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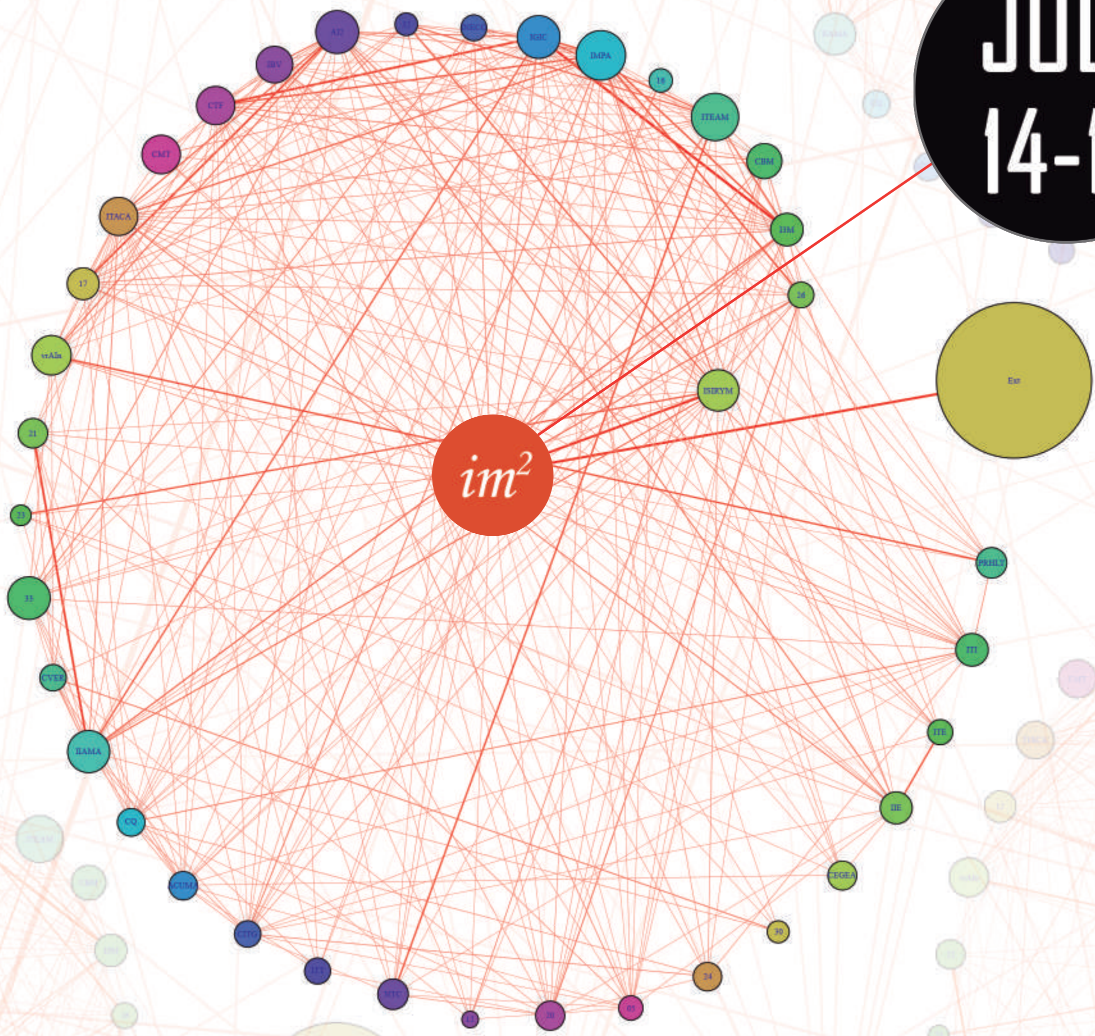

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Solving random fractional second-order linear equations via the mean square Laplace transform

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1 Introduction

In this contribution, the authors extend the Laplace transform in mean square (m.s.) sense to solve random fractional differential equations. Concretely they solve the following random fractional initial value problem (RFIVP).

