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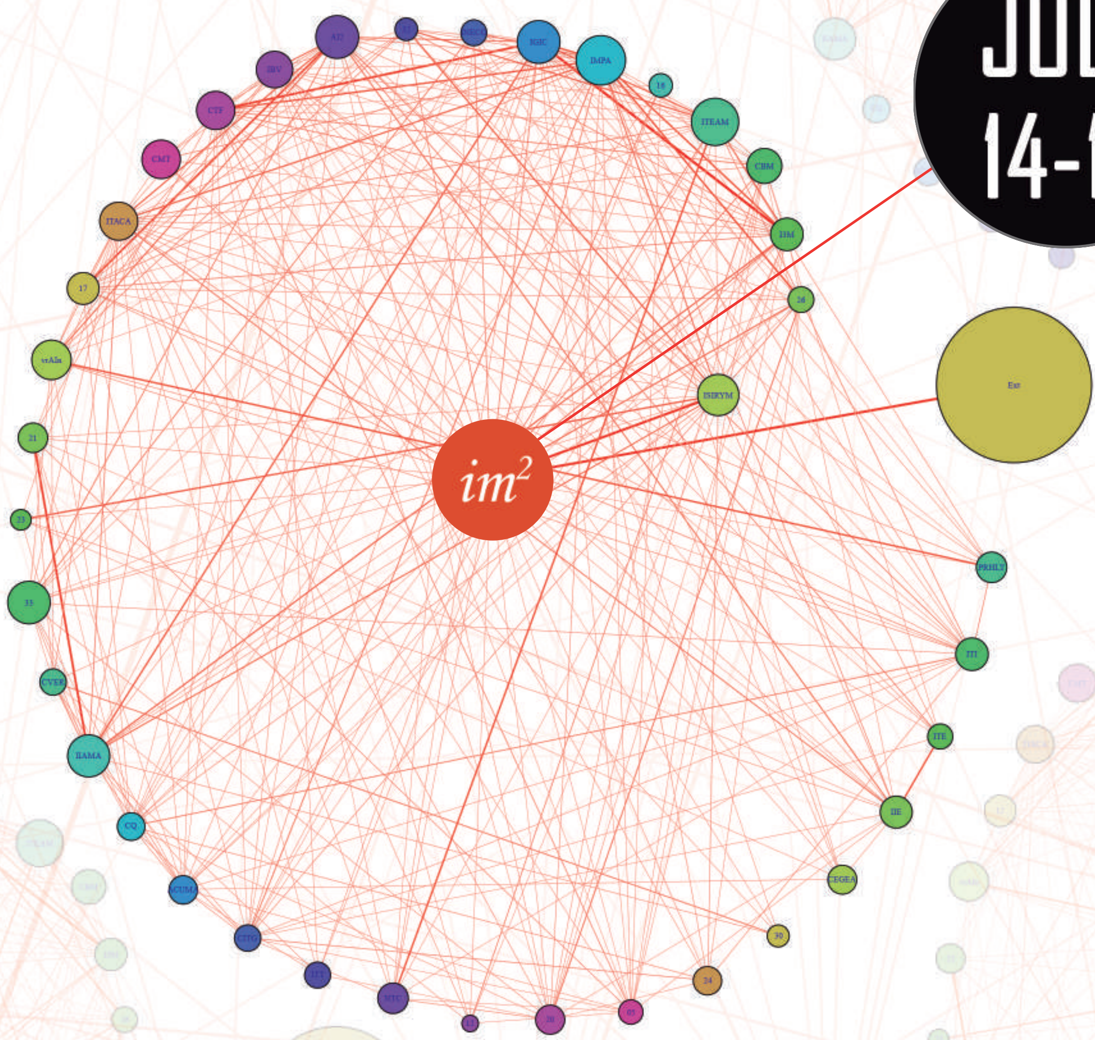

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Solving non homogeneous linear second order difference equations with random initial values: Theory and simulations

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1 Introduction

Difference equations and differential equations are powerful equations for modeling the dynamics of real world phenomena. The former are suitable for modeling the evolution of species whose generations do not overlap and grow at regular intervals over fixed periods of time (e.g., annual) or for modeling the dynamics of economic quantities that are evaluated in discrete periods by connecting the present value with previous capitalized values (in a general sense). For its part, differential equations have been successfully applied to describe the dynamics of quantities from their instantaneous change. When both types of equations are applied to real problems, it is often necessary to take into account the inherent uncertainty of the phenomena under study or the sampled data errors required to determine the data specifying these models.

