Some classes of topological spaces related to zero-sets

F. Golrizkhatami and A. Taherifar

Department of Mathematics, Yasouj University, Yasouj, Iran (F.golrizkhatami@stu.yu.ac.ir, ataherifar@yu.ac.ir)

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Abstract

An almost P-space is a topological space in which every zero-set is regular-closed. We introduce a large class of spaces, C-almost P-space (briefly CAP-space), consisting of those spaces in which the closure of the interior of every zero-set is a zero-set. In this paper we study CAP-spaces. It is proved that if X is a dense and $Z^{\#}$-embedded subspace of a space T, then T is CAP if and only if X is a CAP and CRZ-extended in T (i.e, for each regular-closed zero-set Z in X, $\mathrm{cl}_T Z$ is a zero-set in T). In 6P.5 of [8] it was shown that a closed countable union of zero-sets need not be a zero-set. We call X a CZ-space whenever the closure of any countable union of zero-sets is a zero-set. This class of spaces contains the class of P-spaces, perfectly normal spaces, and is contained in the cozero complemented spaces and CAP-spaces. In this paper we study topological properties of CZ (resp. cozero complemented)-space and other classes of topological spaces near to them. Some algebraic and topological equivalent conditions of CZ (resp. cozero complemented)-space are characterized. Examples are provided to illustrate and delimit our results.

1. Introduction

The set of zero-sets in a topological space X, $Z[X]$, need not be closed under infinite union. Even a countable union of zero-sets need not be a zero-set. For

example, every one-element set in $\mathbb{R}$ is a zero-set in $\mathbb{R}$, but $\mathbb{Q} = \bigcup_{r \in \mathbb{Q}} \{r\}$ is not a zero-set. First, we call a countable subfamily of $Z[X]$ a CZ-family if the union of its elements is a zero-set (cf. Definition 3.1). A question for us was:

When is any countable subfamily of $Z[X]$ a CZ-family?

We observe that in a space X every countable subfamily is a CZ-family if and only if X is a P-space (cf. Proposition 3.2). In 5.15 of [8], it is shown that if a countable union of zero-sets belongs to a real z-ultrafilter $\mathcal{A}$, then at least one of them belongs to $\mathcal{A}$. But we need the converse of this fact for our aims. It is shown that for an ideal I of $C(X)$, $\bigcup_{i=1}^{\infty} Z(f_i) \in Z[I]$ implies $f_i \in I$ for some $i \in \mathbb{N}$ if and only if I is a real maximal ideal (cf. Proposition 3.3). We apply this result and prove that for any countable CZ-family $\{Z_1, Z_2, ..., Z_n, ...\}$ of $Z[X]$, $cl_{\beta X}(\bigcup_{i \in \mathbb{N}} Z_i) = \bigcup_{i \in \mathbb{N}} cl_{\beta X} Z_i$ if and only if X is a pseudocompact space (cf. Theorem 3.6).

In a general space, even a closed, countable union of zero-sets need not be a zero-set; see 6P.5 in [8]. This was a motivation for introducing the class of CZ-spaces in this paper. In section 4, we introduce CZ-spaces as those spaces which in the closure of any countable union of zero-sets is a zero-set (cf. Definition 4.1). We observe that a space X is a CZ-space if and only if the set of basic z-ideals is closed under countable intersection, i.e., for every countable subset $f_1, f_2, ..., f_n, ...$ of $C(X)$ there exists $f \in C(X)$ such that $\bigcap_{i \in \mathbb{N}} M_{f_i} = M_f$ (cf. Lemma 4.3). Every open z-embedded subset of a CZ-space (hence open C^*-embedded and cozero-sets) is a cz-space (cf. Proposition 4.4).

In section 5, we give some new equivalent conditions algebraic and topological for the class of cozero complemented spaces. It is proved that a space X is cozero complemented if and only if the set of basic z^o-ideals is closed under countable intersection, i.e., for every countable subset $f_1, f_2, ..., f_n, ...$ of $C(X)$ there exists $f \in C(X)$ such that $\bigcap_{i \in \mathbb{N}} P_{f_i} = P_f$ if and only if for each $f \in C(X)$ there exists a $g \in C(X)$ such that $\text{Ann}(f) = P_g$ (cf. Theorem 5.1). A topological space X is called CAP-space if the closure of the interior of every zero-set in X is a zero-set (cf. Definition 5.3). This class of spaces contains the class of almost P-spaces and perfectly normal spaces. We conclude that every CZ-space is a cozero complemented space and a CAP-space (cf. Proposition 5.6). Examples are given to show that the converse need not be true. We also call a topological space X, CRZ (resp., CZ)-extended in a space T containing it if for each regular-closed zero-set (resp., zero-set) $Z \in Z[X]$, $\text{cl}_T Z$ is a zero-set in T (cf. Definition 5.7). Examples of CRZ (resp., CZ)-extended are given (cf. Example 5.8). We prove that for a dense and $Z^{\#}$-embedded space X in a space T, X is CAP and CRZ-extended in T if and only if T is CAP (cf. Theorem 5.13). From this result, we get the following results (cf. Corollary 5.13):

(1) If X is a weakly Lindelöf dense space in a space T, then X is CAP and CRZ-extended in T if and only if T is CAP.

