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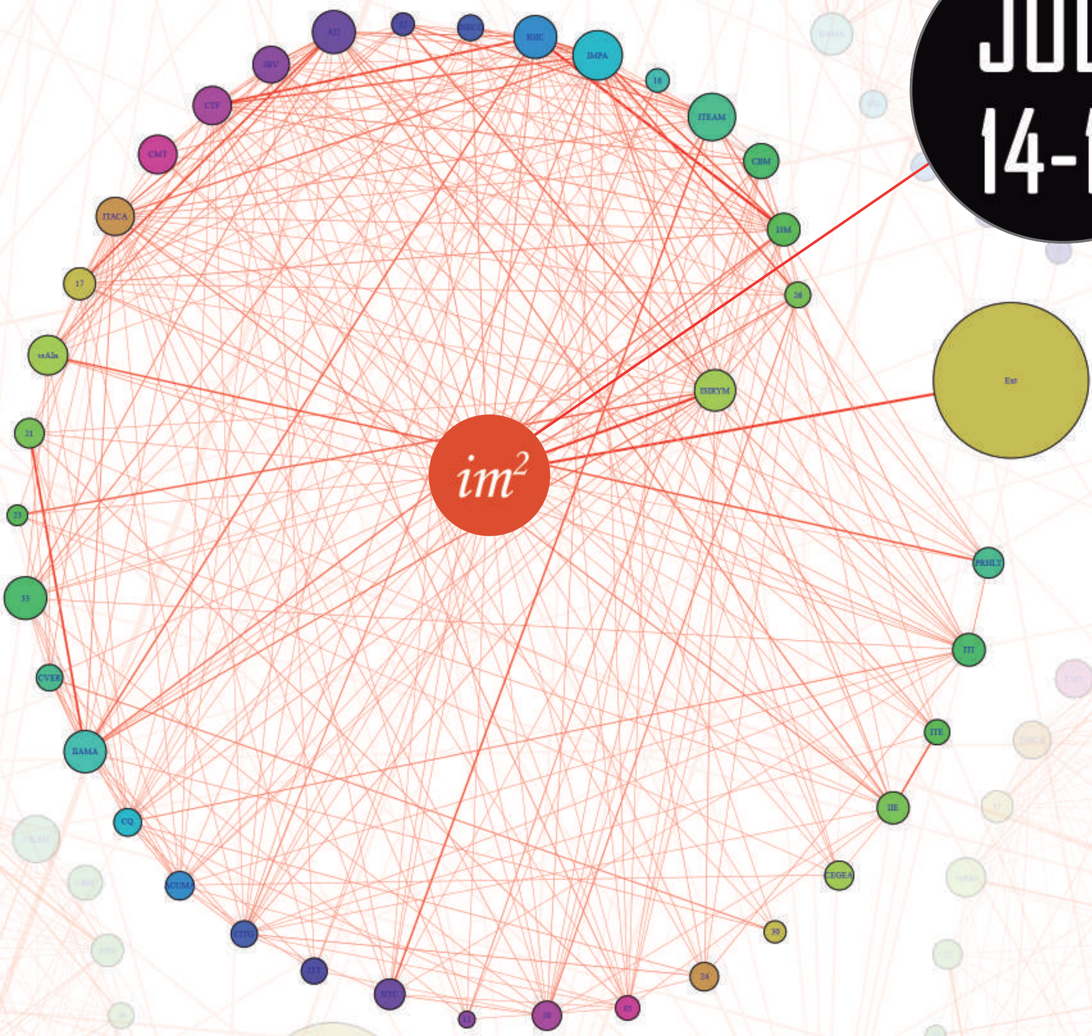

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Introducing a new parametric family for solving nonlinear systems of equations

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1 Introduction

Many problems in Computational Sciences and other disciplines can be stated in the form of a nonlinear equation or nonlinear systems of equations using mathematical modelling. In particular, a large number of problems in Applied Mathematics and Engineering are solved by finding the solutions of these equations. In the literature there are many methods and families of iterative schemes to approximate the simple roots of a nonlinear equation $f(x) = 0$, where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a real function defined in an open interval I. We can find in [1–3] several surveys and overviews of the iterative schemes published in the last years.

