On the effect of the multidimensional weight functions on the stability of iterative processes ☆

Francisco I. Chicharro[a], Alicia Cordero[b], Neus Garrido[a,*], Juan R. Torregrosa[b]

[a] *Escuela Superior de Ingeniería y Tecnología, Universidad Internacional de La Rioja, Logroño, Spain*
[b] *Instituto de Matemáticas Multidisciplinar, Universitat Politècnica de València, València, Spain*

Abstract

In this work, we start from a family of iterative methods for solving nonlinear multidimensional problems, designed using the inclusion of a weight function on its iterative expression. A deep dynamical study of the family is carried out on polynomial systems by selecting different weight functions and comparing the results obtained in each case. This study shows the applicability of the multidimensional dynamical analysis in order to select the methods of the family with the best stability properties.

Keywords: Nonlinear systems; real multidimensional dynamics; stability; basin of attraction.

1. Introduction

Science, Technology, Engineering and Mathematics (STEM) is a novel term used to group a list of academic and research disciplines. Despite the solution of nonlinear equations and systems of nonlinear equations can be classified in the Mathematics subject, the amount of problems that can be solved covers the other three disciplines completely.

An approximation of the solution of the nonlinear problem $F(x) = 0$, where $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 1$, is a nonlinear vectorial function with n unknowns, can be obtained by means of iterative methods, when the analytical solution is not affordable. There are two main issues: the scalar case, for solving nonlinear equations where $n = 1$, and the vectorial case, for solving systems of nonlinear equations where $n > 1$.

Solving a nonlinear equation by means of iterative methods has been a widely discussed problem as can be seen in the overviews [1, 2], although the same is not true for nonlinear systems, which is the case discussed in this manuscript. The designed methods can be classified in terms of different criteria. One of them is the absence or existence of memory, that is, depending whether the method only needs the current iteration for obtaining the following one or it needs more than one previous iterations. Focused on iterative methods without memory, some interesting studies can be found in [3, 4, 5, 6, 7, 8, 9, 10, 11], where iterative schemes for nonlinear systems are designed with the aim of improving the order and the efficiency of Newton's method. For iterative methods with memory, highlighted references are [12, 13, 14]. In some papers, not only have new methods or families of iterative methods been designed, but dynamic studies have been carried out that have made it possible to determine the most stable schemes (see, for example, [15, 16, 17, 18]). As far as we know, [15] is the first paper devoted to analyze the stability of iterative methods for nonlinear systems by using dynamical tools. This analysis plays an important role when we want to select the elements of a family with good stability properties and to refuse the members with chaotic behavior.

In this paper, a complete stability analysis is performed on a family of iterative methods for solving nonlinear systems, constructed in [19], through the procedure of matrix weight functions. This analysis is made for different rational functions resulting from the application of some particular elements of the family under study on polynomials systems of lower degree.

The authors, in [19], by using the weight functions procedure constructed the following class of iterative methods

$$\begin{array}{rcl} y^{(k)} & = & x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}), \\ x^{(k+1)} & = & x^{(k)} - \Gamma(\eta_k)[F'(x^{(k)})]^{-1}F(x^{(k)}), \end{array} \tag{1}$$

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*Corresponding author

Email addresses: francisco.chicharro@unir.net (Francisco I. Chicharro), acordero@mat.upv.es (Alicia Cordero), neus.garrido@unir.net (Neus Garrido), jrtorre@mat.upv.es (Juan R. Torregrosa)

where $\Gamma : X \to X$ is a matrix weight function of variable $\eta_k = [F'(x^{(k)})]^{-1}\left(F'(x^{(k)}) - [y^{(k)}, x^{(k)}; F]\right)$, $X = \mathbb{R}^{n\times n}$ denotes the space of all the real matrices of size $n \times n$, and $[\cdot, \cdot; F]$ stands for the divided difference operator

$$[x, y; F](x - y) = F(x) - F(y),$$

as defined in [20]. The matrix weight function Γ has Frechet derivatives satisfying (see [21]) the following conditions

a) $\Gamma'(u)(v) = G_1 uv$, being $\Gamma' : X \longrightarrow \mathcal{L}(X)$, $G_1 \in \mathbb{R}$ and $\mathcal{L}(X)$ denotes the space of linear mappings from X to itself,

b) $\Gamma''(u, v)(w) = G_2 uvw$, where $\Gamma'' : X \times X \longrightarrow \mathcal{L}(X)$ and $G_2 \in \mathbb{R}$,

c) $\Gamma'''(u, v, w)(t) = G_3 uvwt$, for $\Gamma''' : X \times X \times X \longrightarrow \mathcal{L}(X)$ and $G_3 \in \mathbb{R}$.

The interest of this family lies in its computational efficiency, since all linear systems that we must solve in each iteration have the same matrix of coefficients. On the other hand, many known methods or families of schemes are elements of this class.

It was proven in [19] that the elements of family (1) converges to the solution of $F(x) = 0$ with order 4 when the weight function holds $\Gamma(0) = I$, $G_1 = 1$, $G_2 = 4$ and under standard conditions on F and F'. Section 2 is devoted to perform a deep stability analysis of family (1) depending on the initial estimations. For this purpose, we use some tools of the multidimensional real dynamics, that we recall below. A comprehensive development of these tools can be found in [22].

When an iterative method is applied on a system of polynomials $p(x)$, a rational vectorial operator $R(x)$ as fixed point function is obtained. The orbit of a point $x^{(0)} \in \mathbb{R}^n$ is defined as the set of the successive applications of R, i.e.,

$$\left\{x^{(0)}, R(x^{(0)}), R^2(x^{(0)}), \dots\right\}.$$

The dynamical behavior of the orbit of a point $x^{(0)}$ is set attending to its asymptotic behavior. A point x^T is T-periodic of R if $R^T(x^T) = x^T$ and $R^t(x^T) \neq x^T$, for $t < T$, where T and t are positive integers. For $T = 1$, this point is a fixed point, denoted by x^*. The fixed points different of the roots of $p(x)$ are called strange fixed points. The stability of the periodic points is classified from the next result.

Theorem 1 ([22]). *Let $R : \mathbb{R}^n \to \mathbb{R}^n$ be $\mathcal{C}^2$. Assume x^T is a T-periodic point. Let $\lambda_j, j = 1, \dots, n$, be the eigenvalues of $R'(x^T)$. Then,*

1. *if all the eigenvalues λ_j satisfy $|\lambda_j| < 1$, x^T is attracting,*
2. *if one eigenvalue λ_{j_0} complies with $|\lambda_{j_0}| > 1$, x^T is unstable, that is repelling or saddle,*
3. *if all the eigenvalues λ_j satisfy $|\lambda_j| > 1$, x^T is repelling.*

In addition,

i) if all the eigenvalues λ_j satisfy $|\lambda_j| \neq 1$, then x^T is known as a hyperbolic point,

ii) if x^T is hyperbolic and there exists an eigenvalue $|\lambda_i| < 1$ and and eigenvalue $|\lambda_k| > 1$, then x^T is a saddle point,

iii) if all the eigenvalues are equal to zero, then x^T is superattracting.

The basin of attraction of an attracting fixed point x^*, $\mathcal{A}(x^*)$, is the set of points whose orbits tend to the attracting fixed point x^*, i.e.,

$$\mathcal{A}(x^*) = \{x^{(0)} \in \mathbb{R}^n : R^m(x^{(0)}) \to x^*, m \to \infty\}.$$

The set of points $x^{(0)}$ whose orbit tends to an attracting fixed point shapes the Fatou set $\mathcal{F}(R)$, while its complementary is the Julia set $\mathcal{J}(R)$.

Finally, a point $x^C \in \mathbb{R}^n$ is a critical point of R if the eigenvalues of the Jacobian matrix $R'(x^C)$ are null or all its components satisfy $\dfrac{\partial r_i(x^*)}{\partial x_j} = 0$ for all $i, j \in \{1, \dots, n\}$. A free critical point is a critical point that does not match with the roots of $p(x)$. The interest in analyzing the critical points is based on the classic result of Julia and Fatou which establishes that in every basin of attraction there is at least one critical point.