$$\left\{ \begin{array}{l} {}^{C}D^{\alpha}_{0+}X(t) + A\dot{X}(t) + BX(t) \quad = \quad 0, \qquad t > 0, \quad 1 < \alpha < 2, \\ \\ X(0) = C_0, \quad \dot{X}(0) = C_1, \end{array} \right. \tag{1}$$

where A, B, C_0 and C_1 are second order random variables. Here, ${}^{C}D^{\alpha}_{0+}X(t)$ denotes the mean square Caputo derivative of the stochastic process $X(t)$, [1–3].
Firstly, some important results related with the Laplace transform in mean square sense are presented in order to construct a solution stochastic process of the RFIVP (1). This solution is described via a generalized power series. Mild conditions will be impossed into the random input parameters to guarantee the convergence of the power series solution in the m.s. sense. Once a convergent solution is obtained, we compute approximations for the main statistical moments: the mean and the variance.

2 Main properties of the Laplace transform in m.s. sense

This section is devoted to introduce Laplace transform in m.s. sense and its main properties.

Definition 1. *The Laplace transform of a 2-stochastic process, $\{X(t) : t \geq 0\}$, is defined by*

$$\mathcal{L}\{X(t); s\} := \int_0^\infty \mathrm{e}^{-st} X(t) \mathrm{d}t, \qquad s \in \mathcal{S} \subset \mathbb{R}, \tag{2}$$

provided the improper integral exists in $\mathrm{L}^2(\Omega)$.
Next, we compute the Laplace transform of a power function. This result will be useful to obtain the solution of the RFIVP (1).

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Example 1. *Let $X(t) = Ut^k$ be a 2-SP, then*

$$\mathcal{L}\{X(t); s\} = U\frac{1}{s^{k+1}}\Gamma(k+1), \quad s > 0.$$

So far, the Laplace transform has been presented. Now we compute the Laplace transform for the first, second and Caputo derivatives.

Proposition 1. *Let $\{X(t) : t \geq 0\}$ be a 2-stochastic process satisfying the following conditions:*

i) $X(t)$ is mean square differentiable (so, continuous),

ii) $\dot{X}(t)$ is mean square piecewise continuous,

iii) $X(t)$ is of exponential order $s_0 \geq 0$.

Then,

$$\mathcal{L}\{\dot{X}(t); s\} = s\mathcal{L}\{X(t); s\} - X(0), \quad s > s_0 \geq 0.$$

Remark 8. *If the 2-stochastic process $X(t)$ is twice mean square differentiable such that $X(t)$ and $\dot{X}(t)$ are both mean square continuous and of exponential order and $\ddot{X}(t)$ is mean square piecewise continuous, then applying twice Proposition 1, one obtains*

$$\begin{aligned}\mathcal{L}\{\ddot{X}(t); s\} &= s\mathcal{L}\{\dot{X}(t); s\} - \dot{X}(0) = s(s\mathcal{L}\{X(t); s\} - X(0)) - \dot{X}(0) \\ &= s^2\mathcal{L}\{X(t); s\} - sX(0) - \dot{X}(0). \end{aligned} \tag{3}$$

Proposition 2. *Let $X(t)$ be a 2-stochastic process satisfying the hypotheses of Remark 8. Let ${}^{C}D^{\alpha}_{0+}X(t)$, $1 < \alpha < 2$, denote its mean square Caputo fractional derivative. If the pathwise integral $\int_0^\infty \mathrm{e}^{-s\tau}|\ddot{X}(\tau)|\mathrm{d}\tau$ exists and is finite, then*

$$\mathcal{L}\{{}^{C}D^{\alpha}_{0+}X(t); s\} = s^{\alpha}\mathcal{L}\{X(t); s\} - s^{\alpha-1}X(0) - s^{\alpha-2}\dot{X}(0),$$

and $\mathcal{L}\{{}^{C}D^{\alpha}_{0+}X(t); s\}$ belongs to $\mathrm{L}^2(\Omega)$.

The following Theorem allow us to express the Laplace transform of the solution SP in a generalized power series.