This simple approach leads to the need to examine this type of equations, both from a theoretical and applied point of view, taking into consideration the randomness in their formulation. There are different types of strategies to introduce randomness in the study of these problems, but in the present work we will consider noises defined through random variables or stochastic processes with regular sampling behavior.

This document is organized as follows. In Section 2, we will determine the first probability density function (1-p.d.f.). of the solution of the initial value problem (i.v.p.) bearing in mind the nature of the roots of the associated characteristic equation. In Section 3, we will illustrate the theoretical results developed through examples. The last section draws conclusions.

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2 Computing the 1-p.d.f.

Let us consider the following initial value problem

$$\begin{cases} Z_{n+2} + A_1 Z_{n+1} + A_2 Z_n = B, \quad n = 0, 1, 2, \dots, \\ Z_0 = \Gamma_0, \\ Z_1 = \Gamma_1. \end{cases} \tag{1}$$

In this problem, we will treat the parameters A_1, A_2 and B as constant coefficients, and, without loss of generality, we will treat the initial conditions $\Gamma_0 = \Gamma_0(\omega)$ and $\Gamma_1 = \Gamma_1(\omega)$ as dependent continuous random variables (r.v.'s) defined on a common probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, which have an arbitrary joint probability density function, $f_{\Gamma_0,\Gamma_1}(\gamma_0, \gamma_1)$. In the following, the domains of the input parameters Γ_0, Γ_1 will be denoted as follows

$$D_{\Gamma_0} = \{\gamma_0 = \Gamma_0(\omega), \omega \in \Omega : \gamma_{0,1} \leq \gamma_0 \leq \gamma_{0,2}\},$$
$$D_{\Gamma_1} = \{\gamma_1 = \Gamma_1(\omega), \omega \in \Omega : \gamma_{1,1} \leq \gamma_1 \leq \gamma_{1,2}\},$$

where the endpoints of each interval can take any real value. For notation reasons, the sample dependence for r.v.'s denoted by the w-notation will be omitted.

Thus, since the problem is a non homogeneous linear second order difference equation with random initial conditions, its solution will be a stochastic process, say $\{Z_n : n \geq 0\}$. Therefore, our objective now is to determine the solution of the i.v.p. and, in addition, to calculate the first probability density function since from it we can determine statistical characteristics such as the mean $\mathbb{E}[Z_n]$, the variance $\mathbb{V}[Z_n]$ or any other statistical moment

$$\mathbb{E}[(Z_n)^k] = \int_{-\infty}^{\infty} z^k f_{Z_n}(z) dz, \quad n, k = 0, 1, 2, \dots$$

Notice that this makes a considerable difference from the deterministic scenario.

In order to figure out the 1-p.d.f. of the solution of the i.v.p. we will use the Random Variable Transformation (R.V.T.) technique. Among the numerous versions of this technique, the one we are going to implement throughout this work is found in [2].

Theorem 1. *Let $\boldsymbol{V} = (V_1, ..., V_m)$ be a random vector of dimension m with joint p.d.f. $f_{\boldsymbol{V}}(\boldsymbol{v})$. Let $\boldsymbol{r}: \mathbb{R}^m \to \mathbb{R}^m$ be a one-to-one deterministic map which is assumed to be continuous with respect to each one of its arguments, and with continuous partial derivatives. Then, the joint p.d.f. $f_{\boldsymbol{W}}(\boldsymbol{w})$ of the random vector $\boldsymbol{W} = \boldsymbol{r}(\boldsymbol{V})$ is given by*

$$f_{\boldsymbol{W}}(\boldsymbol{w}) = f_{\boldsymbol{V}}(\boldsymbol{s}(\boldsymbol{w}))\ |J_m|, \tag{2}$$

where $\boldsymbol{s}(\boldsymbol{w})$ is the inverse transformation of $\boldsymbol{r}(\boldsymbol{v}) : \boldsymbol{v} = \boldsymbol{r}^{-1}(\boldsymbol{w}) = \boldsymbol{s}(\boldsymbol{w})$ and J_m is the Jacobian of the transformation, i.e

$$J_m = \det\left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{w}}\right) = \det\left(\frac{\partial \boldsymbol{s}(\boldsymbol{w})}{\partial \boldsymbol{w}}\right) = \det \begin{pmatrix} \frac{\partial s_1(\boldsymbol{w})}{\partial w_1} & \cdots & \frac{\partial s_m(\boldsymbol{w})}{\partial w_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial s_1(\boldsymbol{w})}{\partial w_m} & \cdots & \frac{\partial s_m(\boldsymbol{w})}{\partial w_m} \end{pmatrix},$$

which is assumed to be different from zero.