In this manuscript, we introduce a new parametric family of multistep iterative schemes for solving nonlinear systems of equations as an extension of the family presented in [4] for nonlinear equations. This family is built from the Ostrowski's scheme, adding a Newton step with a "frozen" derivative and using a divided difference operator. Firstly, we design a fourth-order triparametric family that, by holding only one of its parameters, we get to accelerate its convergence and finally obtain a sixth-order uniparametric family. We study the convergence of this last class of iterative schemes, analyzed its stability by means of complex dynamics tools and checked its numerical performance on some test problems. The parameter spaces and dynamical planes presented in [4], which show the complexity of the family for nonlinear equations, are used in the present work for systems of nonlinear equations. From the parameter spaces we have been able to determine different members of the family that have bad convergence properties, since attracting periodic orbits and attracting strange fixed points appear in their dynamical planes. Moreover, this same study has allowed us to detect family members with specially stable behavior and suitable for solving practical problems. Several numerical tests are performed to illustrate the efficiency and stability of the presented family.

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2 New parametric family

The new parametric family object of study in this manuscript has the following iterative expression:

$$\begin{cases} y^{(k)} = x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}) \\ z^{(k)} = y^{(k)} - [2[x^{(k)}, y^{(k)}; F] - F'(x^{(k)})]^{-1}F(y^{(k)}) \\ x^{(k+1)} = z^{(k)} - (\alpha I + \beta u^{(k)} + \gamma v^{(k)})[F'(x^{(k)})]^{-1}F(z^{(k)}) \end{cases}, \tag{1}$$

where $u^{(k)} = I - [F'(x^{(k)})]^{-1}[x^{(k)}, y^{(k)}; F]$, $v^{(k)} = [x^{(k)}, y^{(k)}; F]^{-1}F'(x^{(k)})$, $k = 0, 1, 2, ...$, and α, β and γ are arbitrary parameters. The divided difference operator $[x, y; F]$ is the map $[\cdot, \cdot; F] : D \times D \subset \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n)$, satisfying $[x, y; F](x - y) = F(x) - F(y), \forall x, y \in D$.

Theorem 18 (triparametric family)**.** *Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a sufficiently differentiable function in an open convex set D and $\xi \in D$ a solution of the nonlinear system $F(x) = 0$. Let us suppose that $F'(x)$ is continuous and nonsingular at ξ and $x^{(0)}$ is an initial estimate close enough to ξ. Then, sequence $\{x^{(k)}\}_{k\geq 0}$ obtained by using expression* (1) *converges to ξ with order four, being its error equation*

$$e^{(k+1)} = (1 - \alpha - \gamma)C_2\left(C_2^2 - C_3\right)e^{(k)^4} + \mathcal{O}(e^{(k)^5}),$$

where $e^{(k)} = x^{(k)} - \xi$, $C_q = \frac{1}{q!}[F'(\xi)]^{-1}F^{(q)}(\xi)$ and $q = 2, 3, ...$

From this theorem, it follows that the new triparametric family has an order of convergence of four for any real or complex value of α, β and γ. However, convergence can be speed-up if only one parameter is held and the family is reduced to an uniparametric iterative scheme.

Theorem 19 (uniparametric family)**.** *Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a sufficiently differentiable function in an open convex set D and $\xi \in D$ a solution of the nonlinear system $F(x) = 0$. Let us suppose that $F'(x)$ is continuous and nonsingular at ξ and $x^{(0)}$ is an initial estimate close enough to ξ. Then, sequence $\{x^{(k)}\}_{k\geq 0}$ obtained by using expression* (1) *converges to ξ with order six, provided that $\beta = 1 + \alpha$ and $\gamma = 1 - \alpha$, being its error equation*

$$e^{(k+1)} = \left(6C_2^5 - 7C_2^3C_3 + C_2C_3^2\right)e^{(k)^6} + \mathcal{O}(e^{(k)^7}),$$

where $e^{(k)} = x^{(k)} - \xi$, $C_q = \frac{1}{q!}[F'(\xi)]^{-1}F^{(q)}(\xi)$ and $q = 2, 3, ...$

