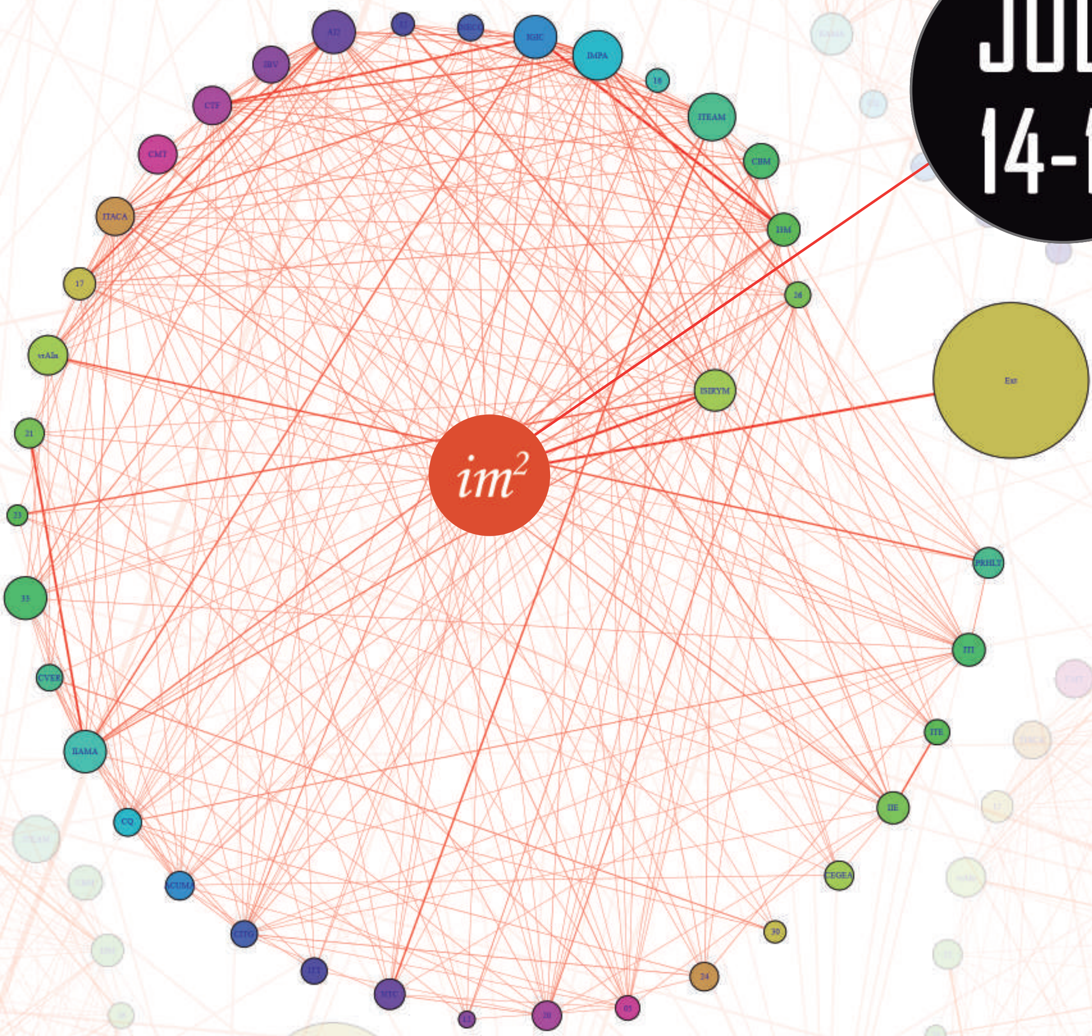
MODELLING FOR ENGINEERING & HUMAN BEHAVIOUR 2021

JULY 14-16



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I.S.B.N.: 978-84-09-36287-5

November 30th, 2021
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Edited by: I.U. de Matemàtica Multidisciplinar, Universitat Politècnica de València.
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Contents

31

Parametric family of root-finding iterative methods

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1 Introduction

A large number of problems in Science and Engineering require the solution of nonlinear equations or systems. In general, these equations cannot be solved analytically and we must resort to iterative methods to approximate their solutions. With the increasing speed of computers, numerical techniques have become indispensable for scientists and engineers. The principle of these methods is to approach the solution of nonlinear equations, of the form $f(x) = 0$, through a sequence of iterations, starting from an initial estimation. The most known and widely used method to solve nonlinear equations is Newton's scheme, whose iterative expression is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, k = 0, 1, 2, ... \tag{1}$$

and presents quadratic convergence. This method requires two functional evaluations ($d = 2$), one of the function and one of its first derivative, per iteration. The Kung-Traub conjecture [4] states that an iterative method without memory for finding a simple zero of a scalar equation is optimal if its order of convergence is equal to 2^{d-1}. Therefore, Newton's method is optimal.
Research interest in iterative multipoint methods has increased due to the limitations of one-step methods, which need higher-order derivatives to increase the order of convergence. We aim to design a new multipoint fixed point class, without memory, that improves or bring together the existing ones, as for example appears in [7, 8], in different areas such as computational efficiency, stability and convergence order. We present a family of multipoint parametric iterative methods in Section 2, whose order of convergence is four, using the weight function technique.