Based on the deterministic theory, it is known that the general solution Z_n of (1) is equal to the sum of the general solution of the associated homogeneous equation and a particular solution of the non-homogeneous equation

$$Z_n^c = Z_n^h + Z_n^p,$$

and depends on the character of the roots of the associated characteristic equation

$$r^2 + A_1 r + A_2 = 0,$$

that is, on the values

$$r_1 = \frac{-A_1 + \sqrt{A_1^2 - 4A_2}}{2}, \qquad r_2 = \frac{-A_1 - \sqrt{A_1^2 - 4A_2}}{2}. \tag{3}$$

The nature of the roots, which can be real or complex, depends on the discriminant

$$\Delta = A_1^2 - 4A_2.$$

So,

1. if the discriminant is positive, $\Delta > 0$, then there are two distinct real roots, i.e.

$$r_1, r_2 \in \mathbb{R} \quad \text{and} \quad r_1 \neq r_2.$$

2. if the discriminant is zero, $\Delta = 0$, then there is a real root of double multiplicity, i.e.

$$r_1 = r_2 = r \in \mathbb{R}.$$

3. if the discriminant is negative, $\Delta < 0$, then there are two distinct complex roots, i.e.

$$r_1 = a + ib, \; r_2 = a - ib \in \mathbb{C} \quad \text{and} \quad r_2 = \overline{r}_1.$$

Regarding the particular solution, which will be the same for the three cases, let us assume that it is a constant discrete function $Z_n = k$, with a constant k to be determined. Given this circumstance, the particular solution obtained is

$$Z_n^p = \frac{B}{1 + A_1 + A_2}, \tag{4}$$

where $1 + A_1 + A_2 \neq 0$.
We will now examine each of the three cases.

2.1 Real and distinct roots

The solution of the (1) is

$$Z_n^c = \frac{r_1 (r_2)^n - r_2 (r_1)^n}{r_1 - r_2} \Gamma_0 + \frac{(r_1)^n - (r_2)^n}{r_1 - r_2} \Gamma_1 + \frac{B}{1 + A_1 + A_2} \left(\frac{r_2 - 1}{r_1 - r_2} (r_1)^n + \frac{1 - r_1}{r_1 - r_2} (r_2)^n + 1 \right),$$

where r_1 and r_2 are given by (3).
Now, we are going to calculate the 1-p.d.f. of the r.v. $Z = Z_n$. To do so, we set n and we apply the Theorem 1 using the following identification: $m = 2$ and

$$\begin{aligned} \mathbf{V} &= (\Gamma_0, \Gamma_1), & f_{\mathbf{V}}(\mathbf{v}) &= f_{\Gamma_0,\Gamma_1}(\gamma_0, \gamma_1), \\ \mathbf{W} &= (W_1, W_2) = \mathbf{r}(\Gamma_0, \Gamma_1). \end{aligned}$$

The inverse mapping is given by

$$W_1 = r_1(\Gamma_0, \Gamma_1) = \tfrac{r_1(r_2)^n - r_2(r_1)^n}{r_1 - r_2}\Gamma_0 + \tfrac{(r_1)^n - (r_2)^n}{r_1 - r_2}\Gamma_1 + \tfrac{B}{1+A_1+A_2}\left(\tfrac{r_2-1}{r_1-r_2}(r_1)^n + \tfrac{1-r_1}{r_1-r_2}(r_2)^n + 1\right)$$

$$\Rightarrow \Gamma_0 = s_1(W_1, W_2) = \left(W_1 - \tfrac{(r_1)^n - (r_2)^n}{r_1 - r_1}W_2 - \tfrac{B}{1+A_1+A_2}\left(\tfrac{r_2-1}{r_1-r_2}(r_1)^n + \tfrac{1-r_1}{r_1-r_2}(r_2)^n + 1\right)\right)\tfrac{r_1-r_2}{r_1(r_2)^n - r_2(r_1)^n},$$

$$W_2 = r_2(\Gamma_0, \Gamma_1) = \Gamma_1 \quad \Rightarrow \quad \Gamma_1 = s_2(W_1, W_2) = W_2.$$

