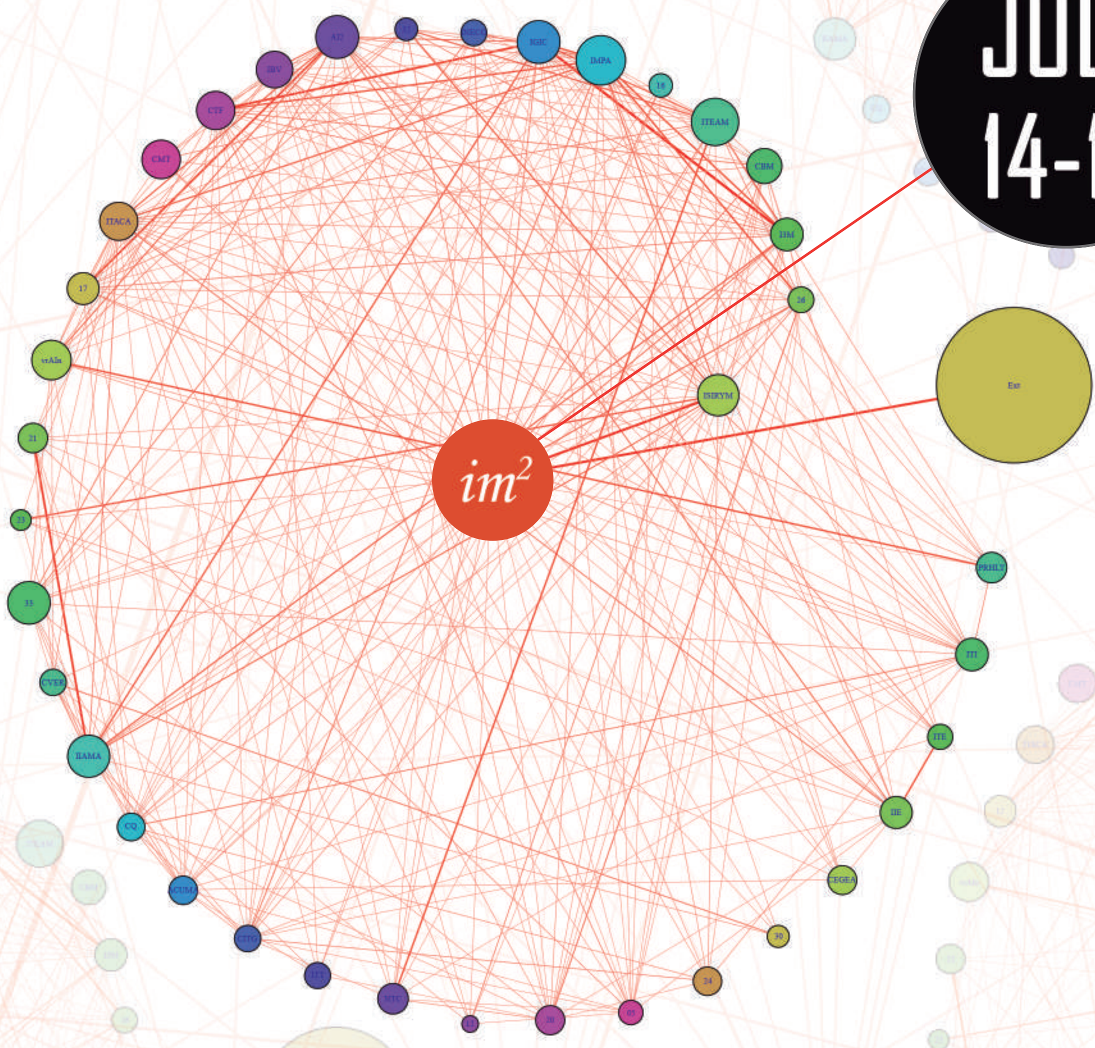
MODELLING FOR ENGINEERING & HUMAN BEHAVIOUR 2021

JULY 14-16



Edited by

Juan Ramón Torregrosa   Juan Carlos Cortés   Antonio Hervás   Antoni Vidal   Elena López-Navarro

UNIVERSITAT POLITÈCNICA DE VALÈNCIA

im2
Instituto Universitario
de Matemática Multidisciplinar

Modelling for Engineering & Human Behaviour 2021

València, July 14th-16th, 2021

This book includes the extended abstracts of papers presented at XXIII Edition of the Mathematical Modelling Conference Series at the Institute for Multidisciplinary Mathematics *Mathematical Modelling in Engineering & Human Behaviour*.