(2) If X is CZ-extended in T, then X is CAP if and only if T is a CAP-space.

(3) For any completely regular space X, βX is a CAP-space if and only if X is a CAP and CRZ-extended in βX.

(4) If X is a CZ-extended in βX, then βX is a CAP-space if and only if X is so.

2. Preliminaries

In this paper, all spaces are completely regular Hausdorff and $C(X)$ ($C^*(X)$) is the ring of all (bounded) real-valued continuous functions on a space X.

For each $f \in C(X)$, the zero-set of f denoted by $Z(f)$ is the set of zeros of f and $\cos f$ is the set $X \setminus Z(f)$ which is called the cozero-set of f. The set of all zero-sets in X is denoted by $Z[X]$ and for each ideal I in $C(X)$, $Z[I]$ is the set of all zero-sets of the form $Z(f)$, where $f \in I$. The support of $f \in C(X)$, is the set $\mathrm{cl}_X(X \setminus Z(f))$.

The space βX is known as the *Stone-Čech compactification* of X. It is characterized as that compactification of X in which X is C^*-embedded as a dense subspace. The space υX is the *real compactification* of X, if X is C-embedded in this space as a dense subspace. For a completely regular Hausdorff space X, we have $X \subseteq \upsilon X \subseteq \beta X$. Whenever $Z = Z(f) \in Z[X]$, we denote $Z(f^\beta)$ with Z^β, where f^β is the extension of f to βX. By a z-ultrafilter on X is meant a maximal z-filter, i.e., one not contained in any other z-filter. When M is a real maximal ideal in $C(X)$, we refer to $Z[M]$ as a real-z-ultrafilter. Thus, the real z-ultrafilters are those with the countable intersection property.

For any $p \in \beta X$, O^p (resp., M^p) is the set of all $f \in C(X)$ for which $p \in \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} Z(f)$ (resp., $p \in \mathrm{cl}_{\beta X} Z(f)$). Also, for $A \subseteq \beta X$, O^A (resp., M^A) is the intersection of all O^p(resp., M^p) where $p \in A$, and whenever $A \subseteq X$, we denote it by O_A (resp., M_A).

The intersection of all minimal prime ideals of $C(X)$ (resp., maximal ideals of $C(X)$) containing f is denoted by P_f (resp., M_f). It is proved in [5] that $P_f = \{g \in C(X) : \mathrm{int}_X Z(f) \subseteq \mathrm{int}_X Z(g)\}$ and $M_f = \{g \in C(X) : Z(f) \subseteq Z(g)\}$. An ideal I of $C(X)$ is a z-ideal if for each $f \in C(X)$, $M_f \subseteq I$. For $f \in C(X)$, $\mathrm{Ann}(f) = \{g \in C(X) : fg = 0\}$ and it is easy to see that $\mathrm{Ann}(f) = M_{X \setminus Z(f)}$. The reader is referred to [8] for more details on $C(X)$.

3. Countable union of zeros-sets

Definition 3.1. A countable subfamily $\{Z_1, Z_2, ..., Z_n, ...\}$ of $Z[X]$ is called a CZ-family if $\bigcup_{i=1}^\infty Z_i$ is a zero-set.

If we consider the real numbers $\mathbb{R}$ with the usual topology and consider a countable zero-set $Z \in Z[\mathbb{R}]$ (e.g., $Z = Z(f)$, where $f(x) = \cos x$), then $\{\{x\} : x \in Z\}$ is a CZ-family and in this space $\{\{x\} : x \in \mathbb{Q}\}$, where $\mathbb{Q}$ is the set of rational numbers, is not a CZ-family of $Z[\mathbb{R}]$.

Recall from [8], a space X is a P-space if $C(X)$ is a regular space, i.e., every zero-set in X is open. The next result shows that whenever X is a non P-space, then there is a countable subfamily of $Z[X]$ which is not a CZ-family.

Proposition 3.2. *The following statements are equivalent.*

   (1) *Every countable subfamily of $Z[X]$ is a CZ-family.*
   (2) *X is a P-space.*
   (3) *For every countable subset $\{Z_1, Z_2, ..., Z_n, ...\}$ of $Z[X]$, $\bigcup_{i \in \mathbb{N}} \operatorname{int}_X Z_i$ is a zero-set.*
   (4) *Every z-ultrafilter on X is closed under countable union.*

Proof. $(1) \Rightarrow (2)$ As every cozero-set is a countable union of zeros-sets, this is evident by [8, 4.J].