The corresponding Jacobian is

$$J_2 = \frac{r_1 - r_2}{r_1(r_2)^n - r_2(r_1)^n} \neq 0.$$

Thus, we obtain the joint p.d.f. of the random vector $\mathbf{W} = (W_1, W_2)$ by applying (2)

$$f_{\mathbf{W}}(\mathbf{w}) = f_{\Gamma_0,\Gamma_1}\left(\left(w_1 - \frac{(r_1)^n - (r_2)^n}{r_1 - r_2}w_2 - \frac{B}{1+A_1+A_2}\left(\frac{r_2-1}{r_1-r_2}(r_1)^n + \frac{1-r_1}{r_1-r_2}(r_2)^n + 1\right)\right)\cdot\right.$$
$$\left.\cdot\frac{r_1-r_2}{r_1(r_2)^n - r_2(r_1)^n}, w_2\right)\cdot\left|\frac{r_1-r_2}{r_1(r_2)^n - r_2(r_1)^n}\right|, \quad w_{i,1} \leq w_i \leq w_{i,2}, \quad 1 \leq i \leq 2.$$

Lastly, taking into account that $Z = W_1$, we obtain the 1-p.d.f. of the solution as follows

$$f_{Z_n}(z) = \int_{\gamma_{1,1}}^{\gamma_{1,2}} f_{\Gamma_0,\Gamma_1}\left(\left(z - \frac{(r_1)^n - (r_2)^n}{r_1 - r_2}\gamma_1 - \frac{B}{1+A_1+A_2}\left(\frac{r_2-1}{r_1-r_2}(r_1)^n + \frac{1-r_1}{r_1-r_2}(r_2)^n + 1\right)\right)\cdot\right.$$
$$\left.\cdot\frac{r_1-r_2}{r_1(r_2)^n - r_2(r_1)^n}, \gamma_1\right)\cdot\left|\frac{r_1-r_2}{r_1(r_2)^n - r_2(r_1)^n}\right| d\gamma_1. \tag{5}$$

To avoid cumbersome notation in this general context, we prefer to leave the explicit definition of the domains in the examples section.

2.2 Real and equal roots

The solution of the (1) is

$$Z_n^c = (1-n)r^n\Gamma_0 + (nr^{n-1})\Gamma_1 + \frac{B}{1+A_1+A_2}\left(nr^n - nr^{n-1} - r^n + 1\right),$$

where $r = r_1 = r_2$.

Now, we are going to calculate the 1-p.d.f. of the r.v. $Z = Z_n$. To do so, we set n and we apply the Theorem 1 using the following identification: $m = 2$ and

$$\mathbf{V} = (\Gamma_0, \Gamma_1), \qquad f_{\mathbf{V}}(\mathbf{v}) = f_{\Gamma_0,\Gamma_1}(\gamma_0, \gamma_1),$$
$$\mathbf{W} = (W_1, W_2) = \mathbf{r}(\Gamma_0, \Gamma_1).$$

The inverse mapping is given by

$$W_1 = r_1(\Gamma_0, \Gamma_1) = (1-n)r^n\Gamma_0 + (nr^{n-1})\Gamma_1 + \tfrac{B}{1+A_1+A_2}\left(nr^n - nr^{n-1} - r^n + 1\right)$$

$$\Rightarrow \Gamma_0 = s_1(W_1, W_2) = \left(W_1 - (nr^{n-1})W_2 - \tfrac{B}{1+A_1+A_2}\left(nr^n - nr^{n-1} - r^n + 1\right)\right)\tfrac{1}{(1-n)r^n},$$

$$W_2 = r_2(\Gamma_0, \Gamma_1) = \Gamma_1 \quad \Rightarrow \quad \Gamma_1 = s_2(W_1, W_2) = W_2.$$