I.S.B.N.: 978-84-09-36287-5

November 30^{th}, 2021
Report any problems with this document to imm@imm.upv.es.

Edited by: I.U. de Matemàtica Multidisciplinar, Universitat Politècnica de València.
J.R. Torregrosa, J-C. Cortés, J. A. Hervás, A. Vidal-Ferràndiz and E. López-Navarro

im2
Instituto Universitario
de Matemática Multidisciplinar

Contents

Subdirect sums of matrices. Definitions, methodology and known results.

F. Pedroche[♮1]

(♮) Institut Universitari de Matemàtica Multidisciplinària, Universitat Politècnica de València
Camí de Vera s/n, 46022, València, España

Abstract

In this communication, we review the concept of subdirect sum of matrices that was introduced in 1999 as an extension of the usual sum of matrices (or as a variation of the direct sum of matrices by allowing overlapping of square blocks). This matrix operation appears naturally when studying matrix completion problems, numerical methods with overlapping blocks to solve linear systems of equations, applications of the finite element method to solve partial differential equations, etc. In this communication, we show the usual technique to analyse the properties of this matrix operation when it is applied to a particular class of matrices (e.g., M-matrices, Doubly Diagonally Dominant matrices, etc.) and the two common types of theoretical results that are being published until present times. Finally, we also comment on some new lines of research.

1 Introduction

It is known that iterative methods to solve linear systems of equations can be solved by using the classical methods of Gauss-Seidel and Jacobi. These methods (and others) can be also implemented by using block matrices and in some circumstances these blocks can show overlapping among them. In these cases (see, e.g., [3]) it is needed to handle sums of matrices that are not of the same size but overlap in an square block. For example, the block matrices

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \tag{1}$$

with A_{22} and B_{11} of size $k \times k$, can be summed in the form

$$C = \begin{bmatrix} A_{11} & A_{12} & O \\ A_{21} & A_{22} + B_{11} & B_{12} \\ O & B_{21} & B_{22} \end{bmatrix} \tag{2}$$

and this sum is called the *k-subdirect sum* (or simply the *subdirect sum*) of A and B, and it is denoted as $C = A \oplus_k B$.
Another application where these sums can appear is in the solution of the Poisson equation by using Finite Elements, in the context of stiffness matrices (see, e.g., [4]). The concept of *k-subdirect*

[1]pedroche@imm.upv.es

sum was introduced in 1999 by Fallat and Johnson in [10], although some results (not using the term *subdirect sum*) where known previously (see [9]).

2 Methodology

For some matrix classes one needs to put additional conditions on the entries of the matrices in order to establish a result about the subdirect sum of these classes. Sometimes these conditions only affect the overlapping blocks A_{22} and B_{11}. To manage these situations it is a standar procedure to use the following notation. Assuming that A and B are square matrices of sizes n_1 and n_2, respectively, it is useful to define the sets

$$\begin{aligned} S_1 &= \{1, 2, \ldots, n_1 - k\} \\ S_2 &= \{n_1 - k + 1, n_1 - k + 2, \ldots, n_1\} \\ S_3 &= \{n_1 + 1, n_1 + 2, \ldots, n\} \end{aligned} \tag{3}$$

and denoting $C = (c_{ij})$ the elements of $C = \oplus_k B$, and $t = n_1 - k$, one can write

$$c_{ij} = \begin{cases} a_{ij} & i \in S_1, \quad j \in S_1 \cup S_2 \\ 0 & i \in S_1, \quad j \in S_3 \\ a_{ij} & i \in S_2, \quad j \in S_1 \\ a_{ij} + b_{i-t,j-t} & i \in S_2, \quad j \in S_2 \\ b_{i-t,j-t} & i \in S_2, \quad j \in S_3 \\ 0 & i \in S_3, \quad j \in S_1 \\ b_{i-t,j-t} & i \in S_3, \quad j \in S_2 \cup S_3 \end{cases} \tag{4}$$