$(2) \Rightarrow (3)$ By [8, 4.J], every zero-set is open, so this is obvious.

$(3) \Rightarrow (4)$ Let $Z[M^p]$ be a z-ultrafilter on X. By hypothesis and the fact that every cozero-set is a countable union of the interior of zero-sets, every cozero-set is a zero-set. That is every zero-set is a cozero-set. Now assume $\{Z(f_i)|i \in \mathbb{N}\}$ is a countable subset of $Z[M^p]$. Then for each $i \in \mathbb{N}$, there exists a cozero-set $X \setminus Z(g_i)$ such that $Z(f_i) = X \setminus Z(g_i)$. Hence, $\bigcup_{i \in \mathbb{N}} Z(f_i) = \bigcup_{i \in \mathbb{N}} (X \setminus Z(g_i)) = X \setminus \bigcap_{i \in \mathbb{N}} Z(g_i) = X \setminus Z(g)$, for some $g \in C(X)$. Again by hypothesis, $X \setminus Z(g)$ is a zero-set, so $\bigcup_{i \in \mathbb{N}} Z(f_i) \in Z[M^p]$.

$(4) \Rightarrow (1)$ Consider a countable subset $S = \{Z_1, Z_2, ..., Z_n, ...\}$ of $Z[X]$. For each $i \in \mathbb{N}$, define $Z_i' = Z_1 \cup Z_2 \cup ... \cup Z_i$. Then $S' = \{Z_i' : i \in \mathbb{N}\}$ has the finite intersection property and $\bigcup S = \bigcup S'$. Now, consider the collection of all zero-sets in X that contains finite intersections of the members of S'. This is a proper z-filter in X. Thus this z-filter is contained in a unique z-ultrafilter, say, $Z[M^p]$ for some $p \in \beta X$. For each $i \in \mathbb{N}$, we have $Z_i' \in Z[M^p]$. Thus by hypothesis, $\bigcup_{i \in \mathbb{N}} Z_i = \bigcup_{i \in \mathbb{N}} Z_i' \in Z[M^p]$. So S is a CZ-family. $\qquad \square$

Proposition 3.3. *Let I be an ideal of $C(X)$. Then $\bigcup_{i \in \mathbb{N}} Z(f_i) \in Z[I]$ implies $f_i \in I$ for some $i \in \mathbb{N}$ if and only if I is a real maximal ideal.*

Proof. The sufficiency follows from [8, 5.15(a)].

Necessity. First, trivially I is a z-ideal. Next, for each $n \in \mathbb{N}$, put $Z_n = \{x \in X : |f(x)| \geq 1/n\}$ and suppose that $f \notin I$. We have $Z(f) \cup (\bigcup_{n \in \mathbb{N}} Z_n) = X \in Z[I]$. Thus $Z_n \in Z[I]$, for some $n \in \mathbb{N}$. But Z_n is of the form $Z(1 - gf)$ for some $g \in C(X)$, see Lemma 2.1 in [2]. Thus $Z(1 - gf) \in Z[I]$ for some $g \in C(X)$. This shows that $1 - gf \in I$, i.e., I is a maximal ideal. Now we want to prove I is a real ideal. By [8, Theorem 5.14], it is enough to show that I has the countable intersection property. To see this, let for each $n \in \mathbb{N}$, $Z_n \in Z[I]$ and $\bigcap_{n \in \mathbb{N}} Z_n = \varnothing$. Then $\bigcup_{n \in \mathbb{N}} (X \setminus Z_n) = X \in Z[I]$. As every $X \setminus Z_n$ is a countable union of zero-sets, we have a zero-set $Z \in Z[I]$ contained in some $X \setminus Z_n$. This contradicts with $Z_n \in Z[I]$. $\qquad \square$

It is well known that $\upsilon X = \{p \in \beta X : M^p \text{ is real }\}$. Now by using the above theorem we obtain the following result.

Corollary 3.4. *Let $p \in \beta X$. Then $p \in \upsilon X$ if and only if for each CZ-family $\{Z(f_i) : i \in \mathbb{N}\}$, $p \in \operatorname{cl}_{\beta X}(\bigcup_{i=1}^{\infty} Z(f_i))$ implies $p \in \operatorname{cl}_{\beta X} Z(f_i)$ for some $i \in \mathbb{N}$.*

We again apply Proposition 3.3 for proving the next results.

Corollary 3.5. *Let $\{Z_1, Z_2, ..., Z_n, ...\}$ be a CZ-family in $Z[X]$.*

(1) $\mathrm{cl}_{\upsilon X}(\bigcup_{i=1}^{\infty} Z_i) = \bigcup_{i=1}^{\infty} \mathrm{cl}_{\upsilon X} Z_i$.