From this theorem, it follows that if we only hold α in (1), the new triparametric family is reduced to an uniparametric family with an order of convergence of six, for any real or complex value of α, as long as $\beta = 1 + \alpha$ and $\gamma = 1 - \alpha$.
So, the iterative expression of the new uniparametric family, dependent only of α and which we will call CMT(α) family, is defined as

$$\begin{cases} y^{(k)} = x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}) \\ z^{(k)} = y^{(k)} - [2[x^{(k)}, y^{(k)}; F] - F'(x^{(k)})]^{-1}F(y^{(k)}) \\ x^{(k+1)} = z^{(k)} - (\alpha I + (1+\alpha)u^{(k)} + (1-\alpha)v^{(k)})[F'(x^{(k)})]^{-1}F(z^{(k)}) \end{cases}, \tag{2}$$

where $u^{(k)} = I - [F'(x^{(k)})]^{-1}[x^{(k)}, y^{(k)}; F]$, $v^{(k)} = [x^{(k)}, y^{(k)}; F]^{-1}F'(x^{(k)})$, $k = 0, 1, 2, ...$, and α is an arbitrary parameter.

3 Numerical results

In this section, we perform several numerical tests to illustrate the efficiency and stability of the presented family. We consider two members of the family as representatives. One of them is for $\alpha = 0$, whose value is inside the stability region of the parameter spaces shown in [4], that is, it is in the red area. The other member is for $\alpha = 200$, whose value is outside the stability region of the same parameter spaces, located in the black area. These methods are applied on two nonlinear test systems, whose expressions and corresponding roots are shown in Table 1.

Table 1: Nonlinear test systems and corresponding roots.

Nonlinear test system	Roots
$F_1(x_1, x_2) = (e^{x_1}e^{x_2} + x_1\cos(x_2), x_1 + x_2 - 1)$	$\xi \approx (3.4706, -2.4706)^T$
$F_2(x_1, x_2) = \left(\ln(x_1^2) - 2\ln(\cos(x_2)), x_1 \tan\left(\frac{x_1}{\sqrt{2}} + x_2\right) - \sqrt{2}\right)$	$\xi \approx (0.9548, 6.5850)^T$

Thus, in Table 2 we show the numerical performance of CMT(0) for initial estimates very close to the solution ($x^{(0)} \approx \xi$). Also, we introduce a comparative analysis between this method and three others obtained from the literature: Newton of order 2 in [5], Ostrowski of order 4 in [6], and HMT of order 6 proposed in [7].
The calculations have been developed in Matlab R2020b programming package using variable precision arithmetics with 200 digits of mantissa. For each method, we analyze the number of iterations (iter) required to converge to the solution, so that the stopping criteria $||x^{(k+1)} - x^{(k)}|| < 10^{-100}$ or $||F(x^{(k+1)})|| < 10^{-100}$ are satisfied. Note that $||x^{(k+1)} - x^{(k)}||$ represents the error estimation between two consecutive iterations and $||F(x^{(k+1)})||$ is the residual error of the nonlinear test system.
To check the theoretical order of convergence of the methods, we calculate the approximate computational order of convergence (ACOC) given in [8]. In the numerical results, if the ACOC vector inputs do not stabilize their values throughout the iterative process, it is marked as '-'; and, if any of the methods used does not reach convergence in a maximum of 50 iterations, it is marked as 'nc'.
In Table 2, we notice that CMT(0) always converges to the solution even with fewer iterations than the other methods. But, what about the dependence of CMT(0) on initial estimations? To answer this question, we analyze this method for initial estimates near and far from the solution, that is, for $x^{(0)} = 2\xi$ and $x^{(0)} = 10\xi$, respectively. The results can be observed in Tables 3 and 4.
The results shown in Tables 3 and 4 are encouraging because we can notice that CMT(0) always converges to the solution in the two nonlinear test systems, regardless of the initial estimates used. Therefore, we verify this method is robust, according to the stability results shown in [4].
Now, we are going to analyze the CMT(200) method. Its numerical performance, for initial estimations very close to ($x^{(0)} \approx \xi$) and near to the solution ($x^{(0)} \approx 2\xi$), is shown in Tables 5 and 6.