2 Methods

In our research, we have used the weight function [8] and composition techniques to design a new parametric family of two-steps, whose members reach the fourth-order of convergence. This technique allows to significantly increase the order of convergence without excessively increasing the number of functional evaluations. In addition to the order of convergence, we are interested in methods that are less demanding on the initial estimate. We analyze the stability, using tools of complex dynamics [3, 4], and we select the values of the free parameter α which converge to

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the roots with a wider set of initial estimations. We can choose the most efficient members of the family, by studying the behaviour of the fixed points of the operator associated [2] to the iterative method on quadratic polynomials, which, when applied to the nonlinear function to be solved, provides us with important information about its stability and reliability.

Theorem 20. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function in an open interval I and $x^* \in I$ a simple root of equation $f(x) = 0$. Let $G(\eta)$ be a real function satisfying $G(1) = 1$, $G'(1) = \frac{-3}{4}$, $G''(1) = \frac{9}{4}$, and $|G'''(1)| < +\infty$. If $\gamma = \frac{2}{3}$ and we choose an initial approximation x_0 close enough to x^*, then iterative family defined by*

$$
\begin{aligned}
y_k &= x_k - \gamma \frac{f(x_k)}{f'(x_k)}, \\
x_{k+1} &= x_k - G(\eta)\frac{f(x_k)}{f'(x_k)}, \qquad k = 0, 1, 2, ...,
\end{aligned} \tag{2}
$$

satisfying the error equation below:

$$e_{k+1} = \Big(\Big(5 + \frac{32G'''(1)}{81}\Big)c_2^3 - c_2c_3 + \frac{c_4}{9}\Big)e_k^4 + O(e_k^5),$$

where $c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)}$, $\eta = \frac{f'(y_k)}{f'(x_k)}$, $k = 2, 3, ...$ and $e_k = x_k - \alpha$ and therefore converges to x^* with order of convergence four.

Belonging to (1.2) class, we introduce a new parametric family (3), with a given function $G(\eta)$, which verifies the conditions set out in Theorem 1,

$$
\begin{aligned}
y_k &= x_k - \frac{2}{3}\frac{f(x_k)}{f'(x_k)}, \\
x_{k+1} &= x_k - \left(1 - \frac{3}{4}(\eta - 1) + \frac{9}{8}(\eta - 1)^2 + \alpha(\eta - 1)^3\right)\frac{f(x_k)}{f'(x_k)}, \qquad k = 0, 1, 2, \ldots,
\end{aligned} \tag{3}
$$

where α is a free parameter and $\eta = \dfrac{f'(y_k)}{f'(x_k)}$.

Furthermore, we analyse the dynamical behaviour of the methods, with the representation of basins of attraction. In order to study the dynamic behaviour of these iterative methods, it is necessary to recall a few concepts: Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function, R is obtained by applying an iterative method on a quadratic polynomial. The orbit of point $z_0 \in \hat{\mathbb{C}}$ is defined as the successive application of the operator R on that point, such that it is determined by the set $[1, 2]$:

$$\{z_0, R(z_0), R^2(z_0), \ldots, R^n(z_0), \ldots\},$$

where $R^n(z_0)$ is the application of the R operator for n iterations.
Thus, a fixed point $z^F \in \hat{\mathbb{C}}$ is one which is kept invariant by the operator R, it is one that satisfies the equation $R(z_F) = z_F$. All the roots of the quadratic polynomial are fixed points of the operator R.

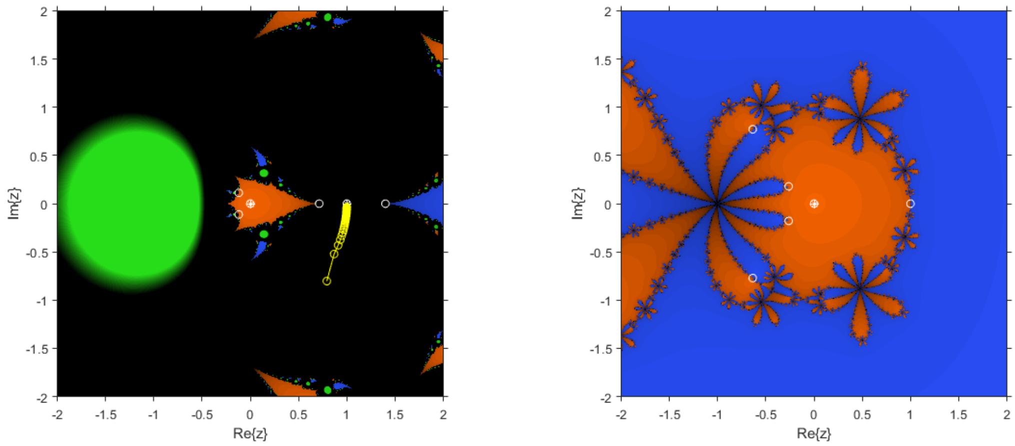
The dynamic stability of fixed points can be classified as follows:

- Attractive if $|R'(z^F)| < 1$,
- Superattractive if $R'(z^F) = 0$,
- Repulsive if $|R'(z^F)| > 1$,
- Parabolic or Neutral if $|R'(z^F)| = 1$.

