The random variable transformation method to solve some randomized first-order linear control difference equations

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1 Introduction

It is well known that Control Theory is a branch of Mathematics with many applications in Engineering. Some authors of this contribution have recently studied a complete probabilistic solution to random continuous first-order control problems (differential equations) [1]. Many control theory problems can be modelled by systems of first-order difference equations. In practice, model parameters are set by means of measurements that usually involve errors, so uncertainties. In this contribution, we are interested in the stochastic study of the following discrete control system

$$\begin{aligned} &\vec{x}(k+1) = A\vec{x}(k) + \vec{b}\,u(k), \quad k = 0,1,2,\ldots,K-1, \\ &\vec{x}(0) = \vec{x}_0, \end{aligned} \tag{1}$$

where, for each step $k = 0,1,2,\ldots,K$, $\vec{x}(k) \in \mathbb{R}^n$ represents the state of the system, A is a $n \times n$ matrix containing the free dynamics part, $\vec{b}$ is a $n \times 1$ vector, and $u(k)$ is a scalar control. Given an initial state $\vec{x}_0$, we are interested in exact controllable systems. That is, those systems in which a given final state $\vec{x}_1 \in \mathbb{R}^n$ can be reached from every initial state $\vec{x}_0$ in a fixed number of steps $K < \infty$, i.e. given any initial condition $\vec{x}_0$,

$$\vec{x}(K) = \vec{x}_1. \tag{2}$$

In particular, we study a complete stochastic analysis considering that initial value $\vec{x}_0$ and final target $\vec{x}_1$ in (1)–(2) are random vectors.
Finally, we apply our findings in a numerical example.

2 Methods

In Theorem 1, we provide the solution $\vec{x}(k)$ of problem (1) in a compact expression that will simplify our investigation. It is important to notice that this solution is a stochastic process.

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Theorem 1. *The solution of the initial value problem* (1) *for a fixed step* $k = 1, 2, ..., K$ *is*

$$\vec{x}(k) = A^k \vec{x}_0 + \sum_{j=1}^{k} A^{k-j} \vec{b}\, u(j-1) = A^k \vec{x}_0 + \mathcal{U}_k \vec{u}_k, \tag{3}$$

where each component of the vector $\vec{u}_k$ *is the scalar control evaluated in* $0, 1, 2, ..., k-1$, *i.e.,* $\vec{u}_k = [u(0), u(1), \ldots, u(k-1)]^\top$, *being the superscript* $^\top$ *the transpose operator.* $\mathcal{U}_k$ *is a* $n \times k$ *matrix defined as* $\mathcal{U}_k = \left[A^{k-1}\vec{b} \,|\, \cdots \,|\, A\vec{b} \,|\, \vec{b}\right]$.

Proof We prove it by induction.

- If $k = 1$, then by Eq. (1)

$$\vec{x}(1) = A\vec{x}(0) + \vec{b}\, u(0) \overset{\text{Eq. (1)}}{=} A\vec{x}_0 + \vec{b}\, u(0) = A^1 \vec{x}_0 + \mathcal{U}_1 \vec{u}_1.$$

- If $k = 2$, then by Eq. (1)

$$\begin{aligned} \vec{x}(2) &= A\vec{x}(1) + \vec{b}\, u(1) \overset{\text{Step } k=1}{=} A\left(A\vec{x}_0 + \vec{b}\, u(0)\right) + \vec{b}\, u(1) \\ &= A^2\vec{x}_0 + A\vec{b}\, u(0) + \vec{b}\, u(1) = A^2\vec{x}_0 + \left[A\vec{b} \,|\, \vec{b}\right] \begin{bmatrix} u(0) \\ u(1) \end{bmatrix} = A^2\vec{x}_0 + \mathcal{U}_2 \vec{u}_2. \end{aligned}$$