The corresponding Jacobian is

$$J_2 = \frac{1}{(1-n)r^n} \neq 0.$$

Thus, we obtain the joint p.d.f. of the random vector $\mathbf{W} = (W_1, W_2)$ by applying (2)

$$f_{\mathbf{W}}(\mathbf{w}) = f_{\Gamma_0,\Gamma_1}\left(\left(w_1 - (nr^{n-1})w_2 - \frac{B}{1+A_1+A_2}\left(nr^n - nr^{n-1} - r^n + 1\right)\right)\cdot \right.$$
$$\left. \cdot\frac{1}{(1-n)r^n}, w_2\right) \cdot \left|\frac{1}{(1-n)r^n}\right|, \quad w_{i,1} \leq w_i \leq w_{i,2}, \quad 1 \leq i \leq 2.$$

Lastly, taking into account that $Z = W_1$, we obtain the 1-p.d.f. of the solution as follows

$$f_{Z_n}(z) = \int_{\gamma_{1,1}}^{\gamma_{1,2}} f_{\Gamma_0,\Gamma_1}\left(\left(z - (nr^{n-1})\gamma_1 - \frac{B}{1+A_1+A_2}\left(nr^n - nr^{n-1} - r^n + 1\right)\right)\cdot \right.$$
$$\left. \cdot\frac{1}{(1-n)r^n}, \gamma_1\right) \cdot \left|\frac{1}{(1-n)r^n}\right| d\gamma_1.$$

To avoid cumbersome notation in this general context, we prefer to leave the explicit definition of the domains in the examples section.

2.3 Complex roots

The solution of the (1) is

$$Z_n^c = \frac{R^n\sin(\theta(1-n))}{\sin(\theta)}\Gamma_0 + \frac{R^{n-1}\sin(\theta n)}{\sin(\theta)}\Gamma_1 + \frac{B}{1+A_1+A_2}\left(\frac{R^n\sin(\theta(n-1))}{\sin(\theta)} - \frac{R^{n-1}\sin(\theta n)}{\sin(\theta)} + 1\right),$$

where $R = |r_1| = |r_2| = \sqrt{a^2+b^2}$ and $\theta = \arctan(\frac{b}{a})$.
Now, we are going to calculate the 1-p.d.f. of the r.v. $Z = Z_n$. To do so, we set n and we apply the Theorem 1 using the following identification: $m = 2$ and

$$\mathbf{V} = (\Gamma_0, \Gamma_1), \qquad f_{\mathbf{V}}(\mathbf{v}) = f_{\Gamma_0,\Gamma_1}(\gamma_0, \gamma_1),$$
$$\mathbf{W} = (W_1, W_2) = \mathbf{r}(\Gamma_0, \Gamma_1).$$

The inverse mapping is given by

$$W_1 = r_1(\Gamma_0, \Gamma_1) = \tfrac{R^n\sin(\theta(1-n))}{\sin(\theta)}\Gamma_0 + \tfrac{R^{n-1}\sin(\theta n)}{\sin(\theta)}\Gamma_1 + \tfrac{B}{1+A_1+A_2}\left(\tfrac{R^n\sin(\theta(n-1))}{\sin(\theta)} - \tfrac{R^{n-1}\sin(\theta n)}{\sin(\theta)} + 1\right)$$

$$\Rightarrow \Gamma_0 = s_1(W_1, W_2) = \left(W_1 - \tfrac{R^{n-1}\sin(\theta n)}{\sin(\theta)}W_2 - \tfrac{B}{1+A_1+A_2}\left(\tfrac{R^n\sin(\theta(n-1))}{\sin(\theta)} - \tfrac{R^{n-1}\sin(\theta n)}{\sin(\theta)} + 1\right)\right)\tfrac{\sin(\theta)}{R^n\sin(\theta(1-n))},$$

$$W_2 = r_2(\Gamma_0, \Gamma_1) = \Gamma_1 \quad \Rightarrow \quad \Gamma_1 = s_2(W_1, W_2) = W_2.$$

The corresponding Jacobian is

$$J_2 = \frac{\sin(\theta)}{R^n \sin(\theta(1-n))} \neq 0.$$

Thus, we obtain the joint p.d.f. of the random vector $\mathbf{W} = (W_1, W_2)$ by applying (2)

$$f_{\mathbf{W}}(\mathbf{w}) = f_{\Gamma_0,\Gamma_1}\left(\left(w_1 - \frac{R^{n-1}\sin(\theta n)}{\sin(\theta)}w_2 - \frac{B}{1+A_1+A_2}\left(\frac{R^n\sin(\theta(n-1))}{\sin(\theta)} - \frac{R^{n-1}\sin(\theta n)}{\sin(\theta)} + 1\right)\right)\cdot\right.$$
$$\left.\cdot\frac{\sin(\theta)}{R^n\sin(\theta(1-n))}, w_2\right)\cdot\left|\frac{\sin(\theta)}{R^n\sin(\theta(1-n))}\right|, \quad w_{i,1} \le w_i \le w_{i,2}, \quad 1 \le i \le 2.$$