Theorem 9. *Let A and B be bounded random variables, i.e. there exist $M_A > 0$ and $M_B > 0$ such that $|A(\omega)| \leq M_A$ and $|B(\omega)| \leq M_B$, for all $\omega \in \Omega$. If $1 < \alpha < 2$ and $s > K_1 := \max\left\{M_A^{\frac{1}{\alpha-1}}, (M_A + M_B)^{\frac{1}{\alpha-1}}, 1\right\}$, then for all $\omega \in \Omega$,*

i) $|A(\omega)|s^{1-\alpha} < 1$.

ii) $\left|\frac{B(\omega)s^{-1}}{s^{\alpha-1}+A(\omega)}\right| < 1$.

iii) $\frac{1}{s^{\alpha}+A(\omega)s+B(\omega)} = \sum_{n,m\geq 0} \frac{\Gamma(m+n+1)}{\Gamma(n+1)\Gamma(m+1)}(-A(\omega))^m(-B(\omega))^n s^{-(m(\alpha-1)+\alpha(n+1))}$.

3 Obtaining the solution SP of the RFIVP (1)

This section is devoted to compute the solution SP of the RFIVP (1) using the Laplace transform and the results described in the last section.
Let us consider the RFIPV (1) given by

$${}^{C}D^{\alpha}_{0+}X(t) + A\dot{X}(t) + BX(t). \tag{4}$$

Applying the Laplace transform, using its linearity and applying Propositions 1 and 2, one gets

$$\begin{aligned}&\mathcal{L}\{{}^{C}D^{\alpha}_{0+}X(t)+A\dot{X}(t)+BX(t);s\}\\&=\mathcal{L}\{{}^{C}D^{\alpha}_{0+}X(t);s\}+A\mathcal{L}\{\dot{X}(t);s\}+B\mathcal{L}\{X(t);s\}\\&=s^{\alpha}\mathcal{L}\{X(t);s\}-s^{\alpha-1}X(0)-s^{\alpha-2}\dot{X}(0)+As\mathcal{L}\{X(t);s\}-AX(0)+B\mathcal{L}\{X(t);s\}.\end{aligned} \tag{5}$$

Isolating $\mathcal{L}\{X(t);s\}$ and taking into account that $X(0)=C_0$ and $\dot{X}(0)=C_1$, one gets

$$\mathcal{L}\{X(t);s\}=\frac{s^{\alpha-1}C_0+s^{\alpha-2}C_1+AC_0}{s^{\alpha}+As+B}. \tag{6}$$

Applying Theorem (9) iii), we can express (6) as a power series. Let us define

$$\phi_{n,m}(A,B)=(-B)^n(-A)^m\frac{\Gamma(m+n+1)}{\Gamma(m+1)\Gamma(n+1)}=(-1)^{n+m}B^nA^m\frac{\Gamma(m+n+1)}{m!n!}, \tag{7}$$

so

$$\begin{aligned}\mathcal{L}\{X(t);s\}=&\sum_{n,m\geq 0}\phi_{n,m}(A,B)C_0s^{-((\alpha-1)m+n\alpha+1)}\\&+\sum_{n,m\geq 0}\phi_{n,m}(A,B)C_1s^{-(m(\alpha-1)+n\alpha+2)}\\&+\sum_{n,m\geq 0}\phi_{n,m}(A,B)AC_0s^{-(m(\alpha-1)+(n+1)\alpha)}.\end{aligned} \tag{8}$$

Taking into account Example 1 and applying the inverse of the Laplace transform we can obtain the solution SP which is given by

$$\begin{aligned}X(t)=&\sum_{n,m\geq 0}\phi_{n,m}(A,B)C_0\frac{t^{m\nu+n\alpha}}{\Gamma(m\nu+n\alpha+1)}\\&+\sum_{n,m\geq 0}\phi_{n,m}(A,B)C_1\frac{t^{m\nu+n\alpha+1}}{\Gamma(m\nu+n\alpha+2)}\\&+\sum_{n,m\geq 0}\phi_{n,m}(A,B)AC_0\frac{t^{m\nu+n\alpha+\alpha-1}}{\Gamma(m\nu+n\alpha+\alpha)}.\end{aligned} \tag{9}$$

This solution is m.s. convergent for all $t>K_2$ if the following hypothesis fulfil.