- Let us assume that this is true for k, let us see for $k+1$

$$\begin{aligned} \vec{x}(k+1) &= A\vec{x}(k) + \vec{b}\, u(k) \\ &\overset{\text{Eq. (3)}}{=} A\left(A^k\vec{x}_0 + \mathcal{U}_k \vec{u}_k\right) + \vec{b}\, u(k) \\ &= A\left(A^k\vec{x}_0 + \left[A^{k-1}\vec{b} \,|\, \cdots \,|\, A\vec{b} \,|\, \vec{b}\,\right] \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(k-1) \end{bmatrix}\right) + \vec{b}\, u(k) \\ &= A^{k+1}\vec{x}_0 + \left[A^k\vec{b} \,|\, \cdots \,|\, A^2\vec{b} \,|\, A\vec{b}\,\right] \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(k-1) \end{bmatrix} + \vec{b}\, u(k) \\ &= A^{k+1}\vec{x}_0 + \left[A^k\vec{b} \,|\, \cdots \,|\, A^2\vec{b} \,|\, A\vec{b} \,|\, \vec{b}\,\right] \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(k-1) \\ u(k) \end{bmatrix} \\ &= A^{k+1}\vec{x}_0 + \mathcal{U}_{k+1}\vec{u}_{k+1}. \end{aligned}$$

■

As we are interested in controllable discrete systems, in addition of the stochastic process solution it is also necessary the conditions of controllability. As components of A and $\vec{b}$ are deterministic,

necessary and sufficient condition for the controllability of system (1)–(2) can be found in the literature [2–4], since these conditions do not depend on the initial and final state.
If $K > n$, the stochastic process solution of controllable system (1)–(2) can be expressed as

$$\boxed{\vec{x}(k) = A^k \vec{x}_0 + \mathcal{U}_k \vec{u}_k = G\vec{x}_0 + H\vec{x}_1} \tag{4}$$

where

$$G = A^k - HA^K,$$

$$H = \mathcal{U}_k \begin{bmatrix} I_k & 0_{k\times(K-k)} \end{bmatrix} S,$$

$$\mathcal{U}_k = \left[A^{k-1}\vec{b} \,|\, \cdots \,|\, A\vec{b} \,|\, \vec{b} \,\right],$$

$$S = \mathcal{U}_K^\top L^{-1},$$

$$L = \mathcal{U}_K \mathcal{U}_K^\top.$$

Let us fix $k > 0$. We define the mapping $r : R^{2n} \to \mathbb{R}^{2n}$ whose components are

$$\begin{array}{rcccl} \vec{z}_1 & = & r_1(\vec{x}_0, \vec{x}_1) & = & G\,\vec{x}_0 + H\,\vec{x}_1, \\ \vec{z}_2 & = & r_2(\vec{x}_0, \vec{x}_1) & = & \vec{x}_0, \end{array}$$

The inverse mapping $s = r^{-1}$ is

$$\begin{array}{rcccl} \vec{x}_0 & = & s_1(\vec{z}_1, \vec{z}_2) & = & \vec{z}_2, \\ \vec{x}_1 & = & s_2(\vec{z}_1, \vec{z}_2) & = & H^{-1}\left\{\vec{z}_1 - G\,\vec{z}_2\right\}, \end{array}$$

The absolute value of the Jacobian of $\mathbf{s}$ is

$$|J_{2n}| = \left|\det\left(H^{-1}\right)\right|.$$

Notice that s is well defined and $|J_{2n}| \neq 0$ w.p. 1.
Now, applying the random variable transformation technique [5, Pag.25], the probability density function of the random vector $(\vec{z}_1, \vec{z}_2)$ in function of the joint PDF of the random vector of input parameters $(\vec{x}_0, \vec{x}_1)$, is

$$f_{\vec{z}_1,\vec{z}_2}(\vec{z}_1, \vec{z}_2) = f_0\left(\vec{z}_2, H^{-1}\left\{\vec{z}_1 - G\,\vec{z}_2\right\}\right) \,|\det\left(H^{-1}\right)|. \tag{5}$$

As $\vec{x}(k) = \vec{z}_1$, marginalizing (5) with respect to $\vec{z}_2 = \vec{x}_0$ we obtain the first probability density function

$$\boxed{f_1(\vec{x}, k) = \int_{\mathbb{R}^n} f_0\left(\vec{x}_0, H^{-1}\left\{\vec{x} - G\,\vec{x}_0\right\}\right) \left|\det\left(H^{-1}\right)\right| \,\mathrm{d}\vec{x}_0,} \tag{6}$$

