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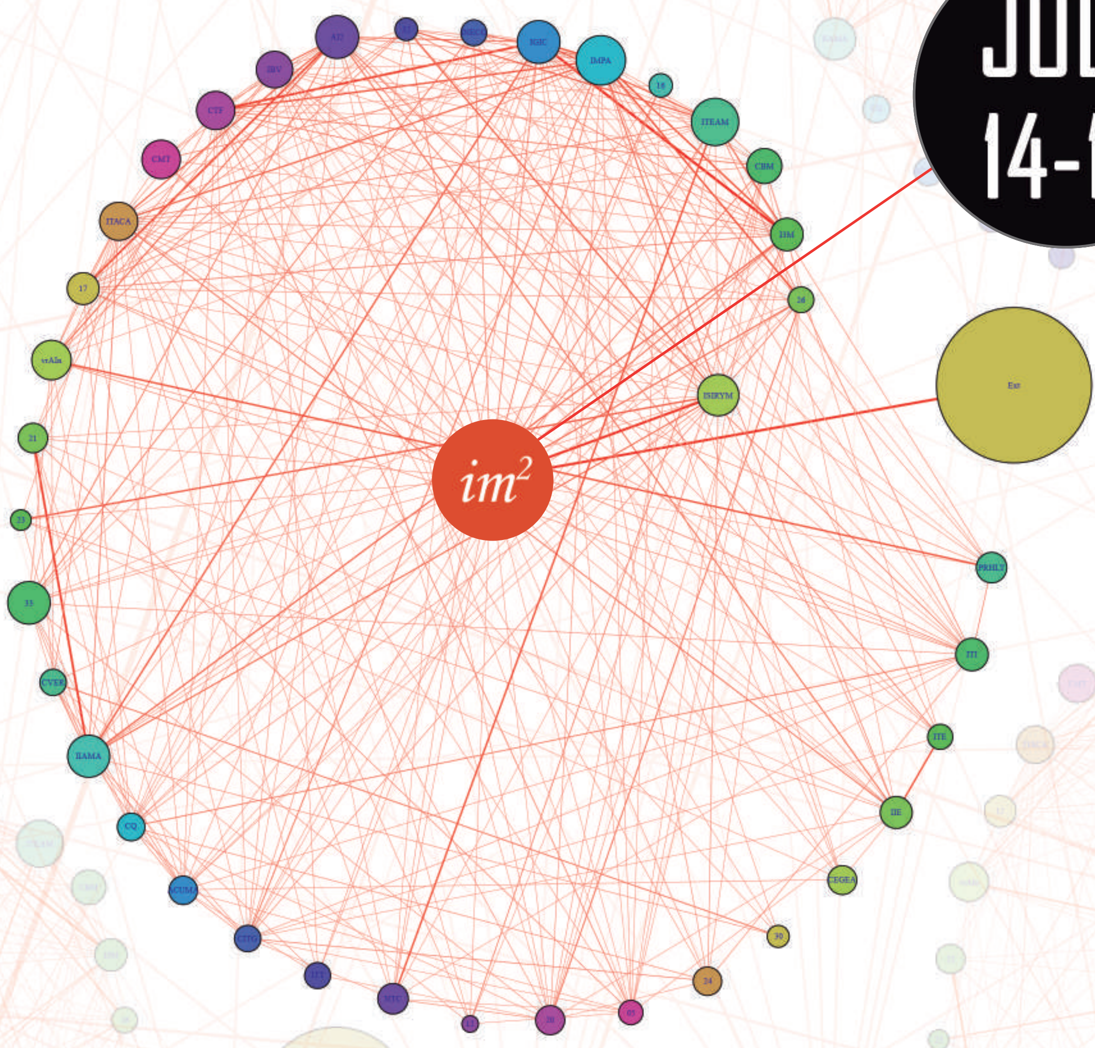

