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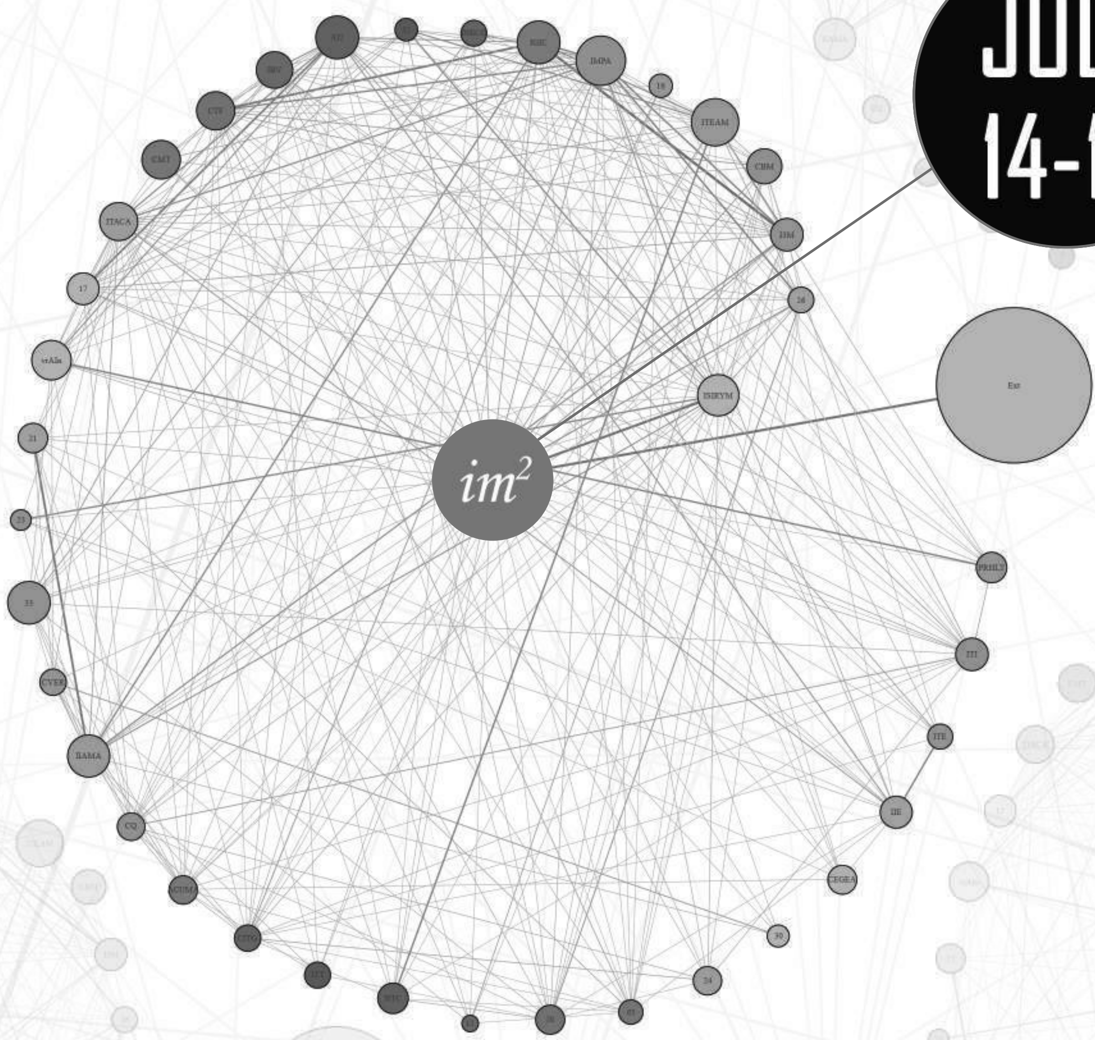

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The effect of the memory on the spread of a disease through the environtment

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1 Introduction

In recent years, several papers have studied how to apply the fractional order derivative to model the spread of an infectious disease. Most of them apply Caputo's definition of fractional derivative in differential equations or systems of differential equations [1]. However, it is interesting to work with discrete systems since statistical data on epidemics are collected at discrete times and it is easier to compare them with the output of discrete-time systems. For the discrete case, the fractional order differential operator is less used. In this line, some papers study discrete linear systems of fractional order using the discrete approximation of the Grünwald-Letnikov fractional derivative, see for instance [2].