Now, we select three different weight matrix functions $\Gamma(\eta)$ which give us elements of family (1), whose stability we analyze in the next section. The idea of how to choose weight functions in the case $n = 1$ appeared in Chun et al. [23]. The first iterative method, named $\mathcal{G}1$, corresponds to the weight matrix function $\Gamma_1(\eta_k) = I + \eta_k + 2\eta_k^2$, whose iterative expression is

$$\begin{array}{rcl} y^{(k)} & = & x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}), \\ x^{(k+1)} & = & x^{(k)} - \left[I + \eta_k + 2\eta_k^2\right][F'(x^{(k)})]^{-1}F(x^{(k)}). \end{array} \tag{2}$$

The second case of study, called $\mathcal{G}2$, corresponds to the weight matrix function $\Gamma_2(\eta_k) = [I - 2\eta_k]^{-1}(I - \eta_k)$, resulting in the iterative expression

$$\begin{array}{rcl} y^{(k)} & = & x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}), \\ x^{(k+1)} & = & x^{(k)} - [I - 2\eta_k]^{-1}(I - \eta_k)[F'(x^{(k)})]^{-1}F(x^{(k)}). \end{array} \tag{3}$$

The third iterative class belonging to family (1), named $\mathcal{G}3$, corresponds to the use of the weight matrix function $\Gamma_3(\eta_k) = I + \eta_k + 2\eta_k^2 + \dfrac{\alpha}{6}\eta_k^3$, $\alpha \in \mathbb{R}$. The iterative expression of the resulting class of iterative methods is

$$\begin{array}{rcl} y^{(k)} & = & x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}), \\ x^{(k+1)} & = & x^{(k)} - \Gamma_3(\eta_k)[F'(x^{(k)})]^{-1}F(x^{(k)}). \end{array} \tag{4}$$

Let us remark that the value $\alpha = 0$ provides the previous method denoted by $\mathcal{G}1$.

The resulting iterative methods and the uniparametric iterative class will be applied on two-dimensional quadratic polynomials in order to depict the main dynamical graphical representations in the real plane. Then, the analytical results can be easily extended to the corresponding n-dimensional ones.

The two-dimensional quadratic polynomial systems that have been considered are

1. $p(x) = \begin{cases} p_1(x) = x_1^2 - 1 \\ p_2(x) = x_2^2 - 1 \end{cases}$,
2. $q(x) = \begin{cases} q_1(x) = x_1x_2 + x_1 - x_2 - 1 \\ q_2(x) = x_1x_2 - x_1 + x_2 - 1 \end{cases}$,

whose roots are $(\pm 1, \pm 1)$ for $p(x)$, and $(1, 1), (-1, -1)$ for $q(x)$. Although these polynomial systems are quite simple, the related multidimensional rational functions are complex enough to observe all the qualitative behavior.

2. Dynamical analysis of $\mathcal{G}1$, $\mathcal{G}2$ and $\mathcal{G}3$

In this section, we analyze the stability of the fixed points of each rational vectorial operator obtained when each method is applied on systems $p(x)$ and $q(x)$. We also study, in each case, the critical points and we represent the dynamical planes.

The dynamical planes show the Fatou set (union of the basins of attraction of all the fixed and periodic points) and the Julia set (its complementary in the plane). They are plotted by using the multidimensional rational function resulting from the application of an iterative method on a polynomial system. A mesh of initial estimations is defined and each one of the points is painted in a color, depending on the attracting point it converges to. Otherwise, the initial guess is represented in black. The routines for the representation follow similar guidelines as in [24].

In this manuscript, every dynamical plane has been generated in Matlab2018b, taking a mesh of 500×500 initial guesses for $(x_1, x_2) \in [-10, 10] \times [-10, 10]$. The algorithm iterates until the maximum number of iterations is reached or the norm between the current and previous iteration is lower than a certain tolerance. The values of the maximum number of iterations is 50 and the tolerance used is 10^{-3}. The attracting fixed points are represented with white stars, and the free critical points are plotted with white circles.

2.1. Analysis of $\mathcal{G}1$

The rational function $R_1(x_1, x_2)$ associated to iterative method (2) applied on $p(x)$ is

$$R_1(x_1, x_2) = \begin{bmatrix} \dfrac{5x_1^6 + 15x_1^4 - 5x_1^2 + 1}{16x_1^5} \\ \dfrac{5x_2^6 + 15x_2^4 - 5x_2^2 + 1}{16x_2^5} \end{bmatrix}. \tag{5}$$

The fixed points of (5) match with the roots of $p(x)$ that are superattracting points, as is shown in the following result.

Proposition 1. *Rational function $R_1(x_1, x_2)$, associated to method $\mathcal{G}1$ applied on the polynomial system $p(x)$, has four fixed points, $(x_1^*, x_2^*) = (\pm 1, \pm 1)$, being all of them superattracting points.*

Proof. The jth-coordinate ($j = 1, 2$) of the rational function $R_1(x_1, x_2)$ is

$$\frac{5x_j^6 + 15x_j^4 - 5x_j^2 + 1}{16x_j^5}.$$

The fixed points are obtained solving $R_1(x_1, x_2) = (x_1, x_2)$, or analogously, solving equations

$$\frac{5x_j^6 + 15x_j^4 - 5x_j^2 + 1}{16x_j^5} = x_j, \qquad j = 1, 2. \tag{6}$$

Developing equation (6), we obtain for $j = 1, 2$

$$\begin{aligned} \frac{5x_j^6 + 15x_j^4 - 5x_j^2 + 1}{16x_j^5} = x_j \quad &\Leftrightarrow \quad -11x_j^6 + 15x_j^4 - 5x_j^2 + 1 = 0 \\ &\Leftrightarrow \quad (x_j - 1)(x_j + 1)(-11x_j^4 + 4x_j^2 - 1) = 0. \end{aligned} \tag{7}$$

The only real roots of the previous sixth-order polynomial are $x_j = \pm 1$. Then, the fixed points of R_1 are $(-1, -1)$, $(-1, 1)$, $(1, -1)$ and $(1, 1)$.

Computing $R_1'(x_1, x_2)$, we get the diagonal matrix

$$R_1'(x_1, x_2) = \begin{bmatrix} \dfrac{5\left(x_1^2 - 1\right)^3}{16x_1^6} & 0 \\ 0 & \dfrac{5\left(x_2^2 - 1\right)^3}{16x_2^6} \end{bmatrix}, \tag{8}$$

whose eigenvalues are $\lambda_1(x_1, x_2) = \dfrac{5\left(x_1^2 - 1\right)^3}{16x_1^6}$ and $\lambda_2(x_1, x_2) = \dfrac{5\left(x_2^2 - 1\right)^3}{16x_2^6}$. The evaluation of the fixed points on the eigenvalues results in $\lambda_1 = \lambda_2 = 0$. Therefore, every fixed point is a superattracting point. ■

Regarding the critical points of $R_1(x_1, x_2)$, every one matches with the roots of the polynomial $p(x)$. Then, there are not free critical points. This fact guarantees the stability of the method, since the only basins of attraction are those associated to the roots of the polynomial.

The dynamical plane of $R_1(x_1, x_2)$ is represented in Figure 1a. The basins of attraction of the roots $(-1, -1)$, $(-1, 1)$, $(1, -1)$, $(1, 1)$ are mapped with colors orange, red, green and blue, respectively. In addition, the lower intensity of the colors is used to indicate the lower number of iterations until convergence to the roots is achieved. The representation shows the good performance of the method for the polynomial system $p(x)$, since every initial guess converges to the closest root.

When the iterative method (2) is applied on $q(x)$, whose variables are not uncoupled, its related rational function is denoted by $S_1(x_1, x_2)$ and its expression is

$$S_1(x_1, x_2) = \begin{bmatrix} \dfrac{x_1^5 x_2 + x_1^4\left(3x_2^2 + 2\right) + x_1^3\left(4x_2^3 + 6x_2\right) + x_1^2\left(x_2^4 + 12x_2^2 - 3\right) + x_1 x_2\left(x_2^4 + 6x_2^2 - 2\right) + 4x_2^4 - 5x_2^2 + 2}{(x_1 + x_2)^5} \\ \dfrac{x_1^5 x_2 + x_1^4\left(3x_2^2 + 2\right) + x_1^3\left(4x_2^3 + 6x_2\right) + x_1^2\left(x_2^4 + 12x_2^2 - 3\right) + x_1 x_2\left(x_2^4 + 6x_2^2 - 2\right) + 4x_2^4 - 5x_2^2 + 2}{(x_1 + x_2)^5} \end{bmatrix}. \tag{9}$$

Let us note that the two components of the rational function are equal. The only fixed points of (9) are $(-1, -1)$, $(1, 1)$, that agree with the roots of $q(x)$ and are superattracting. In this case, the operator associated to method $\mathcal{G}1$ has free critical points. The next result summarizes the dynamical results obtained for $S_1(x_1, x_2)$.