- A and B are bounded random variables, that is, there are positive numbers M_A and M_B such that $|A(\omega)|\leq M_A$ and $|B(\omega)|\leq M_B$, for all $\omega\in\Omega$. So, $A,B\in \mathrm{L}^2(\Omega)$.
- C_0 and C_1 are second order random variables, i.e. $C_0,C_1\in \mathrm{L}^2(\Omega)$.
- C_0, C_1, A and B are independent random variables.
- $K_2:=\max\left\{(2M_B)^{\frac{1}{\alpha}},(2M_A)^{\frac{1}{\alpha-1}},K_1\right\}$, where K_1 is defined in Theorem 9

4 Approximations for the two first statistical moments of the solution SP

So far, a m.s. convergent solution of the RFIVP 1 has been obtained. This section is devoted to obtain approximations for the mean and for the second order moment taking into account the solution SP obtained in (9). As (9) is m.s. convergent, the mean and the variance of the truncated solution converge to the mean and the variance of the limit. Let us consider a trunctation of (9)

$$\begin{aligned} X_{N,M}(t) &= \sum_{n=0}^{N}\sum_{m=0}^{M} \phi_{n,m}(A,B)C_0 \frac{t^{m\nu+n\alpha}}{\Gamma(m\nu+n\alpha+1)} \\ &\quad + \sum_{n=0}^{N}\sum_{m=0}^{M} \phi_{n,m}(A,B)C_1 \frac{t^{m\nu+n\alpha+1}}{\Gamma(m\nu+n\alpha+2)} \\ &\quad + \sum_{n=0}^{N}\sum_{m=0}^{M} \phi_{n,m}(A,B)AC_0 \frac{t^{m\nu+n\alpha+\alpha-1}}{\Gamma(m\nu+n\alpha+\alpha)}. \end{aligned} \tag{10}$$

Applying the mean operator one gets.

$$\begin{aligned} \mathbb{E}[X_{N,M}(t)] &= \sum_{n=0}^{N}\sum_{m=0}^{M} \frac{(-1)^{n+m}\mathbb{E}[B^n]\mathbb{E}[A^m]\Gamma(m+n+1)}{\Gamma(m+1)\Gamma(n+1)} \\ &\quad \cdot \left\{ \mathbb{E}[C_0]\frac{t^{m\nu+n\alpha}}{\Gamma(m\nu+\alpha n+1)} + \mathbb{E}[C_1]\frac{t^{m\nu+n\alpha+1}}{\Gamma(m\nu+\alpha n+2)} \right\} \\ &\quad + \sum_{n=0}^{N}\sum_{m=0}^{M} (-1)^{n+m}\mathbb{E}[B^n]\mathbb{E}[A^{m+1}]\mathbb{E}[C_0] \\ &\quad \cdot \frac{\Gamma(m+n+1)}{\Gamma(m+1)\Gamma(n+1)} \frac{t^{m\nu+\alpha n+\alpha-1}}{\Gamma(m\nu+\alpha n+\alpha)}. \end{aligned} \tag{11}$$

The second order moment is given by the following expression.