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Contents

8

Solving random free boundary problems of Stefan type

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1 Introduction

Free boundary problems describe several phenomena in nature, engineering and society, among others melting and freezing problems [1]. In these problems apart from determining the unknown function of the diffusion partial differential equation, we have an additional challenge concerning the calculus of evolution for the unknown moving boundary. In order to fit more realistically these type of problems, in this work we extend them into a random scenario using the mean square (m.s.) random calculus. We introduce *known* uncertainty, via considering random variables and stochastic processes following a certain probability distribution and depending on a finite degree of randomness [2, p.37]. In this work we study the following semi-infinite single-phase random melting problem for which the corresponding deterministic problem has an available exact solution [3, Chpts. 1 & 3]:

$$\frac{\partial T(x,t,\omega)}{\partial t} = D(\omega)\,\frac{\partial^2 T(x,t,\omega)}{\partial x^2}\,, \quad D(\omega) = \frac{\kappa(\omega)}{c_p(\omega)\,\rho(\omega)}, \qquad 0 < x = x(t,\omega) < s(t,\omega), \quad \omega \in \Omega\,, \tag{1}$$

with the following random boundary and initial conditions

$$T(0,t,\omega) = T_{\mathrm{w}}(\omega)\,, \quad t>0\,,\ \omega\in\Omega \quad \text{(wall temperature)}, \tag{2}$$

$$T(s(t,\omega),t,\omega) = T_{\mathrm{m}}(\omega)\,, \quad t>0\,,\ \omega\in\Omega \quad \text{(melting front temperature)}, \tag{3}$$

$$T(x,0,\omega) = T_{\mathrm{m}}(\omega), \quad x>0\,,\omega\in\Omega \quad \text{(initial temperature)}, \tag{4}$$

$$s(0,\omega) = 0\,, \qquad \omega\in\Omega \quad \text{(initial position of the interface)}, \tag{5}$$

and the velocity of the 2-stochastic process (2-s.p.) interface $s(t,\omega)$ is stated by a random Stefan's condition:

$$\frac{\mathrm{d}s(t,\omega)}{\mathrm{d}t} = -Q(\omega)\,\left.\frac{\partial T(s(t,\omega),t,\omega)}{\partial x}\right|_{x\to s(t,\omega)^-}\,, \quad Q(\omega) = \frac{\kappa(\omega)}{L(\omega)\,\rho(\omega)}\,,\ \omega\in\Omega\,. \tag{6}$$

Here the unknown 2-s.p. $T(x,t,\omega)$, $\omega\in\Omega$, $0<x<s(t,\omega)$, $t>0$, represents the temperature of the material in the liquid phase, $D(\omega)>0$ in (1) represents the diffusivity random variable (r.v.) involving the thermal conductivity r.v. $\kappa(\omega)>0$, the specific heat r.v. $c_p(\omega)>0$ and the density

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r.v. of the material $\rho(\omega) > 0$. The r.v. $Q(\omega) > 0$ appearing in the Stefan condition (6) involves the latent heat of fusion r.v. of the phase change material $L(\omega) > 0$ and the r.v.'s $\kappa(\omega) > 0$ and $\rho(\omega) > 0$. Since our purpose is numerical we assume a realistic random framework. In our models the involved 2-s.p.'s $T(x,t,\omega)$, $C(x,t,\omega)$ and $s(t,\omega)$ are defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and have p degrees of randomness [2, p.37], i.e., they only depend on a finite number p of random variables (r.v.'s)

$$g(x,t,\omega) = g\left(x,t,B_1(\omega),B_2(\omega),\dots,B_p(\omega)\right) , \tag{7}$$

where

$$\left.\begin{array}{l} B_i(\omega)\,,\ 1 \le i \le p\,, \quad \text{are mutually independent r.v.'s,} \\ g \text{ is a differential real function of the variables } x,\ t. \end{array}\right\} \tag{8}$$

For the treatment of the random moving boundary $s(t,\omega)$ we propose a boundary immobilization formulation or random front-fixing method based on a transformation of the original random problem

$$z = \frac{x(t,\omega)}{s(t,\omega)}\,, \quad \omega \in \Omega, \quad t > 0\,, \tag{9}$$

where z becomes the deterministic spatial variable of the immobilized random boundary problem. The new dependent variable

$$u(z,t,\omega) = T(x(t,\omega),t,\omega)\,, \quad \omega \in \Omega\,, \tag{10}$$

is the solution s.p. of the random transformed problem

$$D(\omega)\,\frac{1}{s^2(t,\omega)}\,\frac{\partial^2 u(z,t,\omega)}{\partial z^2} + z\,\frac{s'(t,\omega)}{s(t,\omega)}\,\frac{\partial u(z,t,\omega)}{\partial z} = \frac{\partial u(z,t,\omega)}{\partial t}\,, \quad 0 < z < 1,\ t > 0\,,\ \omega \in \Omega\,, \tag{11}$$