In this paper, we propose a mathematical representation to study the behaviour of the solution of an epidemic model in which the disease is transmitted through the environment. We focus on an epidemic process that includes indirect transmission of the disease when the population comes into contact with the underlying contamination in space. Then, in addition to susceptible and infected individuals, we consider a new variable representing the amount of contaminant in the enclosure. We propose a discrete-time model based on fractional calculus where the state depends on the current state and previous ones. This model uses a discrete version of the Grünwald-Letnikov fractional derivative operator with truncated memory. This follows from the properties of the coefficients of our operator since we show that they can be negligible beyond a certain step. This justifies taking a truncated operator as a good approximation of the proposed process.

We obtain the equilibrium points of the proposed model and perform a thorough analysis of their stability. To determine whether the system is stable or not we define some parameters depending on the fractional order and the memory steps considered. In the analysis we observe that the population size plays an important role in ensuring the disappearance or permanence of the disease.

Finally, we illustrate different properties of the proposed model by analyzing different examples as a function of the fractional order and the number steps in which we truncate.

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2 Methods

We consider an epidemic model that represents an outbreak of an indirect transmission infectious disease. In this model, the state variables are: the susceptible population $S(t)$ with p survival rate, the infected population $I(t)$ with q survival rate and the amount of contaminant $C(t)$ remaining in the environment, whose survival rate is s. We denote by β the amount of contaminant produced by each infected individual and the incidence of the disease is given by the parameter σ, which represents the rate of indirect contact-based transmission. Note that $0 < p, q, s, \sigma < 1$ and $\beta \geq 0$. The mathematical representation of this infectious process is given by the following system

$$\begin{aligned} S(t+1) &= pS(t) - \sigma C(t)S(t) + \mu(t)P \\ I(t+1) &= qI(t) + \sigma C(t)S(t) \\ C(t+1) &= sC(t) + \beta I(t), t \geq 0; \end{aligned} \tag{1}$$

Note that this system is a discrete time nonlinear system relating the state at time t with the state at time $t+1$. Some results about this model can be found in [3–5].
In this work, we introduce a new model using a fractional order operator which consider the states at previous times. Concretely, we use the discrete-time fractional order Grünwald-Letnikov operator Δ^α [2]. Moreover, we will truncate the operator in order to describe a short-term memory process in a similar way as [6, 7]. Then, the k-truncated discrete-time fractional order Grünwald-Letnikov operator in k steps is given by:

$$\Delta_k^\alpha x(t) = \sum_{j=0}^{k} a_j^\alpha x(t-j). \tag{2}$$

where the fractional order α satisfies $0 < \alpha \leq 1$ and

$$a_j^\alpha = (-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix}, \ j \geq 0, \tag{3}$$

with

$$\begin{pmatrix} \alpha \\ j \end{pmatrix} = \begin{cases} 1 & j = 0 \\ \dfrac{\alpha(\alpha-1)...(\alpha-j+1)}{j!} & j > 0. \end{cases} \tag{4}$$

Using this k-truncated operator we introduce the new model as follows

$$\begin{aligned} \Delta_k^\alpha S(t+1) &= (p-1)S(t) - \sigma C(t)S(t) + \mu(t)P \\ \Delta_k^\alpha I(t+1) &= (q-1)I(t) + \sigma C(t)S(t) \\ \Delta_k^\alpha C(t+1) &= (s-1)C(t) + \beta I(t), t \geq 0; \end{aligned} \tag{5}$$

which is clearly reduced to the system (1) when $\alpha = 1$. This is a model with k steps of memory since the state at time $t+1$ depends on the states at time $t, \ t-1, \ldots, \ t-k+1$.
We assume that the size of the population always remains constant, that is $P = S(t) + I(t)$, $\forall t \geq 0$, with the replacement of dead individuals by new susceptible individuals given by $\mu(t)P$. This condition of constant population P at any time $t \geq 0$ implies that the addition of the two first equations of system (5) gives

$$\mu(t)P = (q-p)S(t) + P(1 - q + \Sigma_k^\alpha), \quad t \geq 0,$$

with $\Sigma_k^\alpha = \sum_{j=0}^k a_j^\alpha, \quad k \geq 0,\ 0 < \alpha \leq 1$. So, applying this condition to our model (5), it can be rewritten as

$$\begin{aligned} S(t+1) &= (q-1+\alpha))S(t) - \sum_{j=2}^{k} a_j^\alpha S(t+1-j) - \sigma S(t)C(t) + P(1-q+\Sigma_k^\alpha) \\ I(t+1) &= (q-1+\alpha)I(t) - \sum_{j=2}^{k} a_j^\alpha I(t+1-j) + \sigma S(t)C(t) \\ C(t+1) &= (s-1+\alpha)C(t) - \sum_{j=2}^{k} a_j^\alpha C(t+1-j) + \beta I(t), \end{aligned} \tag{6}$$

We have to assure that, under the conditions of no infection and no recruitment of population, then both, individuals and contaminant, disappear. So, in order to get consistency with this particular case, the conditions $0 \leq 1-\alpha < q$, and $0 \leq 1-\alpha < s$ have to be hold.
In the Results section we compute the equilibrium points to study the behavior of this discrete time fractional order system. To analyze the evolution of the disease, we will obtain a linear approximation of the model around the disease-free equilibrium point. We will define and obtain the basic reproduction number and study how it relates with the order of the fractional derivative α and the memory steps k.

