×10^{-14}	5.3589×10^{-805}	7.98
$SH4$	0.04914	8.5965×10^{-4}	1.9487×10^{-13}	2.4682×10^{-724}	7.97
$FZ1$	0.04953	4.6574×10^{-4}	3.2414×10^{-16}	4.0100×10^{-908}	7.99
$FZ2$	0.04952	4.7560×10^{-4}	4.2256×10^{-16}	2.0788×10^{-900}	7.99

Example 5. *Vander der Waals Equation of State:*

Consider the function given below,

$$f_5(t) = (t^3 - 5.22t^2 + 9.0825t - 5.2675)^4$$

Taking this case two times, where x represents the fractional conversion of a specie in a chemical reactor, yields the multiple roots $\alpha = 1.75$ and $\alpha = 1.72$ with multiplicity $m = 8$. Taking the initial guess $t_0 = 1.8$ gives the numerical results in the Table 5.
It is apparent from the construction and numerical results that our proposed family is optimal and efficient in terms of small residual errors.

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