$$\begin{aligned} \mathbb{E}[(X_{N,M}(t))^2] = \sum_{n=0}^{N} &\left(\sum_{m=0}^{M} \left(\frac{(n+m)!}{n!m!}\right) \mathbb{E}[B^{2n}] \left[\mathbb{E}[A^{2m}] \left\{ \mathbb{E}[C_0^2] \left(\frac{t^{m\nu+\alpha n}}{\Gamma(m\nu+\alpha n+1)}\right)^2 \right.\right.\right. \\ &+ \mathbb{E}[C_1^2] \left(\frac{t^{m\nu+\alpha n+1}}{\Gamma(m\nu+\alpha n+2)}\right)^2 \\ &+ 2\mathbb{E}[C_0]\mathbb{E}[C_1] \left(\frac{t^{m\nu+\alpha n}}{\Gamma(m\nu+\alpha n+1)}\right)\left(\frac{t^{m\nu+\alpha n+1}}{\Gamma(m\nu+\alpha n+2)}\right) \Bigg\} \\ &+ \mathbb{E}[A^{2m+2}]\mathbb{E}[C_0^2] \left(\frac{t^{m\nu+\alpha n+\alpha-1}}{\Gamma(m\nu+\alpha n+\alpha)}\right)^2 \\ &+ 2\mathbb{E}[C_0^2]\mathbb{E}[A^{2m+1}] \left(\frac{t^{m\nu+\alpha n}}{\Gamma(m\nu+\alpha n+1)}\right)\left(\frac{t^{m\nu+\alpha n+\alpha-1}}{\Gamma(m\nu+\alpha n+\alpha)}\right) \\ &+ 2\mathbb{E}[C_0]\mathbb{E}[C_1]\mathbb{E}[A^{2m+1}] \left(\frac{t^{m\nu+\alpha n+1}}{\Gamma(m\nu+\alpha n+2)}\right)\left(\frac{t^{m\nu+\alpha n+\alpha-1}}{\Gamma(m\nu+\alpha n+\alpha)}\right) \Bigg] \\ &+ 2\sum_{m_1=1}^{M}\sum_{m_2=0}^{m_1-1} (-1)^{2n+m_1+m_2}\mathbb{E}[B^{2n}] \frac{\Gamma(n+m_1+1)}{\Gamma(n+1)\Gamma(m_1+1)} \frac{\Gamma(n+m_2+1)}{\Gamma(n+1)\Gamma(m_2+1)} \\ &\cdot \Bigg[\mathbb{E}[A^{m_1+m_2}] \Bigg\{ \mathbb{E}[C_0^2] \left(\frac{t^{m_1\nu+\alpha n}}{\Gamma(m_1\nu+\alpha n+1)}\right)\left(\frac{t^{m_2\nu+\alpha n}}{\Gamma(m_2\nu+\alpha n+1)}\right) \\ &+ \mathbb{E}[C_0]\mathbb{E}[C_1] \left(\frac{t^{m_1\nu+\alpha n}}{\Gamma(m_1\nu+\alpha n+1)}\right)\left(\frac{t^{m_2\nu+\alpha n+1}}{\Gamma(m_2\nu+\alpha n+2)}\right) \\ &+ \mathbb{E}[C_0]\mathbb{E}[C_1] \left(\frac{t^{m_1\nu+\alpha n+1}}{\Gamma(m_1\nu+\alpha n+2)}\right)\left(\frac{t^{m_2\nu+\alpha n}}{\Gamma(m_2\nu+\alpha n+1)}\right) \end{aligned}$$