or, more graphically

$$C = \begin{array}{c} \begin{array}{ccc} \overleftrightarrow{\quad S_1 \quad} & \overleftrightarrow{\qquad\qquad S_2 \qquad\qquad} & \overleftrightarrow{\quad S_3 \quad} \end{array} \\ \left(\begin{array}{ccccccc} a_{11} & \cdots & a_{1p} & \cdots & a_{1,n_1} & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{p,1} & \cdots & a_{p,p} + b_{1,1} & \cdots & a_{p,n_1} + b_{1,n_1-t} & \cdots & b_{1,n-t} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{n_1,1} & \cdots & a_{n_1,p} + b_{n_1-t,1} & \cdots & a_{n_1,n_1} + b_{n_1-t,n_1-t} & \cdots & b_{n_1-t,n-t} \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & b_{n-t,1} & \cdots & b_{n-t,n_1-t} & \cdots & b_{n-t,n-t} \end{array} \right) \end{array} \begin{array}{l} \updownarrow S_1 \\ \\ \updownarrow S_2 \\ \\ \updownarrow S_3 \end{array}$$

Figure 1: Sets for the subdirect sum $C = A \oplus_k B$, with $t = n_1 - k$ and $p = t + 1$.

Given a particular matrix class, the questions that are usually addressed are the following.

1. If A and B belong to a class, does $A \oplus_1 B$ belong to the same class?

2. A matrix of the form

$$C = \begin{bmatrix} C_{11} & C_{12} & O \\ C_{21} & C_{22} & C_{23} \\ O & C_{32} & C_{33} \end{bmatrix}$$

   that belongs to a class, can be written as $A \oplus_1 B$ with A and B in the same class as C?

3. Question 1 with $k > 1$.

4. Question 2 with $k > 1$.

Clearly, when $k = 1$ things are simpler. For example, it is easy to prove that [10]

$$det(A \oplus_1 B) = det\, A_{11}\, det\, B + det\, A\, det\, B_{33}$$

Sometimes it is needed to put additional conditions on the diagonal terms of the overlapping. See, for example, [5], [6], [2].

3 Results

The most important results about subdirect sum are summarized below, in chronological order, by the year of publication. In each case we detail the class matrix that has been analysed (to see the particular results for each class we refer the reader to the original paper).

- Some positivity classes of matrices and M-matrices are studied in [10]. In particular it is shown that for $k > 1$ the subdirect sum of M-matrices is not in the class. We recall that A is a M-matrix when all the principal minors are positive and $a_{ij} \leq 0$ for $i \neq j$. They also study: positive definite matrices ($\mathbf{x}^T A\mathbf{x} > 0$) for all $\mathbf{x} \neq \mathbf{0}$); semi positive definite matrices (changing the strict inequality in the previous class for ≥ 0; P-matrices (all principal minors are positive); P_0-matrices (all principal minors are nonnegative);totally nonnegative matrices (all minors are nonnegative); completely positive matrices (of the form BB^T with $B \geq 0$); doubly nonnegative matrices (positive semidefinite and with all the terms nonnegative) and inverse of M-matrices.

- In [5] some conditions are shown such that the subdirect sum of inverse of M-matrices are in the class, jointly with related questions.

- In [6] the authors study the class of S-Strictly Diagonally Dominant matrices (A verifies $|a_{ii}| > \sum_{i \neq j} |a_{ij}|$ for all i, when $j \in S$ and another (intricate) condition when $j \notin S$)

- In [24] it is studied the class of doubly diagonally dominant matrices (the definition is a little bit intricate, see the paper for details) and related problems.

- In [12] the authors consider P-matrices that are also strictly diagonally dominant and they also answer question 4 for P_0 matrices

- In [25] the authors show conditions such that the subdirect sum of H-matrices is in the class, and related results.