Proposition 2. *The fixed points of the rational vectorial function $S_1(x_1, x_2)$, obtained from the application of method $\mathcal{G}1$ on polynomial system $q(x)$, are $(-1, -1)$ and $(1, 1)$, being superattracting points. In addition, the rational operator has free critical points whose approximated values are $(-1.07142, -2.55449)$, $(0.07142, -0.93485)$, $(-0.13412, -1.22734)$ and $(1.14102, 0.57816)$, lying in the basins of attraction of the roots.*

Proof. The two components of the fixed points are obtained by solving $S_1(x_1,x_2)=(x_1,x_2)$, that is,

$$\frac{x_1^5x_2+x_1^4\left(3x_2^2+2\right)+x_1^3\left(4x_2^3+6x_2\right)+x_1^2\left(x_2^4+12x_2^2-3\right)+x_1x_2\left(x_2^4+6x_2^2-2\right)+4x_2^4-5x_2^2+2}{(x_1+x_2)^5}=x_j,$$

for $j=1,2$. Equivalently, by solving the equations

$$\begin{aligned}\frac{5x_1^6+15x_1^4-5x_1^2+1}{16x_1^5} &= x_1,\\ x_1 &= x_2,\end{aligned}$$

that is,

$$\begin{aligned}\frac{(x_1^2-1)(11x_1^4-4x-1^2+1)}{16x_1^5} &= 0,\\ x_1 &= x_2,\end{aligned}$$

where polynomial $11x_1^4-4x-1^2+1$ has not real roots.

From the solutions of equation (7), the fixed points of $S_1(x_1,x_2)$ are $(-1,-1)$ and $(1,1)$. On the other hand, the eigenvalues of $S_1'(x_1,x_2)$ are $\lambda_1(x_1,x_2)=0$ and

$$\begin{aligned}\lambda_2(x_1,x_2)=\frac{1}{(x_1+x_2)^6}&\left(x_1^6+2x_1^5x_2+x_1^4(5x_2^2-6)+(13x_2^4-36x_2^2+22)\right.\\&\left.-2x_1x_2^3(x_2^2-2)+x_2^6-18x_2^4+38x_2^2-20-4x_1^3x_2+x_1^2\right).\end{aligned}$$

Then, the asymptotic behavior of the fixed points depends on the second eigenvalue. In this case, $\lambda_2(-1,-1)=\lambda_2(1,1)=0$, so both fixed points are superattracting.

The critical points are calculated by solving $\lambda_2(x_1,x_2)=0$. Then, four real points different from the roots are obtained, so they are free critical points. Their value, obtained by using Mathematica software, are approximately $(-1.07142,-2.55449)$, $(0.07142,-0.93485)$, $(-0.13412,-1.22734)$ and $(1.14102,0.57816)$. It can be seen in the dynamical plane in Figure 1b that all the free critical points remain in the basins of attraction of the roots, so there is no other different behavior than the convergence to them. ■

The dynamical plane of $S_1(x_1,x_2)$ is represented in Figure 1b. The basin of attraction of the root $(-1,-1)$ is mapped with color orange, while the associated to the root $(1,1)$ is mapped with color blue. In addition, the four free critical points are represented in the plane with white circles and they remain in the basins of attraction of the fixed points. In this case, the basins of attraction do not have the same appearance as in the previous case or Newton's method. However, there are not regions of divergence.

2.2. Analysis of $\mathcal{G}2$

The rational function $R_2(x_1,x_2)$ associated to iterative method (3) applied on $p(x)$ is

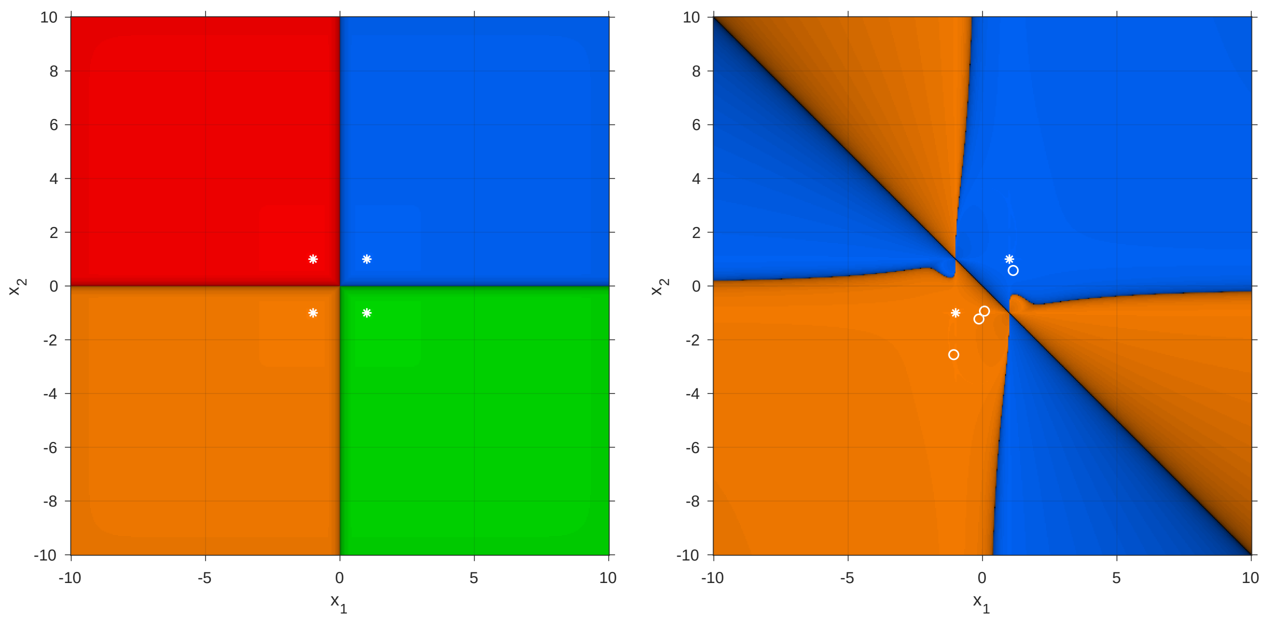
$$R_2(x_1,x_2)=\begin{bmatrix}\dfrac{x_1^4+6x_1^2+1}{4\left(x_1^3+x_1\right)}\\[2mm]\dfrac{x_2^4+6x_2^2+1}{4\left(x_2^3+x_2\right)}\end{bmatrix}. \tag{10}$$

Its dynamical behavior, shown in the following result, is similar to the case of method $\mathcal{G}1$ on $p(x)$.

Proposition 3. *The only fixed points of $R_2(x_1,x_2)$, the resulting operator from the application of method $\mathcal{G}2$ on the polynomial system $p(x)$, match with the roots of the polynomial and they are superattracting.*

The proof of Proposition 3 follows the same guidelines than in Propositions 1 and 2, so we avoid showing it. Moreover, there is no presence of free critical points for $R_2(x_1,x_2)$. This means that there are no basins of attraction different of the corresponding to the roots of polynomial system $q(x)$.

The dynamical plane of $R_2(x_1,x_2)$ is represented in Figure 2a. The mapping of the colors associated with the basins of attraction of the roots is the same than in Figure 1a. Once again, the behavior of the method for this polynomial system is good because every initial guess converges to one of the roots and this root is the nearest one.



(a) $R_1(x_1, x_2)$ (b) $S_1(x_1, x_2)$

Figure 1: Dynamical planes of rational functions related to method $\mathcal{G}1$

The rational function $S_2(x_1, x_2)$ results from applying $\mathcal{G}2$ on $q(x)$. Its expression is

$$S_2(x_1, x_2) = \left[\begin{array}{c} \dfrac{x_1^3 x_2 + x_1^2\left(x_2^2+2\right) - x_1 x_2\left(x_2^2-4\right)+1}{\left(x_1+x_2\right)\left(x_1^2+2x_1x_2-x_2^2+2\right)} \\ \dfrac{x_1^3 x_2 + x_1^2\left(x_2^2+2\right) - x_1 x_2\left(x_2^2-4\right)+1}{\left(x_1+x_2\right)\left(x_1^2+2x_1x_2-x_2^2+2\right)} \end{array} \right]. \tag{11}$$

Let us note again that both components of $S_2(x_1, x_2)$ are equal. The following result gathers the number of fixed and critical points of $S_2(x_1, x_2)$, and the stability of the first ones.

Proposition 4. *The fixed points of* $S_2(x_1, x_2)$ *agree with the roots of the polynomial system* $q(x)$*, that is,* $(-1, -1)$ *and* $(1, 1)$*. They are superattracting points. Moreover, the rational operator has four critical points, given by* $(-1.74193, -3.50389)$, $(0.16533, -0.80333)$, $(-0.25033, -1.39188)$ *and* $(1.22803, 0.70137)$.