Table 2: Numerical performance of CMT(0) and existing methods on test problems for $x^{(0)} \approx \xi$.

System	Method	$\|\|x^{(k+1)} - x^{(k)}\|\|$	$\|\|F(x^{(k+1)})\|\|$	iter	ACOC
F_1	CMT(0)	3.1879e-97	7.7869e-208	3	5.4332
$x^{(0)} = (3, -2)^T$	Newton	2.1273e-99	1.6685e-198	7	2
	Ostrowski	1.395e-49	3.7549e-115	4	2.6275
	HMT	2.0527e-31	7.0573e-187	4	5.7098
F_2	CMT(0)	1.8998e-92	3.2794e-207	5	5.701
$x^{(0)} = (1, 6)^T$	Newton	8.8873e-66	6.0249e-130	9	2.0531
	Ostrowski	1.09e-54	1.9374e-124	6	2.7599
	HMT	8.0798e-36	4.1348e-207	6	5.889

Table 3: Numerical performance of CMT(0) on test problems for $x^{(0)} \approx 2\xi$.

System	$x^{(0)}$	$\|\|x^{(k+1)} - x^{(k)}\|\|$	$\|\|F(x^{(k+1)})\|\|$	iter	ACOC
F_1	$(6, -4)^T$	6.4031e-71	8.0537e-173	5	3.1906
F_2	$(2, 12)^T$	6.6576e-68	1.4261e-166	6	3.6518

Table 6: Numerical performance of CMT(200) on test problems for $x^{(0)} \approx 2\xi$.

System	$x^{(0)}$	$\|\|x^{(k+1)} - x^{(k)}\|\|$	$\|\|F(x^{(k+1)})\|\|$	iter	ACOC
F_1	$(6, -4)^T$	nc	nc	nc	nc
F_2	$(2, 12)^T$	nc	nc	nc	nc

Note that the results shown in Tables 5 and 6 also corroborate the stability analysis performed in [4]. The CMT(200) presents convergence problems even for estimates very close to the root ($x^{(0)} \approx \xi$), this method does not converge to the solution in one of two cases. Furthermore, for estimations near to the root ($x^{(0)} \approx 2\xi$), it does not converge to the solution in all cases, establishing a dependency on the initial estimates used. Therefore, the instability of this method is verified.

4 Conclusions

A highly efficient family of iterative methods CMT(α) has been designed to solve nonlinear systems. This family has an excellent numerical performance considering stable members as representatives. Numerical experiments confirm the theoretical results. The order of convergence is verified by the ACOC, which is close to 6. In general, this family has low errors and number of iterations to converge to the solution.
The method for $\alpha = 0$, value inside the stability region of the parameter spaces referred to in this

Table 4: Numerical performance of CMT(0) on test problems for $x^{(0)} \approx 10\xi$.

System	$x^{(0)}$	$\|\|x^{(k+1)} - x^{(k)}\|\|$	$\|\|F(x^{(k+1)})\|\|$	iter	ACOC
F_1	$(30, -20)^T$	6.7666e-41	3.4488e-113	4	6.4467
F_2	$(10, 60)^T$	3.9e-70	5.5314e-172	48	3.4528

Table 5: Numerical performance of CMT(200) on test problems for $x^{(0)} \approx \xi$.

System	$x^{(0)}$	$\|\|x^{(k+1)} - x^{(k)}\|\|$	$\|\|F(x^{(k+1)})\|\|$	iter	ACOC
F_1	$(3, -2)^T$	8.1881e-94	1.5574e-207	4	2.5252
F_2	$(1, 6)^T$	nc	nc	nc	nc

manuscript, proved to be robust. The method for $\alpha = 200$, value outside the stability region of the same parameter spaces, proved to be unstable and cannot converge to the solution according to the initial estimate and the nonlinear system used.

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