$$
\begin{aligned}
&+\mathbb{E}[C_1^2]\left(\frac{t^{m_1\nu+\alpha n+1}}{\Gamma(m_1\nu+\alpha n+2)}\right)\left(\frac{t^{m_2\nu+\alpha n+1}}{\Gamma(m_2\nu+\alpha n+2)}\right)\qquad\Bigg\}\\
&+\mathbb{E}[A^{m_1+m_2+1}]\mathbb{E}[C_0^2]\left(\frac{t^{m_1\nu+\alpha n}}{\Gamma(m_1\nu+\alpha n+1)}\right)\left(\frac{t^{m_2\nu+\alpha n+\alpha+1}}{\Gamma(m_2\nu+\alpha n+\alpha)}\right)\\
&+\mathbb{E}[C_0]\mathbb{E}[C_1]\mathbb{E}[A^{m_1+m_2+1}]\left(\frac{t^{m_1\nu+\alpha n+1}}{\Gamma(m_1\nu+\alpha n+2)}\right)\left(\frac{t^{m_2\nu+\alpha n+\alpha-1}}{\Gamma(m_2\nu+\alpha n+\alpha)}\right)\\
&+\mathbb{E}[C_0^2]\mathbb{E}[A^{m_1+m_2+1}]\left(\frac{t^{m_1\nu+\alpha n+\alpha-1}}{\Gamma(m_1\nu+\alpha n+\alpha)}\right)\left(\frac{t^{m_2\nu+\alpha n}}{\Gamma(m_2\nu+\alpha n+1)}\right)\\
&+\mathbb{E}[C_0]\mathbb{E}[C_1]\mathbb{E}[A^{m_1+m_2+1}]\left(\frac{t^{m_1\nu+\alpha n+\alpha-1}}{\Gamma(m_1\nu+\alpha n+\alpha)}\right)\left(\frac{t^{m_2\nu+\alpha n+1}}{\Gamma(m_2\nu+\alpha n+2)}\right)\\
&+\mathbb{E}[C_0^2]\mathbb{E}[A^2]\left(\frac{t^{m_1\nu+\alpha n+\alpha-1}}{\Gamma(m_1\nu+\alpha n+\alpha)}\right)\left(\frac{t^{m_2\nu+\alpha n+\alpha-1}}{\Gamma(m_2\nu+\alpha n+\alpha)}\right)\qquad\Bigg]\Bigg)\\
&+2\sum_{n_1=1}^{N}\sum_{n_2=0}^{n_1-1}\sum_{m_1=0}^{M}\sum_{m_2=0}^{M}(-1)^{n_1+m_1+n_2+m_2}\mathbb{E}[B^{n_1+n_2}]\frac{\Gamma(m_1+n_2+1)}{\Gamma(m_1+1)\Gamma(n_1+1)}\\
&\cdot\frac{\Gamma(m_2+n_2+1)}{\Gamma(m_2+1)\Gamma(n_2+1)}\Bigg[\mathbb{E}[A^{m_1+m_2}]\Bigg\{\mathbb{E}[C_0^2]\left(\frac{t^{m_1\nu+\alpha n_1}}{\Gamma(m_1\nu+\alpha n_1+1)}\right)\left(\frac{t^{m_2\nu+\alpha n_2}}{\Gamma(m_2\nu+\alpha n_2+1)}\right)\\
&+\mathbb{E}[C_0]\mathbb{E}[C_1]\left(\frac{t^{m_1\nu+\alpha n_1}}{\Gamma(m_1\nu+\alpha n_1+1)}\right)\left(\frac{t^{m_2\nu+\alpha n_2+1}}{\Gamma(m_2\nu+\alpha n_2+2)}\right)\\
&+\mathbb{E}[C_0]\mathbb{E}[C_1]\left(\frac{t^{m_1\nu+\alpha n_1+1}}{\Gamma(m_1\nu+\alpha n_1+2)}\right)\left(\frac{t^{m_2\nu+\alpha n_2}}{\Gamma(m_2\nu+\alpha n_2+1)}\right)\\
&+\mathbb{E}[C_1^2]\left(\frac{t^{m_1\nu+\alpha n_1+1}}{\Gamma(m_1\nu+\alpha n_1+2)}\right)\left(\frac{t^{m_2\nu+\alpha n_2+1}}{\Gamma(m_2\nu+\alpha n_2+2)}\right)\qquad\Bigg\}\\
&+\mathbb{E}[A^{m_1+m_2+1}]\Bigg\{\mathbb{E}[C_0^2]\left(\frac{t^{m_1\nu+\alpha n_1}}{\Gamma(m_1\nu+\alpha n_1+1)}\right)\left(\frac{t^{m_2\nu+\alpha n_2+\alpha-1}}{\Gamma(m_2\nu+\alpha n_2+\alpha)}\right)\\
&+\mathbb{E}[C_0]\mathbb{E}[C_1]\left(\frac{t^{m_1\nu+\alpha n_1+1}}{\Gamma(m_1\nu+\alpha n_1+2)}\right)\left(\frac{t^{m_2\nu+\alpha n_2+\alpha-1}}{\Gamma(m_2\nu+\alpha n_2+\alpha)}\right)\\
&+\mathbb{E}[C_0^2]\left(\frac{t^{m_1\nu+\alpha n_1+\alpha-1}}{\Gamma(m_1\nu+\alpha n_1+\alpha)}\right)\left(\frac{t^{m_2\nu+\alpha n_2}}{\Gamma(m_2\nu+\alpha n_2+1)}\right)\\
&+\mathbb{E}[C_0]\mathbb{E}[C_1]\left(\frac{t^{m_1\nu+\alpha n_1+\alpha-1}}{\Gamma(m_1\nu+\alpha n_1+\alpha)}\right)\left(\frac{t^{m_2\nu+\alpha n_2+1}}{\Gamma(m_2\nu+\alpha n_2+2)}\right)\qquad\Bigg\}\\
&+\mathbb{E}[A^{m_1+m_2+2}]\mathbb{E}[C_0^2]\left(\frac{t^{m_1\nu+\alpha n_1+\alpha-1}}{\Gamma(m_1\nu+\alpha n_1+\alpha)}\right)\left(\frac{t^{m_2\nu+\alpha n_2+\alpha-1}}{\Gamma(m_2\nu+\alpha n_2+\alpha)}\right)\qquad\Bigg].
\end{aligned}
\tag{12}
$$