(2) *The set $\{Z_1^{\upsilon}, Z_2^{\upsilon}, ..., Z_n^{\upsilon}, ...,\}$ is a CZ-family in $Z[\upsilon X]$.*

Proof. (1) Trivially, $\bigcup_{i=1}^{\infty} \mathrm{cl}_{\upsilon X} Z_i \subseteq \mathrm{cl}_{\upsilon X}(\bigcup_{i=1}^{\infty} Z_i)$. Now for the proof of the other inclusion, let $p \in \mathrm{cl}_{\upsilon X}(\bigcup_{i=1}^{\infty} Z_i)$. Then this and hypothesis imply that $\bigcup_{i=1}^{\infty} Z_i \in Z[M^p]$. Hence there exists $i \in \mathbb{N}$ such that $Z_i \in Z[M^p]$, by Proposition 3.3. This shows that $p \in \mathrm{cl}_{\upsilon X} Z_i$, and so $p \in \bigcup_{i=1}^{\infty} \mathrm{cl}_{\upsilon X} Z_i$. So we are done.

(2) Let $\bigcup_{i=1}^{\infty} Z_i = Z$, where $Z \in Z[X]$. By (1), $\bigcup_{i=1}^{\infty} Z_i^{\upsilon} = \bigcup_{i=1}^{\infty} \mathrm{cl}_{\upsilon X} Z_i = \mathrm{cl}_{\upsilon X}(\bigcup_{i=1}^{\infty} Z_i) = \mathrm{cl}_{\upsilon X} Z = Z^{\upsilon}$. $\square$

The CZ-family condition for $\{Z_1, Z_2, ..., Z_n, ...\}$ in Part 1 of the above result is necessary. For, consider $\mathbb{Q}$ as a subspace of $\mathbb{R}$ with the usual topology. As mentioned the set $\{\{x\} : x \in \mathbb{Q}\}$ is not a CZ-family and for each $x \in \mathbb{Q}$, $\{x\}$ is a zero-set. However, $\mathrm{cl}_{\mathbb{R}} \bigcup_{x \in \mathbb{Q}} \{x\} \neq \bigcup_{x \in \mathbb{Q}} \{x\}$.

Theorem 3.6. *The following statements hold.*

(1) *For every CZ-family $\{Z_1, Z_2, ..., Z_n, ...\}$ of $Z[X]$, we have $\mathrm{cl}_{\beta X}(\bigcup_{i=1}^{\infty} Z_i) = \bigcup_{i=1}^{\infty} \mathrm{cl}_{\beta X} Z_i$ if and only if X is pseudocompact.*

(2) *For every countable subset $\{Z_1, Z_2, ..., Z_n, ...\}$ of $Z[X]$, we have $\mathrm{int}_{\beta X} \mathrm{cl}_{\beta X}(\bigcup_{i=1}^{\infty} Z_i) = \bigcup_{i=1}^{\infty} \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} Z_i$, if and only if X is finite.*

(3) *For every countable subset $\{Z_1, Z_2, ..., Z_n, ...\}$ of $Z[X]$, we have $\mathrm{int}_X(\bigcup_{i=1}^{\infty} Z_i) = \bigcup_{i=1}^{\infty} \mathrm{int}_X Z_i$ if and only if X is a P-space.*

Proof. (1) Necessity. Let $p \in \beta X \setminus \upsilon X$. Then for each $n \in \mathbb{N}$, there exists $Z_n \in Z[M^p]$ such that $\bigcap_{n=1}^{\infty} Z_n = \varnothing$. Thus $\bigcup_{n=1}^{\infty}(X \setminus Z_n) = X$. As each $X \setminus Z_n$ is a countable union of zero-sets say $\bigcup_{m=1}^{\infty} Z_{mn}$, so we have $\bigcup_{n=1}^{\infty}(\bigcup_{m=1}^{\infty} Z_{mn}) = X$. This shows that the set $\{Z_{mn} : m, n \in \mathbb{N}\}$ is a CZ-family in $Z[X]$. So, by hypothesis, $\bigcup_{n=1}^{\infty}(\bigcup_{m=1}^{\infty} \mathrm{cl}_{\beta X} Z_{mn}) = \beta X$. Thus there exist $m, n \in \mathbb{N}$ such that $Z_{mn} \subseteq X \setminus Z_n$ and $p \in \mathrm{cl}_{\beta X} Z_{mn}$. This shows that $Z_{mn} \in Z[M^p]$, a contradiction.

Sufficiency, we have $\beta X = \upsilon X$, so this follows from Corollary 3.5.

(2) Necessity. Suppose that $p \in \beta X$ and $\bigcup_{n=1}^{\infty} Z_n \in Z[O^p]$. Then

$$p \in \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X}(\bigcup_{i=1}^{\infty} Z_i) = \bigcup_{i=1}^{\infty} \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} Z_i.$$

Thus there exists $n \in \mathbb{N}$ such that $p \in \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} Z_n$, i.e., $Z_n \in Z[O^p]$. By Proposition 3.3, $O^p = M^p$ and $p \in \upsilon X$ (i.e., $\beta X = \upsilon X$). Thus by [8, 7L], X is a P-space and $\beta X = \upsilon X$ implies X is pseudocompact, by [8, 8A.2]. Hence by [8, 4K.2], X is finite.