- In [2] it is shown conditions such that the subdirect sum of Σ-strictly diagonally dominant matrices is in the class. The class of Σ-SDD matrices is a generalization of S-SDD matrices, and it is also a subclass of H-matrices.

- The subdirect sums of accretive, dissipative and Benzi-Golub matrices are treated in [13]. These matrix classes are based upon the decomposition of a complex matrix as a sum of a real matrix and an imaginary matrix combined with the definition of positive (semi)definiteness.

- In [1] the authors give conditions such that the questions 3 and 4 have positive answer for inverse-positive matrices. A matrix is inverse-positive when all the elements of its inverse are nonnegative.

- In [14] the authors give conditions such that such that the questions 3 and 4 have positive answer for inverse-positive matrices, generalizing the results of [1].
- In [21] it is shown some results about B-matrices and doubly B-matrices (that are subclasses of P-matrices).
- In [15] the authors we generalize the definition of subdirect sum to the set of all bounded linear operators on Hilbert spaces. They also deal with the class of all bounded linear operators with non-negative Moore–Penrose inverse. They obtain previous known results as corollaries of their results.
- In [17] the authors give conditions such that the subdirect sum of Nekrasov matrices is in the class. Nekrasov matrices are a generalization of SDD matrices.
- In [18] the authors give sufficient conditions such that the subdirect sum of two weakly chained diagonally dominant matrices is in the class. WCDD matrices are a subset of diagonally dominant matrices.
- In [8] it is shown some conditions such that the subdirect sums of SDD_1 matrices lays in the class. SDD_1 matrices are a generalization of SDD matrices.
- In [11] the authors show some sufficient conditions for the subdirect sum of QN-matrices to lay in the class. QN-matrices are a generalization of Nekrasov matrices. As a particular case they give results on Nekrasov matrices.
- In [20] the authors show some conditions such that the subdirect sums of p-norm strictly diagonally dominant matrices is in the class. The class of p-norm SDD was introduced in [16].
- In [19] the authors give several results about the subdirect sum of Doubly strictly diagonally dominant matrices and related results (e.g., the sum of an SDD matrix and a DSDD matrix, with some conditions).

4 Conclusions and future works

We have presented a literature review of the concept of subdirect sum of matrices that was introduced in [10] in 1999. We have shown that a broad range of matrix classes have been studied in the literature. A close look at the references show that up to 9 different groups of researches from 7 countries (China, India, Portugal, Russia, Serbia, Spain and USA) have been involved in these studies about subdirect sums.

It is important to remark that some of the matrix classes that have been studied (e.g., S-SDD, Σ-SDD, B-matrices, etc.) are related to the problem of localization of eigenvalues, see, e.g., [22]. Also, while most studies are centered on the question whether the subdirect sum of two matrices is in the class (or the reciprocal problem) there is one paper (see [15]) that extends the concept of subdirect sum to Hilbert spaces, opening new topics of research. Another new lines of research on this topic come related to its use in the discipline of complex networks. For example, the role of subdirect sums in problems of overlapping graphs in multilayer networks, its connection with the concept of simplicial complexes [7], network motifs [23], etc.

References

[1] Abad, M.F., Gassó, M.T., Torregrosa, J.R., Some results about inverse-positive matrices, *Applied Mathematics and Computation*, 218(1): 130–139, 2011.

[2] Bru, R., Cvetković, L., Kostić, V., Pedroche, F., Sums of Σ-Strictly diagonally dominant matrices, *Linear and Multilinear Algebra*, 58(1):75–78, 2010.

[3] Bru, R., Pedroche, F., Szyld, D. B., Additive Schwarz iterations for Markov chains, *SIAM Journal on Matrix Analysis and Applications*, 27(2):445–458, 2005.

[4] Bru, R., Pedroche, F., Szyld, D. B., Sumas subdirectas de M-matrices invertibles. XIX Congreso de ecuaciones diferenciales y aplicaciones. IX congreso de matemática aplicada. Actas- ISBN 84-689-7726-8. Leganés, Espanya. 2005.