5 Numerical example

The aim of this Section is to illustrate the previous theoretical findings with a numerical example. Let us consider that the order of the derivative $\alpha = 1.5$. C_0 and C_1 are 2-RVs with mean 0.5 and second order moment 0.5. A and B are beta random variables, i.e. $A \sim Be(10, 20)$ and $B \sim Be(20, 30)$.

To calculate the approximations of the mean and the variance of $X(t)$, we will apply expressions

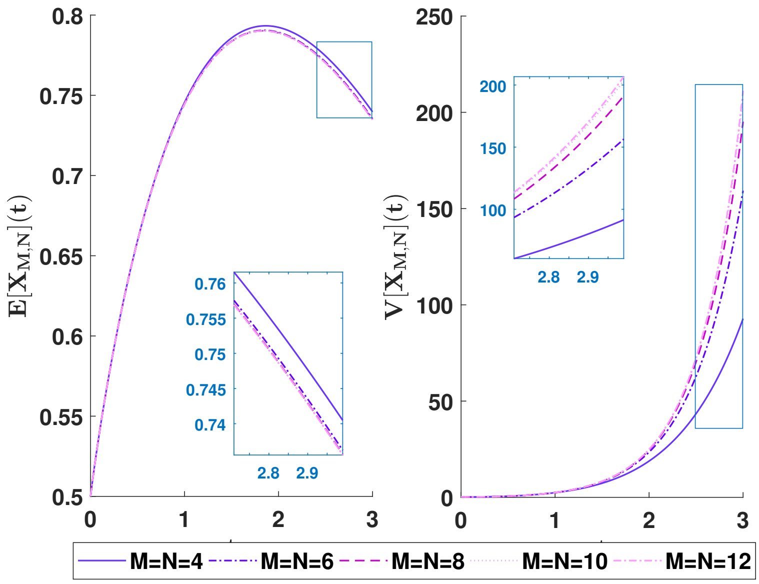

Figure 1: Approximations of the mean (left panel) and for the variance (right panel) of the solution stochastic process of the RIVP (1) considering different order of truncation (M, N).

(11) and (12) taking into account the following expressions for the moments of A and B [4]

$$\mathbb{E}[A^k] = \prod_{r=0}^{k-1} \frac{10+r}{10+20+r}, \quad \mathbb{E}[B^k] = \prod_{r=0}^{k-1} \frac{20+r}{20+30+r}, \quad k = 1, 2, \ldots. \tag{13}$$

Figure 1 shows the approximations for the mean and for the variance of the truncated solution (10), $X_{N,M}(t)$, considering different values of M and N in the time interval $[0, 3]$. To easily visualize the convergence, we zoom-up the results on the right-piece of the interval where it is supposed the accuracy of approximations becomes worse. However, the plots show very good results even taking small order of truncation $M = N = 12$.

Acknowledgments

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