The proof of Proposition 4 follows the same steps than in the proof of Proposition 2. It is based on the real solutions of equation $S_2(x_1, x_2) = (x_1, x_2)$ and the eigenvalues of $S_2'(x_1, x_2)$, being $\lambda_1(x_1, x_2) = 0$ and

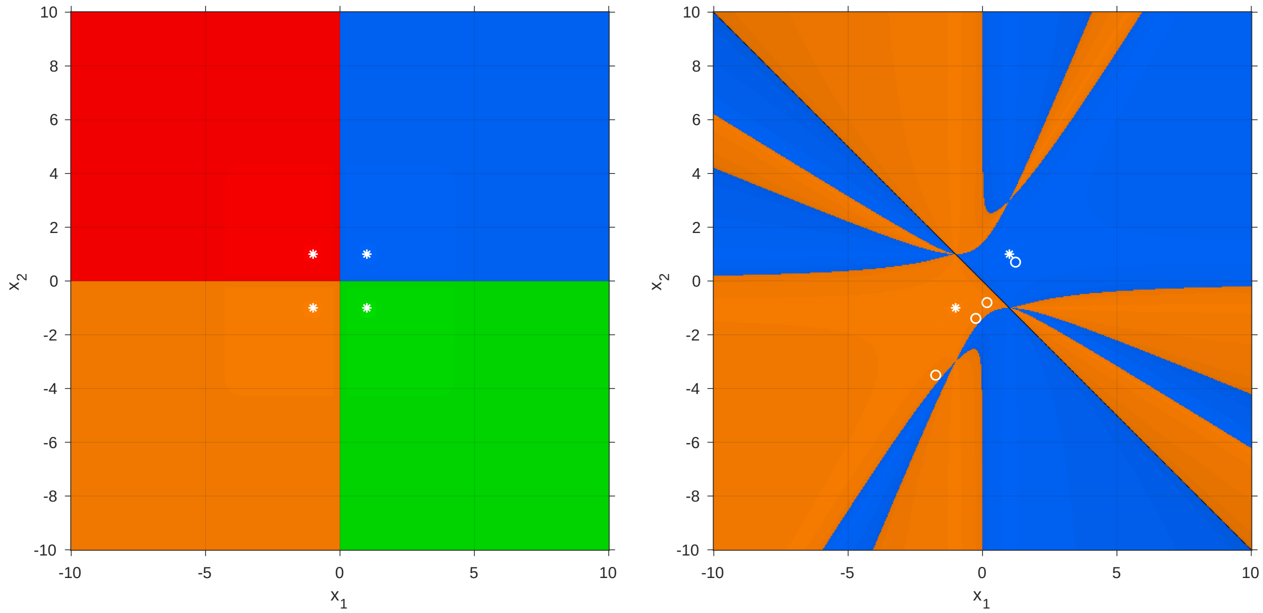
$$\lambda_2(x_1, x_2) = \frac{1}{\left(x_1+x_2\right)^2\left(x_1^2+2x_1x_2-x_2^2+2\right)^2}\left(x_1^6 + 2x_1^5x_2 + x_1^4(x_2^2-2) - 4x_1^3x_2 + x_1^2(x_2^4 - 4x_2^2 + 2)\right.$$
$$\left. -2x_1x_2^3(x_2^2-2) + x_2^6 - 6x_2^4 + 10x_2^2 - 4\right).$$

The fixed points are superattracting, as $\lambda_2(-1, -1) = \lambda_2(1, 1) = 0$. The solution of $\lambda_2(x_1, x_2) = 0$ provides four points different from the roots, so they are free critical points whose values are $(-1.74193, -3.50389)$, $(0.16533, -0.80333)$, $(-0.25033, -1.39188)$ and $(1.22803, 0.70137)$.

The dynamical plane of (11) is represented in Figure 2b. The basins of attraction of $(-1, -1)$ and $(1, 1)$ are mapped with color orange and blue, respectively. It can be observed again that all the free critical points converge to one of the roots of polynomial $q(x)$. Let us remark that every initial guess tend to an attracting point, so the behavior is as good as expected.

2.3. *Analysis of* $\mathcal{G}3$

Let us remark that family $\mathcal{G}3$ includes a parameter on its iterative expression, so its stability will depend on the value of the parameter. Furthermore, when $\alpha = 0$, family $\mathcal{G}3$ turns into method $\mathcal{G}1$ whose dynamics have been analyzed previously.



(a) $R_2(x_1, x_2)$ (b) $S_2(x_1, x_2)$

Figure 2: Dynamical planes of rational functions related to method $\mathcal{G}2$

When family (4) is applied on the polynomial system $p(x)$, the resulting vectorial rational function is denoted by $R_3(x_1, x_2)$ and can be expressed as

$$R_3(x_1, x_2) = \left[\begin{array}{c} \dfrac{-\alpha + 4(12+\alpha)x_1^2 - 6(40+\alpha)x_1^4 + 4(180+\alpha)x_1^6 + (240-\alpha)x_1^8}{768x_1^7} \\ \dfrac{-\alpha + 4(12+\alpha)x_2^2 - 6(40+\alpha)x_2^4 + 4(180+\alpha)x_2^6 + (240-\alpha)x_2^8}{768x_2^7} \end{array} \right]. \tag{12}$$

The corresponding fixed points are obtained by solving $R_3(x_1, x_2) = (x_1, x_2)$. In adittion, the Jacobian matrix $R_3'(x_1, x_2)$ is the diagonal matrix

$$R_3'(x_1, x_2) = \left[\begin{array}{cc} -\dfrac{\left(x_1^2-1\right)^3 \left(7\alpha + (\alpha-240)x_1^2\right)}{768x_1^8} & 0 \\ 0 & -\dfrac{\left(x_2^2-1\right)^3 \left(7\alpha + (\alpha-240)x_2^2\right)}{768x_2^8} \end{array} \right],$$

so its eigenvalues are

$$\lambda_1(x_1, x_2) = -\frac{\left(x_1^2-1\right)^3 \left(7\alpha + (\alpha-240)x_1^2\right)}{768x_1^8}, \qquad \lambda_2(x_1, x_2) = -\frac{\left(x_2^2-1\right)^3 \left(7\alpha + (\alpha-240)x_2^2\right)}{768x_2^8}. \tag{13}$$

The next result gathers the fixed points for this rational operator and also their stability depending on the value of parameter α.

Proposition 5. *The fixed points of $R_3(x_1, x_2)$ and their asymptotic behavior are*

1. *for all $\alpha \in \mathbb{R}$, the roots of polynomial system $p(x)$, i.e, $(\pm 1, \pm 1)$, are fixed points of $R_3(x_1, x_2)$, being superattracting,*

2. *for $\alpha \in (-\infty, -528) \cup (0, +\infty)$ the rational operator has, in addition to $(\pm 1, \pm 1)$, twelve strange fixed points: $(\pm 1, r_1)$, $(\pm 1, r_2)$, $(r_1, \pm 1)$, $(r_2, \pm 1)$, that are saddle points; and (r_1, r_1), (r_1, r_2), (r_2, r_1), (r_2, r_2), which are repelling, being the values r_1 and r_2 the real roots of the sixth-degree polynomial:*

$$p_6(x) = -\alpha + (48+3\alpha)x^2 + (-192-3\alpha)x^4 + (528+\alpha)x^6.$$

3. *For $\alpha \in [-528, 0]$, $R_3(x_1, x_2)$ does not have strange fixed points and the only fixed points are the roots of $p(x)$.*

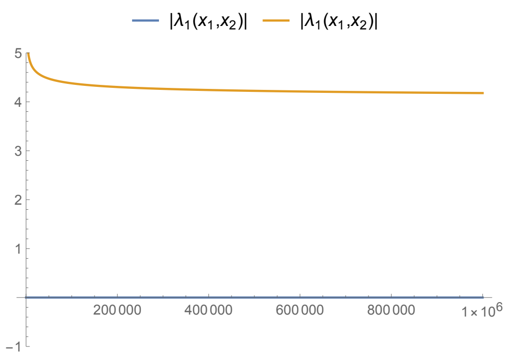
Proof. From (12), the fixed points are the real solutions of equations

$$\frac{-\alpha + 4(12+\alpha)x_j^2 - 6(40+\alpha)x_j^4 + 4(180+\alpha)x_j^6 + (240-\alpha)x_j^8}{768x_j^7} = x_j, \qquad j = 1, 2$$

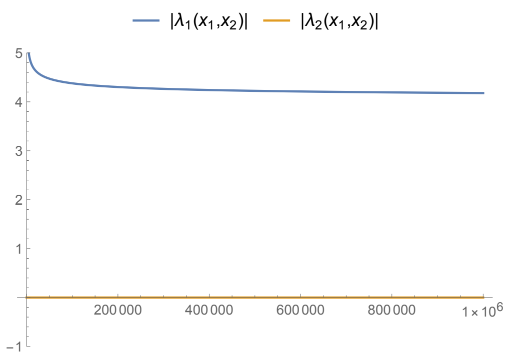
$$\Leftrightarrow \qquad \frac{(x_j^2 - 1)(-\alpha + (48+3\alpha)x^2 + (-192-3\alpha)x^4 + (528+\alpha)x^6)}{768x_j^7} = 0, \qquad j = 1, 2. \tag{14}$$

The product of the terms in the numerator of (14) give the components of the fixed points of $R_3(x_1, x_2)$. From $x_j^2 - 1$ we obtain the fixed points $(\pm 1, \pm 1)$. The other term in the product is a sixth-degree polynomial that only has two real roots, denoted r_1 and r_2, when $\alpha \in (-\infty, -528) \cup (0, +\infty)$. The set of strange fixed points is obtained by all the pair of solutions of (14), that is, $(\pm 1, r_1)$, $(\pm 1, r_2)$, $(r_1, \pm 1)$, $(r_2, \pm 1)$, (r_1, r_2), (r_2, r_1), (r_1, r_1) and (r_2, r_2).