Sufficiency, X is finite, so this is obvious.

(3) Necessity. The proof is similar to the (2).

Sufficiency, X is a P-space. Thus by Proposition 3.2, $\bigcup_{i=1}^{\infty} Z_i$ is a zero-set in X and so $p \in \mathrm{int}_X(\bigcup_{i=1}^{\infty} Z_i)$ implies $\bigcup_{i=1}^{\infty} Z_i \in Z[O_p] = Z[M_p]$. Thus, by Proposition 3.3, $Z_i \in Z[O_p]$, i.e., $p \in \mathrm{int}_X Z_i$, for some $i \in \mathbb{N}$. The other inclusion always holds, so we are done. $\square$

4. CZ-SPACE

Definition 4.1. A topological space X is called a CZ-space if the closure of any countable union of zero-sets is a zero-set.

Example 4.2.

(1) By Proposition 3.2, every P-space is a CZ-space.
(2) Every perfectly normal space (e.g., a metric space) is a CZ-space. So the space of real numbers with usual topology is a CZ-space which is not a P-space.
(3) In [8, 6P.5], $\mathbb{N}$ is a closed discrete subset of the space $\Lambda = \beta\mathbb{R}\setminus(\beta\mathbb{N}\setminus\mathbb{N})$, hence is a closed countable union of zero-sets. However it is not a zero-set. Thus Λ is not a CZ-space.
(4) In [8, 4.N], S is a non-discrete P-space and hence this is a CZ-space. However it contains a closed subset which is not a zero-set. So this is an example of CZ which is not a perfectly normal space.

Let us call a z-ultrafilter $\mathcal{F}$ a CZ-ultrafilter if for each countable subset $\{Z_1, Z_2, ..., Z_n, ...\}$ of $\mathcal{F}$, $\mathrm{cl}_X(\bigcup_{i\in\mathbb{N}} Z_i) \in \mathcal{F}$.

Lemma 4.3. *The following statements are equivalent.*

(1) *The space X is a CZ-space.*
(2) *For every countable subset $\{f_1, f_2, ..., f_n, ...\}$ of $C(X)$ there exists $f \in C(X)$ such that $\bigcap_{i\in\mathbb{N}} M_{f_i} = M_f$.*
(3) *Every z-ultrafilter on X is a CZ-ultrafilter.*

Proof. (1)$\Rightarrow$(2) Consider an arbitrary countable subset $\{f_1, f_2, ..., f_n, ...\}$ of $C(X)$. By hypothesis, there exists $f \in C(X)$ such that $\mathrm{cl}_X(\bigcup_{i\in\mathbb{N}} Z(f_i)) = Z(f)$. Thus we have:

$$M_{\bigcup_{i\in\mathbb{N}} Z(f_i)} = M_{\mathrm{cl}_X(\bigcup_{i\in\mathbb{N}} Z(f_i))} = M_{Z(f)} = M_f.$$

Trivially $M_{\bigcup_{i\in\mathbb{N}} Z(f_i)} = \bigcap_{i\in\mathbb{N}} M_{f_i}$. So we are done.

(2)$\Rightarrow$(3) Let $\{Z(f_1), Z(f_2), ...Z(f_n), ...\}$ be a countable subset of a z-ultrafilter $Z[M^p]$. By hypothesis, $\bigcap_{i\in\mathbb{N}} M_{f_i} = M_f$, for some $f \in C(X)$. This equality shows that $M_{\bigcup_{i\in\mathbb{N}} Z(f_i)} = M_{Z(f)}$. Hence $\mathrm{cl}_X(\bigcup_{i\in\mathbb{N}} Z(f_i)) = Z(f)$. Thus $\mathrm{cl}_X(\bigcup_{i\in\mathbb{N}} Z(f_i)) \in Z[M^p]$.

(3)$\Rightarrow$(1) Consider a countable subset $S = \{Z_1, Z_2, ..., Z_n, ...\}$ of $Z[X]$. For each $i \in \mathbb{N}$, define $Z'_i = Z_1 \cup Z_2 \cup ... \cup Z_i$. Then $S' = \{Z'_i : i \in \mathbb{N}\}$ has the finite intersection property and $\bigcup S = \bigcup S'$. Now, consider the collection of all zero-sets in X that contains finite intersections of members of S'. This is a proper z-filter in X. Thus this z-filter is contained in a unique z-ultrafilter, say, $Z[M^p]$ for some $p \in \beta X$. For each $i \in \mathbb{N}$, we have $Z'_i \in Z[M^p]$. Thus by hypothesis, $\mathrm{cl}_X(\bigcup_{i\in\mathbb{N}} Z_i) = \mathrm{cl}_X(\bigcup_{i\in\mathbb{N}} Z'_i) \in Z[M^p]$. So X is a CZ-space. $\square$

Proposition 3.2 shows that whenever X is a P-space, then every z-ultrafilter is a CZ-ultrafilter. However, if X is a CZ-space which is not a P-space (e.g., $\mathbb{R}$ with usual topology), then there is a CZ-ultrafilter which is not closed under countable union.