where

$$\vec{x}_0 = (x_{01}, \ldots, x_{0n})^\top,$$

$$\mathrm{d}\vec{x}_0 = \prod_{1\le i\le n} \mathrm{d}x_{0i}.$$

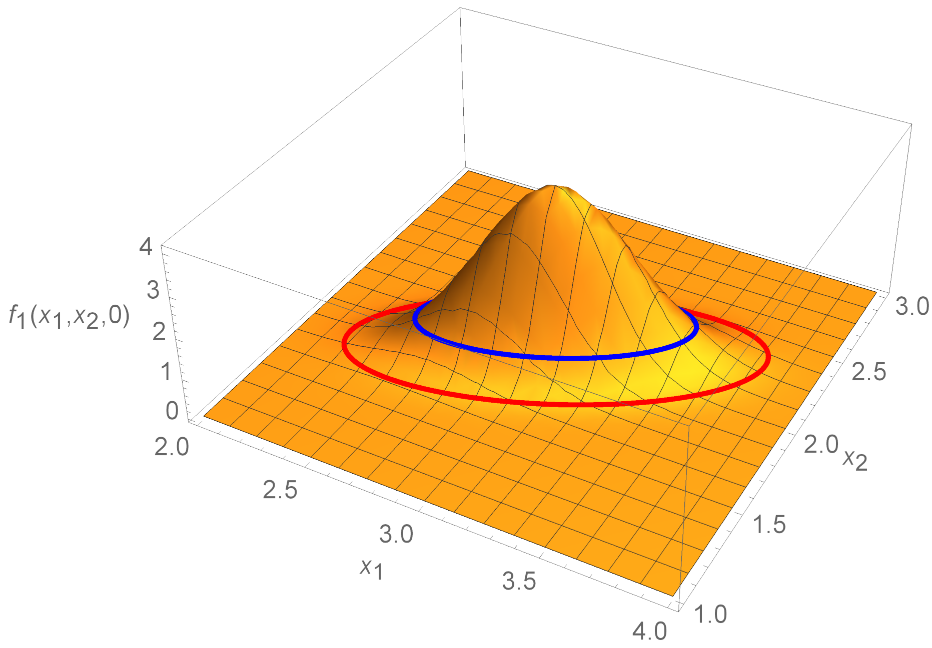

Figure 1: First probability density function, given by (6), of the solution stochastic process of controllable system (1)–(2) at $k = 0$ considering the data of the Example.

3 A numerical example

In this section, we provide a numerical example to Eq. (1)–(2) where final reached target is a deterministic vector coming from an initial random vector. We consider the following deterministic values for parameters and the number of steps, K, to reach the final condition:

$$A = \begin{pmatrix} 1/2 & 1 \\ -1 & 1/4 \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}, \qquad \vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad K = 10 \tag{7}$$

We consider $\vec{x}_0(\omega)$ a multivariate Normal distribution with mean $\vec{\mu} = (3, 2)$ and variance-covariance matrix

$$\Sigma = \begin{pmatrix} 0.08 & 0.03 \\ 0.03 & 0.03 \end{pmatrix}, \qquad \text{i.e.} \qquad \vec{x}_0 = (x_{01}, x_{02}) \sim \mathrm{N}(\vec{\mu}, \Sigma) \tag{8}$$

Notice that

$$\operatorname{rank}(\mathcal{U}_2) = \operatorname{rank}\left(A\vec{b}|\vec{b}\right) = \operatorname{rank}\begin{pmatrix} 5/4 & 1/2 \\ -1/4 & 1 \end{pmatrix} = 2$$

so Kalman's controllability condition is fulfilled and problem (1)–(2) considering (7)–(8) is exactly controllable for $K \geq 2$.

Now, we can determine the first probability density function of the solution stochastic process of (1)–(2) using (6). This probability density function is represented in different steps, k, in particular, $k = 0$ in Figure 1 and $k = 8$ in Figure 2. In this figures, we can observe the evolution in the probability density function of the solution as k increases.

Finally, in Figure 3, we have represented the phase portrait. The expectation of the solution stochastic process follows the shape of an spiral line, being represented in each step, k, by pink points. Also, at each step, confidence regions of 50% and 90% are represented in blue and red lines, respectively.