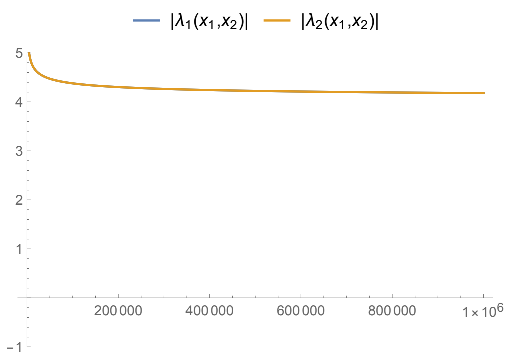
From (13), $\lambda_1(\pm 1, \pm 1) = \lambda_2(\pm 1, \pm 1) = 0$, so the fixed points are superattracting. The asymptotical behavior of the strange fixed points has been proved numerically from the values of $|\lambda_1(x_1, x_2)|$ and $|\lambda_2(x_1, x_2)|$, being (x_1, x_2) the components of the strange fixed points. Figure 3 shows graphically the results obtained for some of the strange fixed points (the same results are obtained for the others). The absolute value of the eigenvalues for each strange fixed point has been represented. We can observe saddle points in Figures 3a and 3b, as one eigenvalue is less than 1 and the other is greater than 1. In Figures 3c and 3d the eigenvalues have the same value, being greater than 1, so the points are repelling.
■



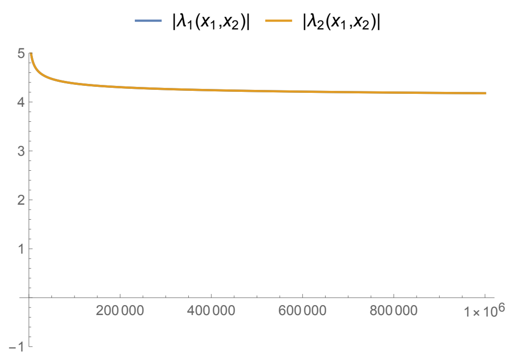
(a) $(x_1, x_2) = (1, r_1)$



(b) $(x_1, x_2) = (r_2, -1)$



(c) $(x_1, x_2) = (r_1, r_1)$



(d) $(x_1, x_2) = (r_1, r_2)$

Figure 3: Eigenvalues of some strange fixed points

After analyzing the eigenvalues of the Jacobian matrix $R_3'(x_1, x_2)$, we summarize the obtained critical points in the following result.

Proposition 6. *The set of the critical points of the rational operator $R_3(x_1, x_2)$ is given by:*

1. *the roots of the polynomial system $p(x)$ for all $\alpha \in \mathbb{R}$,*

2. *the twelve real free critical points* $\left(\pm 1, \pm\sqrt{\dfrac{7\alpha}{240-\alpha}}\right)$, $\left(\pm\sqrt{\dfrac{7\alpha}{240-\alpha}}, \pm 1\right)$ *and* $\left(\pm\sqrt{\dfrac{7\alpha}{240-\alpha}}, \pm\sqrt{\dfrac{7\alpha}{240-\alpha}}\right)$ *for the values of the parameter* $\alpha \in (0,30) \cup (30,240)$.

3. *When* $\alpha \in (-\infty, 0] \cup \{30\} \cup [240, +\infty)$*, the only real critical points agree with the roots of* $p(x)$*, so there are no free critical points.*

Let us note that, from Proposition 6, when $\alpha \in (-\infty, 0] \cup \{30\} \cup [240, +\infty)$ the only basins of attraction are those associated with the roots of $q(x)$.

When the rational operator depends on the value of a parameter, a useful graphical tool is the parameter line. The parameter lines help to select the values of the parameter that provide the methods of the family with better stability. For this representation, each point in the real line corresponds to a value of the parameter, so it represents a particular method belonging to the uniparametric original family.
In this work, the parameter lines are generated in Matlab R2018b following similar routines than in the dynamical planes. The main difference is that in the dynamical lines the starting point of the iterative process is a free critical point, so there are as much parameter lines as free critical points has the family. After the same stopping conditions as in the dynamical planes, the points in the real line are represented in white if the corresponding method of the family has converged to any of the roots, and otherwise, they are represented in black.

The parameter lines of family (4) when it is applied on polynomial $p(x)$ give the same plot for all the free critical points, so we have represented only one of them in Figure 4. The parameter takes values in the interval $[0, 240]$ as it is the only region where the free critical points are real.

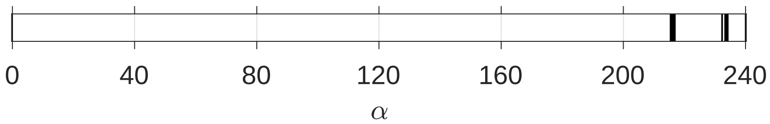


Figure 4: Parameter line of $R_3(x_1, x_2)$

From Figure 4, we can see a wide white region of values of α in the real line. The most stable methods of family $\mathcal{G}3$ are those associated with the values of α in this white region, as there is no other behavior than convergence to the roots. However, we can observe in Figure 4 two narrow black areas that correspond with values of α whose associated methods do not converge to any of the roots.

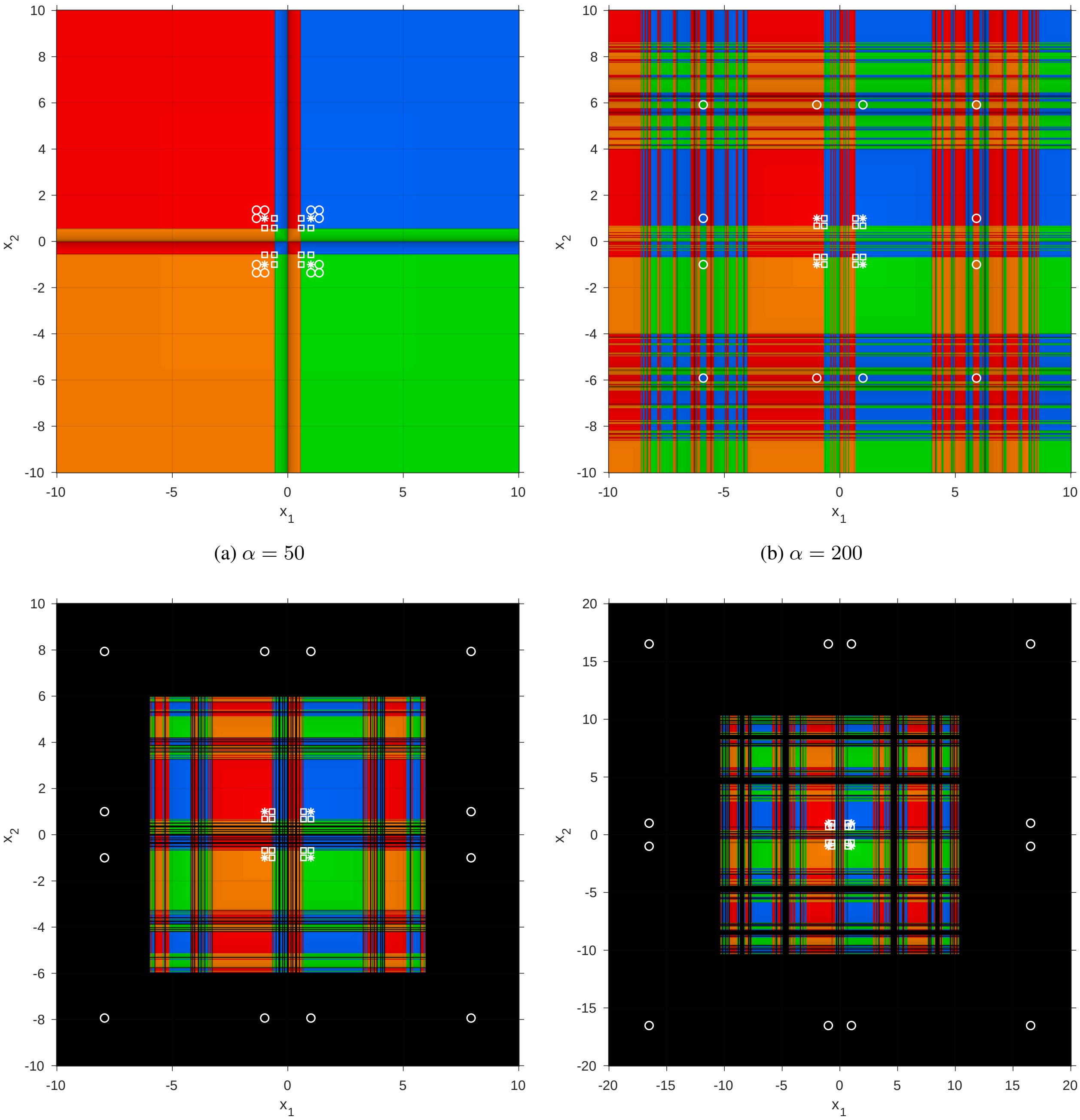
According to the results provided by the parameter lines, we have selected different values of α from the black and white regions in order to represent the associated dynamical planes. Figure 5 shows the dynamical planes of $R_3(x_1, x_2)$ following the same routines in Matlab than Figures 1 and 2. In particular, Figures 5a and 5b correspond to values of α in the white region of the parameter line, while Figures 5c and 5d correspond to the black region. In addition, we have represented with white stars, squares or circles the fixed points, strange fixed points and free critical points, respectively.