Recall from [6], a subset S of the topological space X is z-embedded if each zero-set of S is the restriction to S of a zero-set of X. Now we will see that the open z-embedded subsets inherit the CZ-property from the space.

Proposition 4.4. *The following statements hold.*

(1) *Every open z-embedded subspace of a CZ-space is a CZ-space.*

(2) *Every cozero-set in a CZ-space is a CZ-space.*

(3) *Every open C^*-embedded (resp., C-embedded) subspace of a CZ-space is a CZ-space.*

Proof. (1) Let S be an open z-embedded subspace of X and $\{Z_i : i \in \mathbb{N}\}$ be a countable subset of $Z[S]$. By hypothesis, for each $i \in \mathbb{N}$, there exists $Z_i' \in Z[X]$ such that $Z_i' \cap S = Z_i$. X is a CZ-space, so there exists $Z' \in Z[X]$ with $\mathrm{cl}_X(\bigcup_{i \in \mathbb{N}} Z_i') = Z'$. It is easy to see that

$$\mathrm{cl}_S(\bigcup_{i \in \mathbb{N}} Z_i) = \mathrm{cl}_X(\bigcup_{i \in \mathbb{N}} Z_i) \cap S = \mathrm{cl}_X(\bigcup_{i \in \mathbb{N}} Z_i' \cap S) \cap S = \mathrm{cl}_X(\bigcup_{i \in \mathbb{N}} Z_i') \cap S = Z' \cap S.$$

This shows that S is a CZ-space.

(2) By [6, Proposition 1.1], every cozero-set in X is an open z-embedded. So this follows from (1).

(3) Trivially every open C^*-embedded (resp., C-embedded) subspace is a z-embedded set, so this follows from (1). □

A space X is an F-space (resp., F'-space) if disjoint cozero subsets of X are contained in disjoint zero sets (resp., if disjoint cozero subsets have disjoint closures). As every cozero-set is a countable union of zero-sets, whenever X is a CZ-space, the closure of every cozero subset is a zero-set. Thus we obtain the following result.

Corollary 4.5. *If X is an F'-space and a CZ-space, then it is an F-space.*

In the sequel we characterize some topological properties of the classes of CZ-spaces. Recall from [10] that if $f : X \to Y$ is a continuous surjection map and $f(Z[X]) \subseteq Z[Y]$, then f is said to be zero-set preserving. The following result is Lemma 3.20 of [10].

Lemma 4.6. *An open perfect surjection is zero-set preserving.*

Theorem 4.7. *The following statements hold.*

(1) *If $f : X \to Y$ is open and zero-set preserving and X is CZ, then Y is CZ.*

(2) *If X is compact and $X \times Y$ is CZ, then Y is CZ.*

Proof. (1) Let $\{Z(f_i) : i \in \mathbb{N}\}$ be a countable subset of $Z[Y]$. Then $\{f^{-1}(Z(f_i)) : i \in \mathbb{N}\} \subseteq Z[X]$. Since, for each $i \in \mathbb{N}$, $f^{-1}(Z(f_i)) = Z(f_i \circ f) \in Z[X]$. X is a CZ-space, so there exists $Z(g) \in Z[X]$ such that $\mathrm{cl}_X(\bigcup_{i \in \mathbb{N}} f^{-1}(Z(f_i))) = Z(g)$. We claim that $\mathrm{cl}_Y(\bigcup_{i \in \mathbb{N}} Z(f_i)) = f(Z(g))$, which is a zero-set in Y, by hypothesis. To see this, let $y \in \mathrm{cl}_Y(\bigcup_{i \in \mathbb{N}} Z(f_i))$. We have $y = f(x)$, for some $x \in X$. It is enough to show that $x \in Z(g)$, i.e., $x \in \mathrm{cl}_X(\bigcup_{i \in \mathbb{N}} f^{-1}(Z(f_i)))$. Let U be an