$$u(0,t,\omega) = T_{\mathrm{w}}(\omega)\,, \quad t > 0\,, \quad \omega \in \Omega, \tag{12}$$

$$u(1,t,\omega) = T_{\mathrm{m}}(\omega)\,, \quad t > 0\,, \quad \omega \in \Omega, \tag{13}$$

$$s(0,\omega) = 0\,, \qquad \omega \in \Omega, \tag{14}$$

$$s'(t,\omega) = -\frac{Q(\omega)}{s(t,\omega)}\,\left.\frac{\partial u(z,t,\omega)}{\partial z}\right|_{z \to 1^-}\,, \quad t > 0, \quad \omega \in \Omega, \tag{15}$$

where $s'(t,\omega)$ denotes the first mean square derivative $\frac{\mathrm{d}s(t,\omega)}{\mathrm{d}t}$, $\omega \in \Omega$. The mean square operational calculus developed in (11) and (15) is legitimated when

$$\frac{\partial^2 u(z,t,\cdot)}{\partial z^2}\,, \quad \frac{\partial u(z,t,\cdot)}{\partial z}\,, \quad \frac{s'(t,\cdot)}{s(t,\cdot)}\,, \quad \frac{1}{s(t,\cdot)}\,, \text{ and } \frac{1}{s^2(t,\cdot)} \tag{16}$$

lie in $L_4(\Omega)$, see [4, Sec. 3].

With the immobilised boundary we can use a random finite difference method [5] constructing random difference schemes for both unknowns the temperature s.p. and the melting interface. Both difference schemes will be executed simultaneously because the melting interface is used to compute the temperature. Let us consider the uniform partition of the spatial domain $[0,1]$ taking a step size h in order to obtain equally spaced points $z_i = ih$, $0 \le i \le M$, such that $Mh = 1$. For a fixed time τ and a small initial time $t^0 > 0$, we take a step size k and $N+1$ intermediate time levels are generated $t^n = nk + t^0$, $0 \le n \le N$, with $\tau = Nk + t^0$.

The random difference scheme for determining the approximation $u_i^n(\omega) = u^n(z_i, t^n, \omega)$ to the unknown s.p. temperature $u(z,t,\omega)$, $\omega \in \Omega$, is given by

$$\left.\begin{array}{r} u_i^{n+1}(\omega) = a_i^n(\omega)\, u_{i-1}^n(\omega) + b^n(\omega)\, u_i^n(\omega) + c_i^n(\omega)\, u_{i+1}^n\,, \quad \omega \in \Omega\,, \\ 1 \le i \le M-1,\ 0 \le n \le N-1\,, \\ \\ u_0^n(\omega) = T_{\mathrm{w}}(\omega)\,, \quad u_M^n(\omega) = T_{\mathrm{m}}(\omega)\,, \quad 0 \le n \le N\,, \\ \\ u_i^0(\omega) = \frac{T_{\mathrm{m}}(\omega) - T_{\mathrm{w}}(\omega)}{\mathrm{erf}(\beta)}\, \mathrm{erf}(\beta\, z_i) + T_{\mathrm{w}}(\omega)\,, \quad 0 \le i \le M\,. \end{array}\right\} \tag{17}$$

with the random coefficients

$$\left.\begin{array}{l} a_i^n(\omega) = \dfrac{k}{h^2 (s^n(\omega))^2}\left(D(\omega) + \dfrac{Q(\omega)\, \Delta^n(\omega)}{4} z_i\right) \\ \\ b^n(\omega) = 1 - \dfrac{2kD(\omega)}{h^2\,(s^n(\omega))^2} \\ \\ c_i^n(\omega) = \dfrac{k}{h^2 (s^n(\omega))^2}\left(D(\omega) - \dfrac{Q(\omega)\, \Delta^n(\omega)}{4} z_i\right) \end{array}\right\} \quad 1 \le i \le M-1, \quad 0 \le n \le N-1\,, \tag{18}$$