As it was expected, for the values of $\alpha = 50$ and $\alpha = 200$ represented in the dynamical planes in Figure 5, there is full convergence to any of the roots of polynomial $p(x)$. However, when α is taken from the black region of the parameter line, we can observe wide black regions in Figures 5c and 5d with no convergence to the roots. These black regions corresponds to basins of attraction of periodic points of period four. In this case, there exist six different periodic orbits. Figure 6 shows the orbit of different initial points that tend to each of the six different periodic orbits for $\alpha = 216$, being the results for $\alpha = 234$ completely analogous.

In addition, the dynamical planes associated to $\alpha = -50$ and $\alpha = 250$ are represented in Figure 7. According to Proposition 6, for values of the parameter in $(-\infty, 0] \cup \{30\} \cup [240, +\infty)$ the rational operator $R_3(x_1, x_2)$ does not have any free critical point, so there are only basins of atraction of the roots of $p(x)$ when $\alpha = -50$ and $\alpha = 250$. This fact is observed in both dynamical planes, where all the points converge to a root of $p(x)$, showing the stability of the methods of iterative family $\mathcal{G}3$ associated to the considered parameters. Moreover, in Figure 7a the operator does not have strange fixed points, and the resulting dynamical plane shows that every initial estimation converges to the nearest root.

Finally, the rational function associated to family (4) on $q(x)$ is

$$S_3(x_1, x_2) = \begin{bmatrix} \dfrac{6(x_1+x_2)^2\xi - \alpha\left(x_1^2-1\right)\left(x_2^2-1\right)^3}{6(x_1+x_2)^7} \\ \dfrac{6(x_1+x_2)^2\xi - \alpha\left(x_1^2-1\right)\left(x_2^2-1\right)^3}{6(x_1+x_2)^7} \end{bmatrix},$$



(a) $\alpha = 50$ (b) $\alpha = 200$

(c) $\alpha = 216$ (d) $\alpha = 234$

Figure 5: Dynamical planes of $\mathcal{G}3$ on $p(x)$ for several values of α

where $\xi = \left(2x_1^4 + \rho x_2^4 + 2x_1\left(2x_1^2+3\right)x_2^3 + \left(3x_1^2\rho - 5\right)x_2^2 - 3x_1^2 + x_1\left(x_1^4 + 6x_1^2 - 2\right)x_2 + x_1x_2^5 + 2\right)$ and $\rho = \left(x_1^2 + 4\right)$. As in the previous cases, the number of fixed points of $S_3(x_1, x_2)$ depend on the value of α. The next result shows the intervals where the rational operator has real strange fixed points and also their stability.

Proposition 7. *The roots $(-1,-1)$ and $(1,1)$ of the polynomial system $q(x)$ are superattracting fixed points of $S_3(x_1, x_2)$. In addition to the roots of $q(x)$, the operator has different number of strange fixed points depending on the interval where α is defined, being all of them saddle points. They are summarized in Table 1, where s_i, $i = 1, 2, 3, 4$, denote the real*

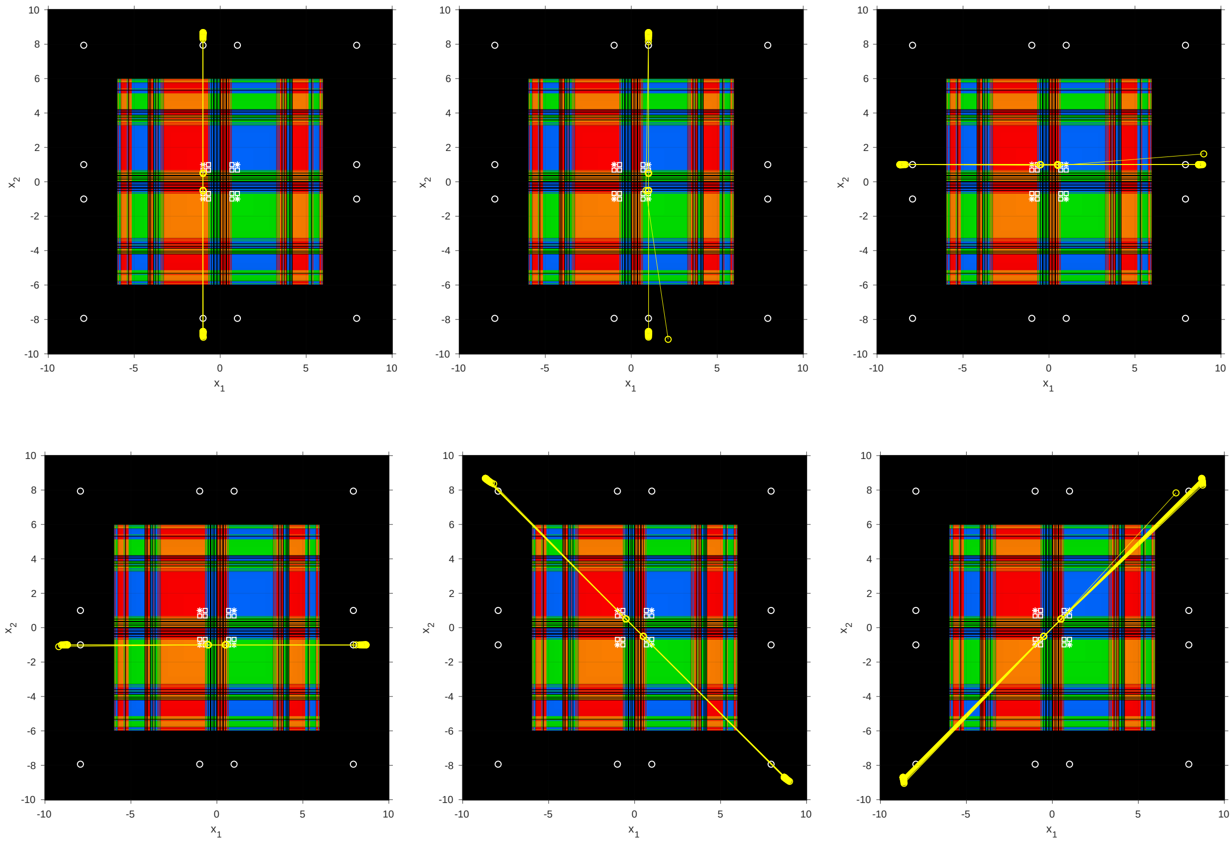
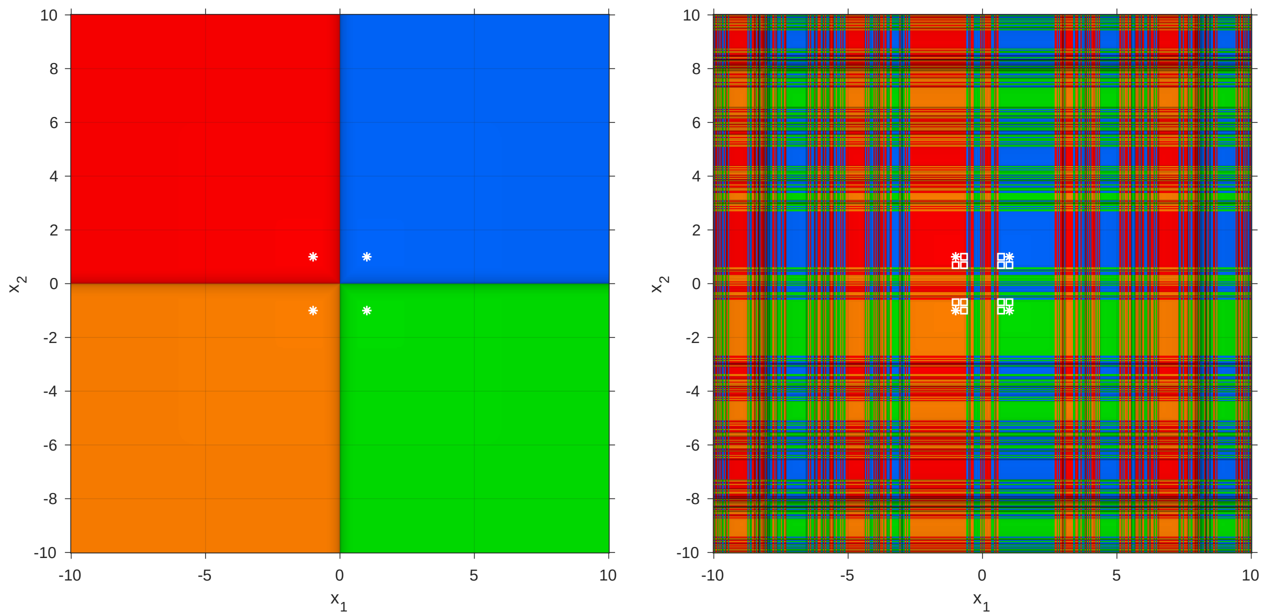