open set in X containing x. Then $f(U)$ is open in Y and containing y and hence $f(U) \cap (\bigcup_{i\in\mathbb{N}} Z(f_i)) \neq \varnothing$. Thus $f(U) \cap Z(f_i) \neq \varnothing$, for some $i \in \mathbb{N}$. This implies $U \cap f^{-1}(Z(f_i)) \neq \varnothing$, for some $i \in \mathbb{N}$. Hence $U \cap (\bigcup_{i\in\mathbb{N}} f^{-1}(Z(f_i))) \neq \varnothing$, i.e., $x \in \mathrm{cl}_X(\bigcup_{i\in\mathbb{N}} f^{-1}(Z(f_i)))$. Now assume $y = f(x) \in f(Z(g))$, where $x \in Z(g)$ and G be an open set in Y containing y. Then $x \in f^{-1}(G)$, which is open in X. Thus $f^{-1}(G) \cap \bigcup_{i\in\mathbb{N}} f^{-1}(Z(f_i)) \neq \varnothing$. Hence $f^{-1}(G) \cap f^{-1}(Z(f_i)) \neq \varnothing$, for some $i \in \mathbb{N}$. This implies $G \cap Z(f_i) \neq \varnothing$, i.e., $G \cap (\bigcup_{i\in\mathbb{N}} Z(f_i)) \neq \varnothing$. Thus $y \in \mathrm{cl}_Y(\bigcup_{i\in\mathbb{N}} Z(f_i))$.

(2) The map $\pi_Y : X \times Y \to Y$ is an open perfect map (since X is compact) and surjective. Thus it is zero-set preserving, by Lemma 4.6. So this follows from (1). $\qquad\square$

As we found algebraic equivalent for a CZ-space in Lemma 4.3 and the fact that $C(X) \simeq C(\upsilon X)$ we obtain the following result.

Proposition 4.8. *The following statements hold.*

(1) *If $C(X)$ is isomorphic with $C(Y)$ (as two rings) and X is a CZ-space, then Y is a CZ-space.*

(2) *X is a CZ-space if and only if υX is a CZ-space.*

(3) *If X is pseudocompact and CZ, then βX is a CZ-space.*

5. CZ-SPACE AND OTHER CLASSES OF TOPOLOGICAL SPACES

A space X is *cozero complemented* if, given any cozero set U, there is a disjoint cozero set V such that $U \cup V$ is dense in X. In [4], this class of space is called m-space, i.e., every prime z^o-ideal of $C(X)$ is minimal. By Proposition 1.5 in [4], X is cozero complemented if and only if for every zero-set $Z \in Z[X]$ there exists a zero-set $F \in Z[X]$ such that $Z \cup F = X$ and $\mathrm{int}_X Z \cap \mathrm{int}_X F = \varnothing$. By Corollary 5.5 in [9], this is equivalent to compactness of the space of minimal prime ideals of $C(X)$. In this section we give some another algebraic and topological equivalent conditions for this class of spaces and conclude that every CZ-space is a cozero complemented space. Some topological properties of cozero complemented spaces are also characterized. We also introduce some other classes of topological spaces which are used in the sequel. In particular, we introduce a large class of topological spaces, which are called CAP-spaces (cf. Definition 4.3), and observe that these spaces, although they are different from the cozero-complemented spaces, behave in a similar manner as the latter ones. We also provide several examples (cf. Examples 5.4 and 5.8).

Theorem 5.1. *The following statements are equivalent.*

(1) *The closure of any countable union of the interior of zero-sets is the closure of the interior of a zero-set.*

(2) *For every countable subset $\{f_1, f_2, ..., f_n, ...\}$ of $C(X)$ there exists $f \in C(X)$ such that $\bigcap_{i\in\mathbb{N}} P_{f_i} = P_f$.*

(3) *For each countably generated ideal I of $C(X)$, there exists $g \in C(X)$ such that $\mathrm{Ann}(I) = P_g$.*

(4) *For each $f \in C(X)$, there exists $g \in C(X)$ such that $\mathrm{Ann}(f) = P_g$.*

(5) X *is a cozero complemented space.*
(6) *Every support in X is the closure of the interior of a zero-set.*

Proof. (1)$\Rightarrow$(2) let $\{f_1, f_2, ..., f_n, ...\}$ be a countable subset of $C(X)$. Clearly,

$$O_{\bigcup_{i\in\mathbb{N}} \text{int}_X Z(f_i)} = \bigcap_{i\in\mathbb{N}} P_{f_i}.$$

By hypothesis, $\text{cl}_X(\bigcup_{i\in\mathbb{N}} \text{int}_X Z(f_i)) = \text{cl}_X(\text{int}_X Z(f))$, for some $f \in C(X)$. Since $\bigcup_{i\in\mathbb{N}} \text{int}_X Z(f_i)$ is open, we have,

$$O_{\bigcup_{i\in\mathbb{N}} \text{int}_X Z(f_i)} = M_{\bigcup_{i\in\mathbb{N}} \text{int}_X Z(f_i)} = M_{\text{cl}_X(\bigcup_{i\in\mathbb{N}} \text{int}_X Z(f_i))} =$$

$$M_{\text{cl}_X(\text{int}_X Z(f))} = M_{\text{int}_X Z(f)} = O_{\text{int}_X Z(f)} = P_f.$$

This implies $\bigcap_{i\in\mathbb{N}} P_{f_i} = P_f$.