being $\Delta^n(\omega) = 3u_M^n(\omega) - 4u_{M-1}^n(\omega) + u_{M-2}^n(\omega)$. The random difference scheme for determining the approximation $s^n(\omega) = s(t^n, \omega)$ to the melting interface s.p. $s(t,\omega)$, $\omega \in \Omega$ takes the form

$$\left.\begin{array}{rcl} s^{n+1}(\omega) & = & s^n(\omega) - \dfrac{k\ Q(\omega)\ \Delta^n(\omega)}{s^n(\omega)\, 2h}\,, \quad 0 \le n \le N-1\,, \\ \\ s^0(\omega) & = & 2\,\beta(\omega)\,\sqrt{D(\omega)\, t^0}\,, \quad t^0 > 0\,, \omega \in \Omega\,, \\ \\ \beta(\omega)\, \mathrm{e}^{\beta(\omega)^2} \mathrm{erf}(\beta(\omega)) & = & \frac{Q(\omega)\,(T_{\mathrm{w}}(\omega) - T_{\mathrm{m}}(\omega))}{D(\omega)\,\sqrt{\pi}}\,. \end{array}\right\} \tag{19}$$

For small enough values of the step-size h together with the hypothesis

$$\frac{k}{h^2} < 2\, t^0\, \beta_{\min}^2\,, \tag{20}$$

where $\beta_{\min} = \min\{\beta(\omega) : \omega \in \Omega\}$, one guarantees the positivity and stability of the solution s.p.'s of the random difference schemes (17)–(18) and the time increasing behaviour of the melting interface s.p. obtained from (19).

In order to compute the mean and the standard deviation of the approximated solutions from (17)–(19) firstly we need to overcome the trouble of solving the random non-linear equation appearing on (19). Then we use a Monte Carlo technique taking a number K of realizations and solve the corresponding sampling deterministic non-linear equations associated. Each sampled solution $\beta(\omega_K)$ will be taken in the difference scheme (19) as well as a number K of realizations of the random data involved in (17)–(19) according to their probability distributions. Finally, the K sampling deterministic difference schemes associated to (17)–(19) will be solved and the mean and the standard deviation of the K results can be computed. Now in order to undo the variable change for computing the mean and the standard deviation of the solution s.p. $T(x,t,\omega)$ of (1) we use the transformation (9) which allows us to compute the mean of the r.v. $x(t,\cdot)$ at a fixed time t,

$$\mu[x(t,\omega)] = z\ \mu[s(t,\omega)]\,, \quad 0 \le z \le 1\,. \tag{21}$$

Then the mean of the temperature s.p. above computed $\mu[u(z,t,\omega)]$ is assigned to the mean of the space variable $\mu[x(t,\omega)]$ given by (21).

In order to illustrate and validate the random solid-liquid phase change simulation results obtained in our study, we are going to consider a block of ice of negligible thickness. The data taken have been considered mutually independent and truncated r.v.'s., see Table 1.

T_{w}	10°C
T_{m}	0°C
Thermal Conductivity ($\kappa(\omega)$)	$\kappa(\omega) \sim N_{[0.5,0.7]}(0.60, 0.10)$ W/m^{o}C
Density of the liquid (ρ)	1 kg/l
Specific heat (c_p)	4.1868 J/g°C
$D(\omega) = \dfrac{\kappa(\omega)}{c_p\,\rho}$	$D(\omega) = 14.3308\,\kappa(\omega)$ mm^2/min
Latent heat of fusion ($L(\omega)$)	$L(\omega) \sim N_{[0.31,0.35]}(0.33, 0.02)$ KJ/g
$Q(\omega) = \dfrac{\kappa(\omega)}{L(\omega)\,\rho}$	$Q(\omega) = 6\,\dfrac{\kappa(\omega)}{L(\omega)}$ mm^2/°C min

Table 1: Thermophysical properties of water and other data of the example.