Figure 6: Dynamical planes of $\mathcal{G}3$ on $p(x)$ for $\alpha = 216$. Attracting periodic orbits



(a) $\alpha = -50$

(b) $\alpha = 250$

Figure 7: Dynamical planes of $\mathcal{G}3$ on $p(x)$ for several values of α

roots of the polynomial

$$\begin{aligned}q_{15}(x) = & -39661043818954752 + 307043333194383360x + 25077472880689152x^2 + 7853236406710272x^3 \\ & -222839021721600x^4 - 190418146541568x^5 - 58994872694784x^6 - 1651783083264x^7 \\ & +678676558464x^8 + 137025732672x^9 + 3521159424x^{10} - 1129826880x^{11} - 67248360x^{12} \\ & -285300x^{13} + 438x^{14} + x^{15}.\end{aligned}$$

Table 1: Strange fixed points of $S_3(x_1, x_2)$

α	**Strange fixed points**
$\alpha < s_1$	(t_1, u_1), (t_2, w_1)
$\alpha = s_1$	(t_1, t_1), (t_2, t_2)
$s_1 < \alpha < -528$	(t_1, u_1), (t_2, w_2)
$-528 \leq \alpha \leq 0$	$\emptyset$
$0 < \alpha < s_2$	(t_1, u_2), (t_2, w_3)
$\alpha = s_2$	(t_1, t_1), (t_2, t_2), $(-1, 1.00731)$, $(1, -1.00731)$, $(-0.990314, 0.997551)$, $(0.990314, -0.997551)$
$s_2 < \alpha < s_3$	(t_1, u_3), (t_2, w_3)
$\alpha = s_3$	(t_1, t_1), (t_2, t_2), $(-1, 1.00731)$, $(1, -1.00731)$, $(-0.992758, 1.00001)$, $(0.992758, -1.00001)$
$s_3 < \alpha < s_4$	(t_1, u_3), (t_2, w_4)
$\alpha = s_4$	(t_1, t_1), (t_2, t_2)
$\alpha > s_4$	(t_1, u_2), (t_2, w_4)

From Table 1, the values t_i ($i = 1, 2$), u_j ($j = 1, 2, 3$) and w_k ($k = 1, 2, 3, 4$) correspond to the real roots of different sixth-degree polynomials. Values t_1 and t_2 are the real roots of $q_6^1(x)$:

$$q_6^1(x) = -\alpha + (48 + 3\alpha)x^2 + (-192 - 3\alpha)x^4 + (529 + \alpha)x^6.$$

We denote by u_1, u_2, u_3 the real roots of the polynomial $q_6^2(x)$:

$$\begin{aligned}q_6^2(x) = & -\alpha + (12 + \alpha)t_1^2 - 18t_1^4 + 12t_1^6 + (24t_1 - 48t_1^3 + 60t_1^5)x + (12 + 2\alpha - (60 + 2\alpha)t_1^2 + 138t_1^4)x^2 \\ & (-48t_1 + 168t_1^3)x^3 + (-18 - \alpha + (108 + \alpha)t_1^2)x^4 + 36t_1x^5 + 6x^6,\end{aligned}$$

and w_1, w_2, w_3, w_4 denote the real roots of polynomial $q_6^3(x)$:

$$\begin{aligned}q_6^3(x) = & -\alpha + (12 + \alpha)t_2^2 - 18t_2^4 + 12t_2^6 + (24t_2 - 48t_2^3 + 60t_2^5)x + (12 + 2\alpha - (60 + 2\alpha)t_2^2 + 138t_2^4)x^2 \\ & (-48t_2 + 168t_2^3)x^3 + (-18 - \alpha + (108 + \alpha)t_2^2)x^4 + 36t_2x^5 + 6x^6.\end{aligned}$$

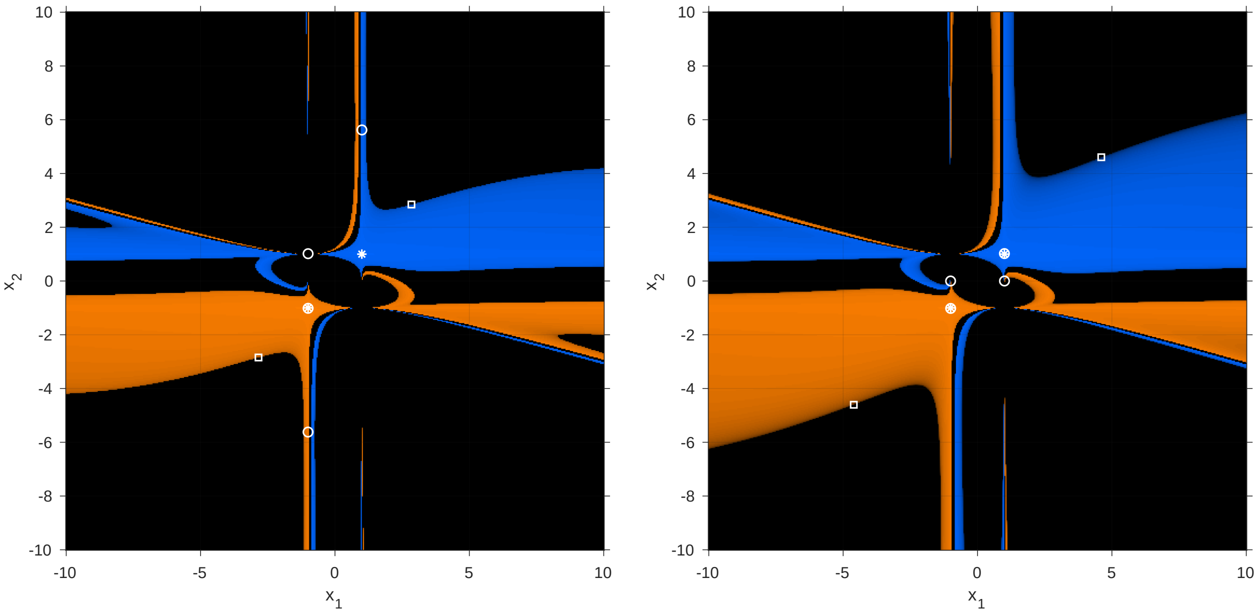
The complicated expression of the critical points of $S_3(x_1, x_2)$ makes not possible their calculation, in general. Then, we set certain values of parameter α to represent the associated dynamical planes and also their fixed and critical points. In particular, we have analyzed the rational operator associated to selected values of α from each of the different subintervals defined in Table 1.

If $\alpha \in]0, s_2[\cup]s_2, s_3[\cup]s_4, +\infty)$, the corresponding methods have dynamical planes with full convergence to the attracting fixed points. This does not happen for the other cases, so we must analyze them.

When $\alpha \in (-\infty, s_1[\cup]s_1, -528[\cup]s_3, s_4[$, the rational operators of the associated iterative schemes of the family have, in addition of two strange fixed points (Table 1), six critical points for each subinterval. Figures 8 and 9 show the dynamical planes associated to $S_3(x_1, x_2)$ for different values of α. The values of the parameter that have been chosen are $\alpha = -750 \in (-\infty, s_1[$ and $\alpha = -600 \in]s_1, -528[$ in Figures 8a and 8b, respectively, and $\alpha = 216 \in]s_3, s_4[$ in Figure 9. We have also represented the strange fixed points with white squares and the critical points with white circles when they belong to the region of the plane that has been represented.

The dynamical behavior shown in Figures 8a and 8b is the same. There are black regions whose initial estimations converge to the infinity and the strange fixed points belong to the Julia set, as they are saddle points. The dynamical planes in Figure 9 show that for $\alpha = 216$ there exist periodic points with period four. Their orbit has been represented in the dynamical planes.