The basins of attraction [5] determine the final state of the orbit of any point in the complex plane after successive application of the operator R. We define the basin of attraction of an attracting fixed point $z^F \in \hat{\mathbb{C}}$ is the set of preimages of any order that meets

$$\mathcal{A}(z^F) = \{z_0 \in \hat{\mathbb{C}} : R^n(z_0) \to z^F, n \to +\infty\}.$$

It can be proven that, the α parameter values -50, -40, -20+45i and -16-45i give us unstable behaviour and the values 1, -20i, 8+20i and -4.5+10i stable behaviour, as it is numerically checked in the following section.



(a) $\alpha = -40$ (b) $\alpha = 1$

Figure 1: Dynamical planes

It can be observed in Figure 1(a) that corresponds to an unstable member of family (1.3) that the basins of attraction of the roots of the polynomial (in orange and blue color in the figure) are very narrow. Moreover, with an initial estimation in the black area, the method converges to $z = 1$ that is an attracting strange fixed point. However, when an stable element of family (1.3) is selected, the only basins of attraction observed are those corresponding to the roots.

3 Results

In this section, we show the behaviour of the new method defined in (3), which is a particular family of the method defined in (2), is compared with different values of the free parameter α and displayed in Table 1 and 2. When the parameter α is within the stable zone, the method achieves better approximation with fewer iterations. Numerical computations have been carried out by using *MATLAB*ő R2019a, by using variable precision arithmetics with 1000 digits of mantissa, on a PC equipped with a *Intel*ő Core™ i5-5200U CPU 2.20GHz. We show the execution time, calculated with the command *cputime*. The stopping criterion used is $|x_{k+1} - x_k| + |f(x_{k+1})| < 10^{-200}$, and the approximated computational order of convergence

(ACOC) [6], has been also shown in the tables, whose expression is:

$$ACOC = \frac{\ln |(x_{k+1} - x_k)/(x_k - x_{k-1})|}{\ln |(x_k - x_{k-1})/(x_{k-1} - x_{k-2})|}, \quad k = 2, 3, \ldots$$

The nonlinear test functions are as follows:

1. $f_1(x) = sin^2 x - x^2 + 1$,
2. $f_2(x) = cosx - xe^x$.

α	$\lvert f(x_k)\rvert$	$\lvert x_{k+1} - x_k\rvert$	Solution	Iteration	ACOC	Time(s)
-50	7.0883e-1007	8.4884e-253	-1.4044	8	4.0	0.2516
-40	0	1.4357e-402	-1.4044	29	4.0	0.7766
-20+45i	-	-	-	-	-	-
-16-45i	2.4842e-1432	5.1013e-425	1.4044	10	4.0	0.4992
1	0	1.8974e-331	1.4044	6	4.0	0.2258
0-20i	7.0302e-1505	8.8753e-498	1.4044	7	4.0	0.2617
8+20i	1.2843e-1457	1.4709e-450	1.4044	7	4.0	0.2750
-4.5+10i	3.5457e-891	1.0501e-223	1.4044	6	4.0	0.3719

Table 1: $f_1(x) = sin^2 x - x^2 + 1, \quad x_0 = 2$

α	$\lvert f(x_k)\rvert$	$\lvert x_{k+1} - x_k\rvert$	Solution	Iteration	ACOC	Time(s)
-50	1.0082e-1007	4.6325e-399	-14.137	8	4.0	0.3758
-40	-	-	-	5	-	-
-20+45i	2.9787e-1009	2.2945e-324	- 17.278	12	4.0	0.4570
-16-45i	d	-	-	-	-	-
1	0	5.7578e-315	0.517	6	4.0	0.3391
0-20i	1.9236e-1366	6.3944e-359	0.517	7	4.0	0.3391
8+20i	5.1542e-1269	2.9387e-318	0.517	7	4.0	0.3305
-4.5+10i	3.0281e-1747	1.7704e-740	0.517	7	4.0	0.3852

Table 2: $f_2(x) = cosx - xe^x, \quad x_0 = 1$

4 Conclusions

We have presented a new fourth-order parametric family of iterative methods for solving nonlinear equation $f(x) = 0$. The class has been generated using composition and weight functions techniques. With the help of dynamical analysis we select the most stable methods by choosing some values of a free parameter. The described numerical examples allow us to confirm the theoretical results corresponding to the proposed convergence and stability. On the other hand, when the parameter alpha is unstable, the method needs more iterations to converge or fails to converge. These numerical tests confirm that the new family of methods is suitable for solving non-linear equations, when the adequate values of the free disposable parameter are used.

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