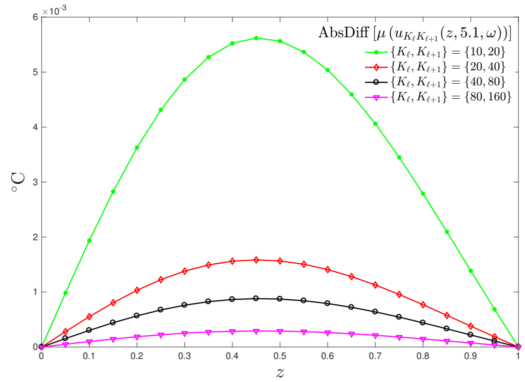
The study of the numerical convergence of these approximations has been treated by means of the analysis of their absolute errors in two stages at a fixed time τ. Firstly, we have fixed the step-sizes (h, k) verifying the sufficient stability condition (20) and we have varied the number K of Monte Carlo realizations comparing their absolute differences, AbsDiff, between two successive realizations $\{K_\ell, K_{\ell+1}\}$ using the following expressions

$$\begin{aligned}
\mathrm{AbsDiff}\left[\mu\left(u_{K_\ell K_{\ell+1}}(z_i,\tau,\omega)\right)\right] &= \left|\mathrm{AbsErr}\left[\mu\left(u_{K_{\ell+1}}(z_i,\tau,\omega)\right)\right] - \mathrm{AbsErr}\left[\mu\left(u_{K_\ell}(z_i,\tau,\omega)\right)\right]\right|,\\
\mathrm{AbsDiff}\left[\sigma\left(u_{K_\ell K_{\ell+1}}(z_i,\tau,\omega)\right)\right] &= \left|\mathrm{AbsErr}\left[\sigma\left(u_{K_{\ell+1}}(z_i,\tau,\omega)\right)\right] - \mathrm{AbsErr}\left[\sigma\left(u_{K_\ell}(z_i,\tau,\omega)\right)\right]\right|,\\
\mathrm{AbsDiff}\left[\mu\left(s_{K_\ell K_{\ell+1}}(t^n,\omega)\right)\right] &= \left|\mathrm{AbsErr}\left[\mu\left(s_{K_{\ell+1}}(t^n,\omega)\right)\right] - \mathrm{AbsErr}\left[\mu\left(s_{K_\ell}(t^n,\omega)\right)\right]\right|,\\
\mathrm{AbsDiff}\left[\sigma\left(s_{K_\ell K_{\ell+1}}(t^n,\omega)\right)\right] &= \left|\mathrm{AbsErr}\left[\sigma\left(s_{K_{\ell+1}}(t^n,\omega)\right)\right] - \mathrm{AbsErr}\left[\sigma\left(s_{K_\ell}(t^n,\omega)\right)\right]\right|,
\end{aligned} \tag{22}$$

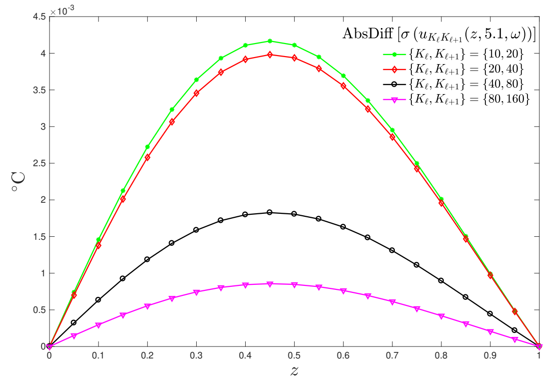
where AbsErr represents the absolute error of the mean and the standard deviation between the exact values, $u(z_i, \tau, \omega)$ and $s(t^n, \omega)$, and the approximated ones denoted by $u_K(z_i, \tau, \omega)$ and $s_K(t^n, \omega)$. Figure 1 shows how the successive absolute differences (22) decrease as the number of Monte Carlo realizations $K_\ell \in \{10, 20, 40, 80, 160\}$ increases for the fixed step-sizes $(h, k) = (0.05, 8\mathrm{e}-04)$. In the second stage about the study of the convergence of the approximations to the both statistical moments, we have taken a fixed number of Monte Carlo realizations K, $K = 1280$, and we have refined the step-sizes (h, k) according to the stability condition (20). The approximations getting better due to the decreasing of the absolute errors as step-sizes decreasing up to the values $(h, k) = (0.025, 2\mathrm{e}-04)$. Table 2 collects the maximum value for these absolute errors.