Finally, the dynamical planes associated with values of α that provide the members of family $\mathcal{G}3$ with good stability properties are represented in Figure 10, as the operator $S_3(x_1, x_2)$ does not have strange fixed points when $\alpha \in [-528, 0]$.



(a) $\alpha = -750$ (b) $\alpha = -600$

Figure 8: Dynamical planes of $\mathcal{G}3$ on $q(x)$ showing divergence areas

For $\alpha = -50$ and $\alpha = -200$ the operator has six free critical points, represented with white circles when they belong to the region that has been represented. The dynamical planes in Figure 10 show that all the initial estimations belong to the basin of attraction of $(-1, -1)$ or $(1, 1)$. Although in Figure 10a there are regions represented in black, they correspond to initial points that require more iterations until convergence to the attracting fixed points is achieved. Therefore, there are no initial guesses with divergence.

The dynamical analysis of family $\mathcal{G}3$ performed in this section shows that the values of $\alpha \in [-528, 0]$ provide the schemes of this iterative class with the best stability properties. The only fixed and critical points of the vectorial operator R_3 in this interval are the roots of $p(x)$. As a consequence, all the initial estimations represented in the dynamical planes belong to the basin of attraction of a root. On the other hand, although the multidimensional function S_3 does not have any strange fixed point when $\alpha \in [-528, 0]$, there exist six free critical points. However, they remain in the basins of attraction of the roots of $q(x)$. In particular, the scheme of family $\mathcal{G}3$ associated with $\alpha = -50$ presents the best properties in terms of stability for both polynomials, showing wide basins of attraction of the roots in the dynamical planes in Figures 7a and 10b.

3. Conclusions

The stability of different iterative classes for solving nonlinear systems of equations is studied in this paper. The analyzed schemes are members of the same iterative family with order four, but are obtained from different weight functions which results in different iterative methods and even parametric subfamilies. The dynamical study performed shows the stability of the initial family applied to two polynomial systems and different weight functions. Moreover, the best results in terms of stability are provided both by selected particular cases, corresponding to weight functions defined by a quadratic polynomial and a rational function, and by a wide range of schemes designed by a parametric weight function.

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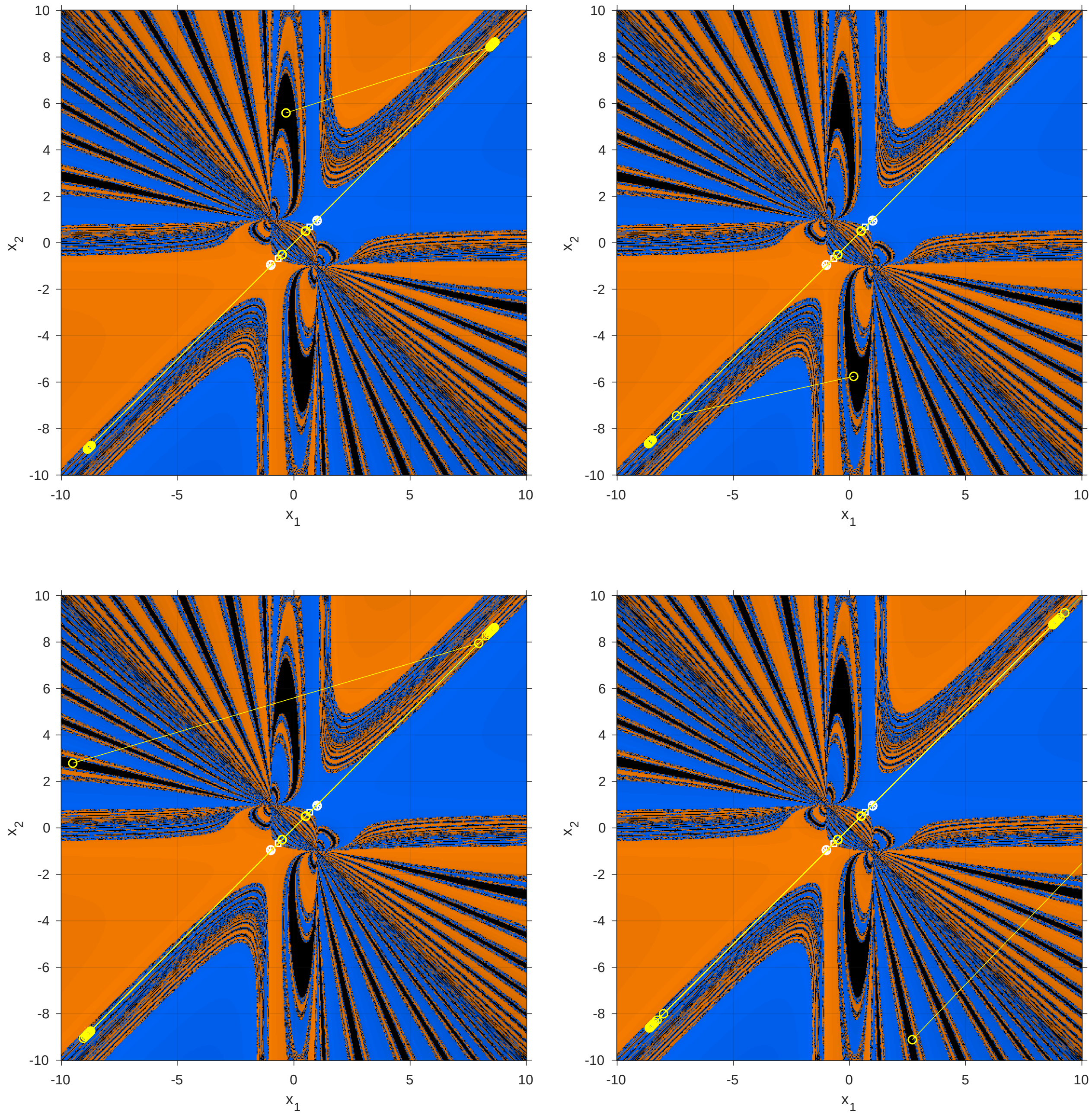

Figure 9: Dynamical planes of $\mathcal{G}3$ on $q(x)$ for $\alpha = 216$ showing attracting periodic orbits

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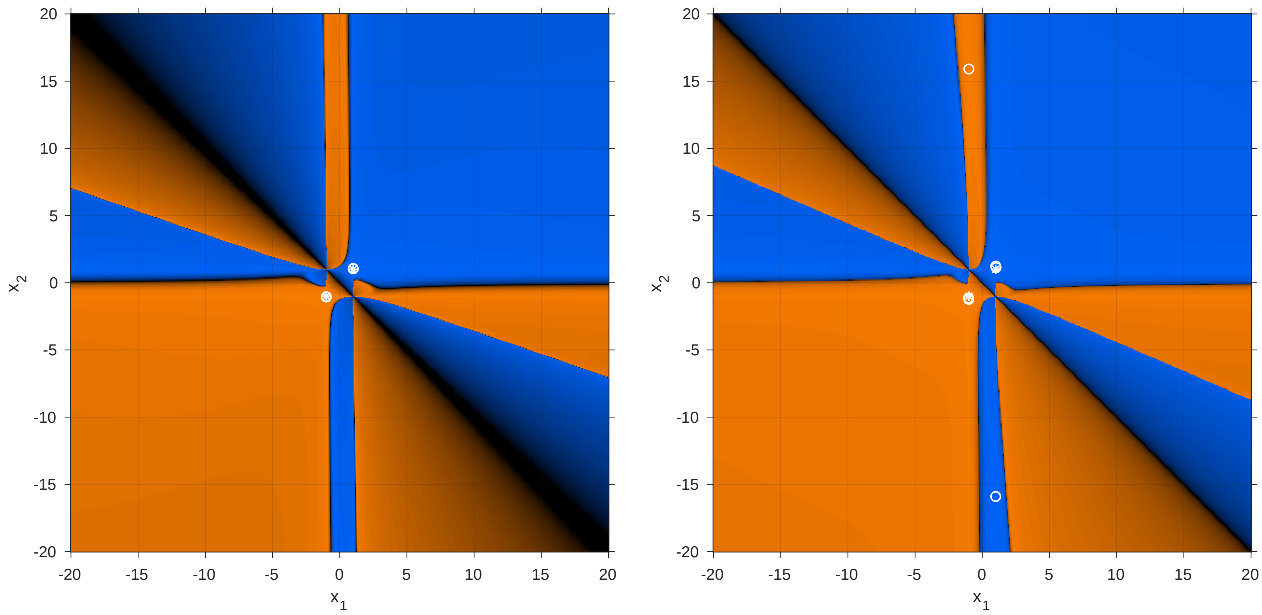

(a) $\alpha = -200$ (b) $\alpha = -50$

Figure 10: Dynamical planes of $\mathcal{G}3$ on $q(x)$ for different values of α

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