[5] Bru, R., Pedroche, F., Szyld, D. B., Subdirect sums of nonsingular M-matrices and of their inverses, *Electron. J. Linear Algebra*, 13:162–174, 2005.

[6] Bru, R., Pedroche, F., Szyld, D. B., Subdirect sums of S-Strictly Diagonally Dominant matrices, *Electron. J. Linear Algebra,* 15: 201–209, 2006.

[7] Courtney, O. T., Bianconi, G., Generalized network structures: The configuration model and the canonical ensemble of simplicial complexes. *PHYSICAL REVIEW E* 93, 062311, 2016.

[8] Chen,X., Wang, Y., Subdirect sums of SDD_1 matrices *Journal of Mathematics*, vol. 2020, Article ID 3810423, 20 pages. 2020.

[9] Drew J.H., Johnson C.R., The No Long Odd Cycle Theorem for Completely Positive Matrices. In: Aldous D., Pemantle R. (eds) *Random Discrete Structures. The IMA Volumes in Mathematics and its Applications*, vol 76. Springer, New York, NY. 1996.

[10] Fallat, S. M., Johnson, C. R., Sub-direct sums and positivity classes of matrices. *Linear Algebra and its Applications*, 288:149–173, 1999.

[11] Gao, L., Huang, H., Li, C., Subdirect sums of QN-matrices, *Linear and Multilinear Algebra*, 68(8), 1605–1623. 2020

[12] Huang, T. Z., Mou, G. F., Tian, G. X., Li, Z., Wang, D., Subdirect sums of P (P0)-matrices and totally nonnegative matrices, *Linear Algebra and Its Applications* 429, 1730—1743, 2008

[13] Ikramov, K.D., Sub-direct sums and differences for certain matrix classes, *Computational Mathematics and Mathematical Physics*, 50(1), 7–11, 2010.

[14] Jose, S., Sivakumar, K.C., On inverse-positivity of sub-direct sums of matrices, *Linear Algebra and its Applications*, 439(6), 1670–1677, 2013.

[15] Jose, S.; Sivakumar, K.C., Sub-direct sum of operators on Hilbert spaces and nonnegative Moore–Penrose inverses, *Acta Scientiarum Mathematicarum* 81(1-2), 215–240. 2015.

[16] Kostić, V., On general principles of eigenvalue localizations via diagonal dominance. *Adv. Computat Math.* 41(1),55–75. 2015.

[17] Li, C., Liu, Q., Gao, L., Li, Y., Subdirect sums of Nekrasov matrices, *Linear and Multilinear Algebra*, 64(2):208–218, 2016.

[18] Li, C., Ma, R., Liu, Q., Li, Y. Subdirect sums of weakly chained diagonally dominant matrices, *Linear and Multilinear Algebra*, 65(6), 1220—1231. 2017.

[19] Li, Y., Chen, X., Liu, Y., Gao, L., Wang, Y., Subdirect Sums of Doubly Strictly Diagonally Dominant Matrices, *Journal of Mathematics*, Volume 2021, Article ID 6624695, 2021.

[20] Liu, Q., He, J., Gao, L., Li, C., Note on subdirect sums of SDD(p) matrices, *Linear and Multilinear Algebra*, (online) 2020

[21] Mendes Araújo, C., Torregrosa, J.R., Some results on B-matrices and doubly B-matrices, *Linear Algebra and Its Applications*, 459, 101–120. 2014.

[22] Peña, J.M., A class of P-matrices with applications to the localization of the eigenvalues of a real matrix, *SIAM J. Matrix Anal. Appl.* 22, 1027–1037. 2001.

[23] Ruzzenenti, F., Picciolo, F., Holme, P., Mastrandrea, R., Weighted network motifs as random walk patterns, arXiv:2012.01265v2. 2021.

[24] Zhu, Y. Huang, T. Z., Subdirect Sum of doubly diagonally dominant matrices, *The Electronic Journal of Linear Algebra*, 16(1), 171-–182, 2007.

[25] Zhu, Y., Huang, T., Liu, J. Subdirect Sums of H-matrices. *International Journal of Nonlinear Science*, 8(1), 50–58. 2009.