Lastly, taking into account that $Z = W_1$, we obtain the 1-p.d.f. of the solution as follows

$$f_{Z_n}(z) = \int_{\gamma_{1,1}}^{\gamma_{1,2}} f_{\Gamma_0,\Gamma_1}\left(\left(z - \frac{R^{n-1}\sin(\theta n)}{\sin(\theta)}\gamma_1 - \frac{B}{1+A_1+A_2}\left(\frac{R^n\sin(\theta(n-1))}{\sin(\theta)} - \frac{R^{n-1}\sin(\theta n)}{\sin(\theta)} + 1\right)\right)\cdot\right.$$
$$\left.\cdot\frac{\sin(\theta)}{R^n\sin(\theta(1-n))}, \gamma_1\right)\cdot\left|\frac{\sin(\theta)}{R^n\sin(\theta(1-n))}\right| d\gamma_1.$$

To avoid cumbersome notation in this general context, we prefer to leave the explicit definition of the domains in the examples section.

3 Examples

In this section, we are going to illustrate the theoretical results through numerical examples. The aim is to plot the 1-p.d.f. of the solution of the i.v.p. (1) obtained in the three cases for some values of n.
In the first case we assume that the constant coefficients take the following values

$$A_1 = -4, \quad A_2 = -2 \quad B = 10.$$

Furthermore, we consider that the random inputs follow a joint Gaussian distribution, i.e.

$$(\Gamma_0, \Gamma_1) \sim \mathcal{N}(\mu, \mathbf{\Sigma}), \qquad \mu = (1, 10), \qquad \mathbf{\Sigma} = \begin{pmatrix} 6 & 0.05 \\ 0.05 & 6 \end{pmatrix}.$$

Figure (1) shows the 1-p.d.f. at $n = 0, 1, 2, 3$. It seems to diverge monotonically as n increases.
In the second case we assume that the constant coefficients take the following values

$$A_1 = -1, \quad A_2 = 1/4 \quad B = 1.$$

We also consider a joint Gaussian distribution, but now with the following mean and covariance matrix

$$\mu = (1, 1.5), \qquad \mathbf{\Sigma} = \begin{pmatrix} 0.1 & 0.05 \\ 0.05 & 0.1 \end{pmatrix}.$$

Figure (2) shows the 1-p.d.f. at $n = 0, 1, 2, 3, 4, 5$. It seems to converge monotonically as n increases.
Finally, the values that we assume in the last case are

$$A_1 = 1, \quad A_2 = 1/2 \quad B = 3, \quad \mu = (0.5, 1), \quad \mathbf{\Sigma} = \begin{pmatrix} 0.2 & 0.01 \\ 0.01 & 0.3 \end{pmatrix}.$$

Figure (3) shows the 1-p.d.f. at $n = 0, 1, 2, 3, 4$. It seems to converge oscillating as n increases.
Note that in all cases the z-domain of $f_{Z_n}(z)$ is $-\infty < z < \infty$.