2 Conclusions and Future work

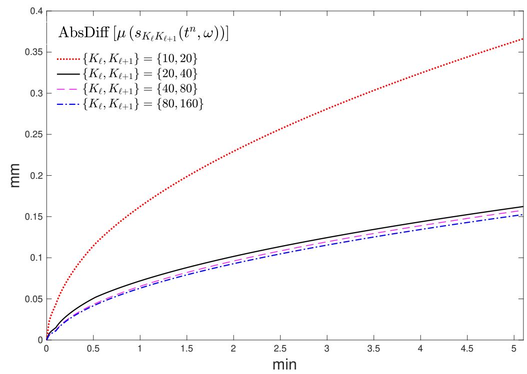
In this work a random free boundary problem has been addressed from the m.s. calculus point of view for the first time to our knowledge. The methodology used combines a random front-fixing method, random finite difference schemes and Monte Carlo technique. The random scheme combined with the Monte Carlo method solves the computational problem associated with random iterative methods as it avoids collapsing in the calculation of symbolic expressions to few temporary steps. In this way, it is possible the computation of the mean and the standard deviation of



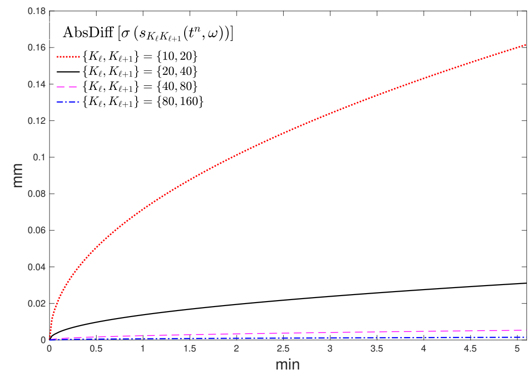
(a) AbsDiff(Mean of temperature s.p.)



(b) AbsDiff(S.Deviation of temperature s.p.)



(c) AbsDiff(Mean of melting interface s.p.)



(d) AbsDiff(S.Deviation of melting interface s.p.)

Figure 1: Absolute differences over the $\tau = 5.1$ minutes for both statistical moments of the approximations s.p. between two successive realizations $\{K_\ell, K_{\ell+1}\}$, $K_\ell \in \{10, 20, 40, 80, 160\}$. The step-sizes $(h, k) = (0.05, 8\text{e}-04)$ are fixed and $t^n = t^0 + nk$, $0 \leq n \leq N = 6250$ in $[t^0 = 0.1, \tau = 5.1]$.

(h,k)	(M,N)	$\|\text{AbsErr}[\mu(u_K(z_i,\tau,\omega))]\|_\infty$ °C	$\|\text{AbsErr}[\sigma(u_K(z_i,\tau,\omega))]\|_\infty$ °C
$(0.1, 3.125\text{e}-03)$	$(10, 1600)$	$1.9464\text{e}-02$	$1.1350\text{e}-03$
$(0.05, 8.0\text{e}-04)$	$(20, 6250)$	$5.7560\text{e}-03$	$5.4689\text{e}-04$
$(0.025, 2.0\text{e}-04)$	$(40, 25000)$	$2.2727\text{e}-03$	$4.0980\text{e}-04$
(h,k)	(M,N)	$\|\text{AbsErr}[\mu(s_K(t^n,\omega))]\|_\infty$ mm	$\|\text{AbsErr}[\sigma(s_K(t^n,\omega))]\|_\infty$ mm
$(0.1, 3.125\text{e}-03)$	$(10, 1600)$	$1.3208\text{e}-01$	$1.9869\text{e}-02$
$(0.05, 8\text{e}-04)$	$(20, 6250)$	$2.0884\text{e}-02$	$1.4453e-02$
$(0.025, 2\text{e}-04)$	$(40, 25000)$	$7.4025\text{e}-03$	$1.3079\text{e}-02$

Table 2: Maximum values of the absolute errors for both statistical moments of the approximate temperature s.p. and the approximate melting interfaces s.p. from $t^0 = 0.1$ up to $\tau = 5.1$ minutes. The step-sizes (h,k) are refined while the number of the Monte Carlo realizations is the fixed value $K = 1280$. The values M and N are the spatial and temporal levels, respectively.

the approximate temperature s.p. and the approximate melting interface s.p. The numerical comparisons with the statistical moments of the exact solutions for the temperature and the melting interface allow to check the reability of the approximations computed. This method is suitable to be used to solve other types of Stefan problems.

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