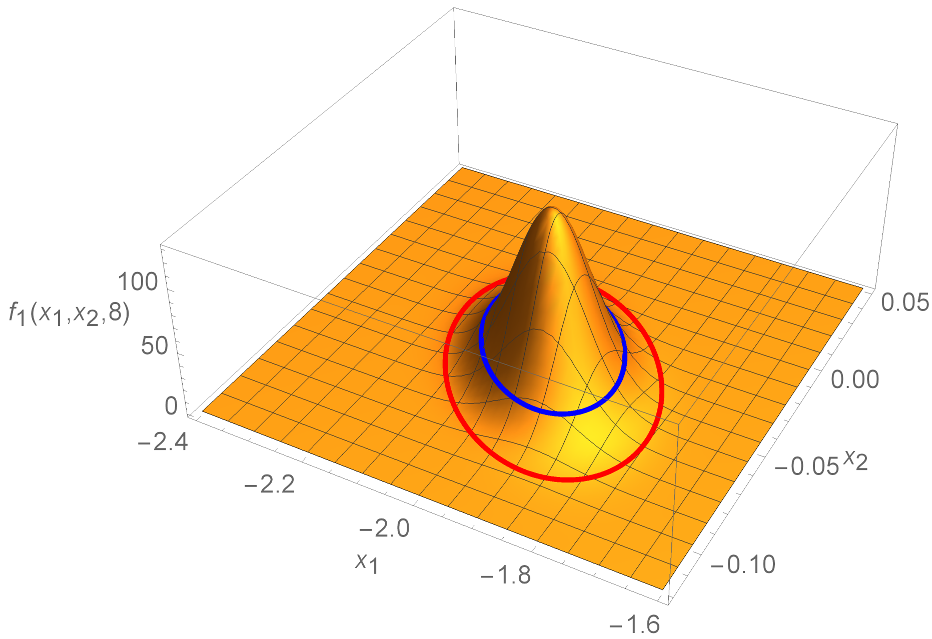

Figure 2: First probability density function, given by (6), of the solution stochastic process of controllable system (1)–(2) at $k = 8$ considering the data of the Example.

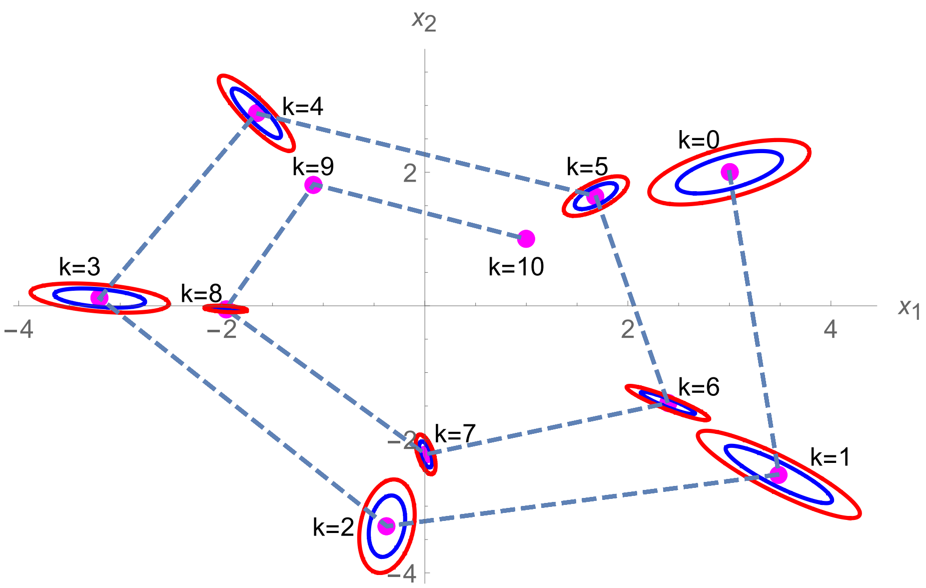

Figure 3: Phase portrait for controllable system (1)–(2). The expectation of the solution is shown in a pink point. 50% (blue) and 90% (red) confidence regions are plotted at all steps k, $k = 0, 1, 2, \dots, 10$, using the data of the Example.

4 Conclusions

In this work, we have determined a complete probabilistic solution of a randomized first-order linear control difference equation. This randomization has been considering the initial and final condition are random vectors. The solution has been provided via the first probability density function of the solution stochastic process of the difference system. This has been possible by applying the random variable transformation method. Finally, a numerical example has been illustrated in order to show the capability of the obtained theoretical results.

Acknowledgements

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