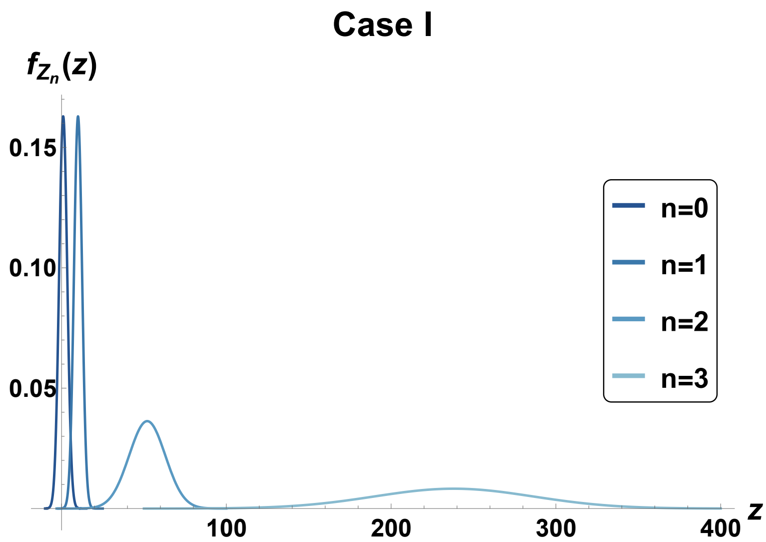

Figure 1: Plot of the 1-p.d.f., $f_{Z_n}(z)$, of the solution, Z_n, in case I at different values of $n = 0, 1, 2, 3$.

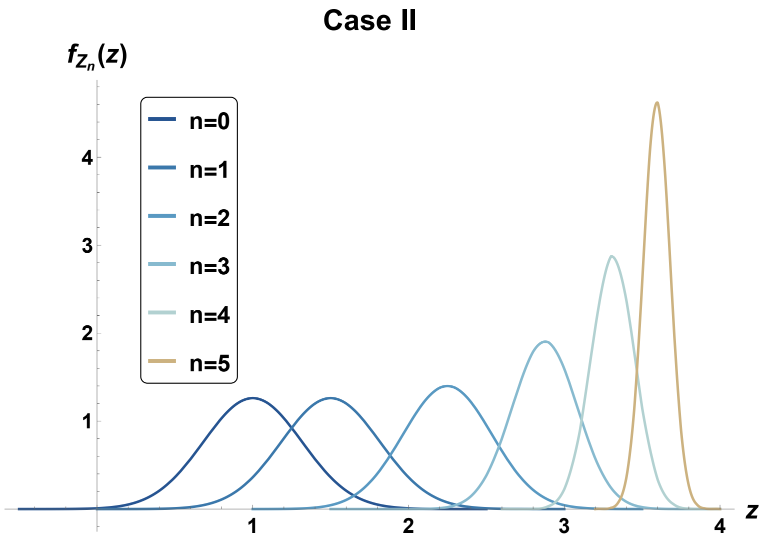

Figure 2: Plot of the 1-p.d.f., $f_{Z_n}(z)$, of the solution, Z_n, in case II at different values of $n = 0, 1, 2, 3, 4, 5$.

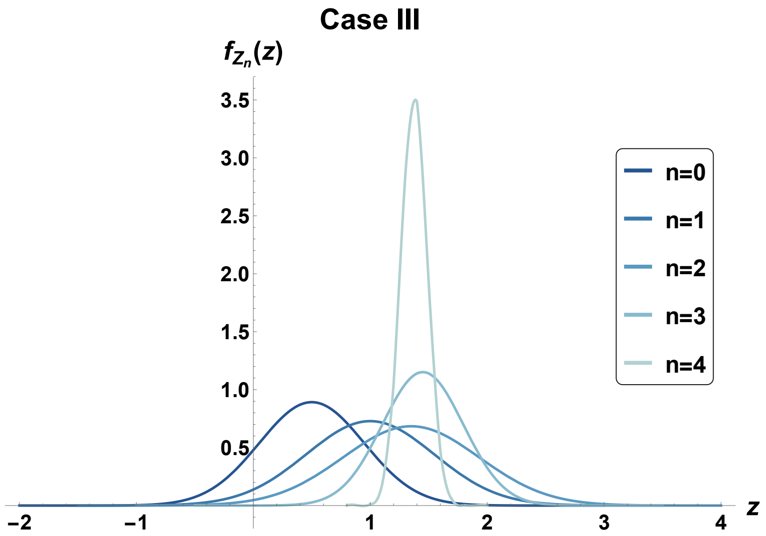

Figure 3: Plot of the 1-p.d.f., $f_{Z_n}(z)$, of the solution, Z_n, in case III at different values of $n = 0, 1, 2, 3, 4$.

4 Conclusions

In this work we have provided a general explicit formula for the 1-p.d.f. of the solution of a non-homogeneous linear second order difference equation with random initial values, which depending on the character of the roots of the associated characteristic equation has one form or another, and where the random inputs involved are statistically dependent. The study has been based on the Random Variable Transformation technique. Moreover, we have shown with examples the theoretical development obtained, in which it can be observed that it is the deterministic counterpart.

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