3 Results

First, we study the biological sense of the truncated discrete time fractional order system (6). It is important to assure that our model is a good representation of the epidemiological process. Taking into account that only nonnegative solutions have biological sense, in the following proposition we establish an upper threshold on the size of the population P in order to assure the nonnegativity condition on the solution.

Proposition 5. *If* $P < \dfrac{(q-1+\alpha)(1-s+\Sigma_k^\alpha)}{\sigma\beta}$ *then the solution of the truncated discrete time fractional order system (6) is nonnegative.*

Next, we study the stability of the system (6) around the disease-free equilibrium point. An easy calculation leads to that this point is given by

$$S_f^* = P, \qquad I_f^* = 0, \qquad C_f^* = 0,$$

for any value $\alpha \in \Lambda_{q,s}$ and $k \geq 1$. Note, that this agrees with the case $\alpha = 1$ when the first order difference is considered.
Let us consider the linearization of system (6) around the disease-free equilibrium point $(P\ 0\ 0)^T$ taking $S(t)C(t) \approx P\ C(t)$. Then, we focus our attention on the equations for the infected individuals and the contaminant of this linearized system

$$\begin{aligned} I(t+1) &= (q-1+\alpha)I(t) - \sum_{j=2}^{k} a_j^\alpha I(t+1-j), \\ C(t+1) &= (s-1+\alpha)C(t) - \sum_{j=2}^{k} a_j^\alpha C(t+1-j) + \beta I(t). \end{aligned} \tag{7}$$

Note that this system can be interpreted as a system with delays and its stability can be studied using a stacked form. That is, the system (7) is a $k-$delayed linear system equivalent to the

$k-$stacked linear system $\hat{x}(t+1) = \mathcal{E}\hat{x}(t)$, with $\hat{x}(t) = (I(t)\ C(t)\ I(t-1)\ C(t-1)\ \ldots\ I(t+1-k)\ C(t+1-k))^T$ and $\mathcal{E}$ given by

$$\mathcal{E} = \begin{pmatrix} \bar{E} & -a_2 I & -a_3 I & \cdots & -a_{k-1} I & -a_k I \\ I & O & O & \cdots & O & O \\ O & I & O & \cdots & O & O \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ O & O & O & \cdots & I & O \end{pmatrix}, \tag{8}$$

where $\bar{E} = \begin{pmatrix} q-1+\alpha & \sigma P \\ \beta & s-1+\alpha \end{pmatrix}$.

The system is asymptotically stable if $\rho(\mathcal{E}) < 1$. So, to determine the spectral radius of $\mathcal{E}$ we need to make an exhaustive study of its eigenvalues. This is given in the following result.

Theorem 11. *The system 6 is asymptotically stable around the disease-free equilibrium point if and only if $P < M(\alpha, k)$, where*

$$M(\alpha, k) = \frac{(1-q+\Sigma_k^\alpha)(1-s+\Sigma_k^\alpha)}{\sigma\beta}, \tag{9}$$

otherwise it is not.

Therefore, we have found a bound for the population as a function of the fractional order α and the selected memory k, given by $M(\alpha, k)$. It is interesting to analyze how this limit varies according to the choice of these parameters to optimize the size of the population according to the available data. From this study we obtain that when considering a greater number of memory steps, stability is maintained with a larger population. On the other hand, in the case of the order of the fractional derivative, if this order is higher, stability is maintained with a larger population size.

Proposition 6. *The population bound given in (9) satisfies that*

$$M(\alpha, k) < M(\alpha, k-1) \quad \textit{and} \quad M(\alpha_2, k) < M(\alpha_1, k)$$

with $0 < \alpha \leq 1$, $k \geq 1$ and $0 < \alpha_1 < \alpha_2 \leq 1$.

4 Conclusions

The spread of a disease transmitted by indirect contact has been mathematically represented by a discrete fractional order model with k memory steps.

In order to assure the nonnegativity of the solution of the system, a threshold for the population size has been obtained.

The stability of the disease free equilibrium point of the model has been analyzed. Using a $k-$stacked form, another bound for the population size has been found, which depends on the fractional order α and the step memory k. This bound increases both, when α decreases for a constant k and when k decreases for a constant α.

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