(2)$\Rightarrow$(3) Let I be an ideal of $C(X)$ generated by $\{f_1, f_2, ..., f_n, ...\}$. For f_i $(1 \leq i \leq n)$, there exists a countable subset $\{f_{1i}, f_{2i}, ..., f_{mi}, ...\}$ of $C(X)$ such that $X \setminus Z(f_i) = \bigcup_{m\in\mathbb{N}} int_X Z(f_{mi})$. Trivially we have,

$$\text{Ann}(I) = M_{\bigcup_{i\in\mathbb{N}}(X\setminus Z(f_i))} = M_{\bigcup_{i\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} int_X Z(f_{mi})} = \bigcap_{i\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} O_{int_X Z(f_{mi})}$$

$$= \bigcap_{i\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} P_{f_{mi}}.$$

By hypothesis, $\bigcap_{i\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} P_{f_{mi}} = P_g$ for some $g \in C(X)$. So we are done.

(3)$\Rightarrow$(4) Trivial.

(4)$\Rightarrow$(5) Let $\text{cl}_X(X\setminus Z(f))$ be a support in X. By hypothesis, $\text{cl}_X(\text{int}_X Z(f)) = \text{cl}_X(X \setminus Z(g))$, for some $g \in C(X)$. Get complement of two hands of the equality, we have $\text{int}_X(\text{cl}_X(X \setminus Z(f))) = \text{int}_X(Z(g))$. Hence $\text{cl}_X(\text{int}_X(\text{cl}_X(X \setminus Z(f))) = \text{cl}_X(\text{int}_X(Z(g)))$. It is easy to see that $\text{cl}_X(\text{int}_X(\text{cl}_X(X \setminus Z(f))) = \text{cl}_X(X \setminus Z(f))$. Thus $\text{cl}_X(X \setminus Z(f)) = \text{cl}_X(\text{int}_X(Z(g)))$.

(5)$\Rightarrow$(1) Let $\{Z(f_1), Z(f_2), ..., Z(f_n)\}$ be a countable subset of $Z[X]$. By hypothesis, for each $i \in \mathbb{N}$, there exists $g_i \in C(X)$ such that $\text{int}_X Z(f_i) = \text{int}_X \text{cl}_X(X \setminus Z(g_i))$. Thus we have,

$$\text{cl}_X(\bigcup_{i=1}^{\infty} \text{int}_X Z(f_i) = \text{cl}_X(\bigcup_{i=1}^{\infty} \text{int}_X \text{cl}_X(X \setminus Z(g_i)) = \text{cl}_X(\bigcup_{i=1}^{\infty}(X \setminus Z(g_i)))$$

$$= \text{cl}_X(X \setminus \bigcap_{i=1}^{\infty} Z(g_i)).$$

There exists $g \in C(X)$ such that $\bigcap_{i=1}^{\infty} Z(g_i) = Z(g)$. Thus $\text{cl}_X(\bigcup_{i=1}^{\infty} \text{int}_X Z(f_i)) = \text{cl}_X(X \setminus Z(g)$, which is the closure of the interior of some zero-set, by hypothesis. $\square$

Proposition 5.2. *The following statements are equivalent.*

(1) *Every support in X is a zero-set.*
(2) *The closure of any countable union of the interior of zero-sets is a zero-set.*

(3) *For every countable subset $\{f_1, f_2, ..., f_n, ...\}$ of $C(X)$ there exists $f \in C(X)$ such that $\bigcap_{i \in \mathbb{N}} P_{f_i} = M_f$.*
(4) *For each countably generated ideal I of $C(X)$, there exists $g \in C(X)$ such that $\mathrm{Ann}(I) = M_g$.*
(5) *For each $f \in C(X)$, there exists $g \in C(X)$ such that $\mathrm{Ann}(f) = M_g$.*

Proof. $(1) \Rightarrow (2)$ Let $\{Z(f_i) : i \in \mathbb{N}\}$ be a countable subset of $Z[X]$. By hypothesis, for each $i \in \mathbb{N}$, there exists a cozero-set $X \setminus Z(g_i)$ such that $\mathrm{int}_X Z(f_i) = X \setminus Z(g_i)$. Thus $\mathrm{cl}_X(\bigcup_{i=1}^{\infty} \mathrm{int}_X Z(f_i)) = \mathrm{cl}_X(\bigcup_{i=1}^{\infty}(X \setminus Z(g_i))) = \mathrm{cl}_X(X \setminus \bigcap_{i=1}^{\infty} Z(g_i))$. There exists $g \in C(X)$ such that $\bigcap_{i=1}^{\infty} Z(g_i) = Z(g)$. Hence $\mathrm{cl}_X(\bigcup_{i=1}^{\infty} \mathrm{int}_X Z(f_i)) = \mathrm{cl}_X(X \setminus Z(g))$, which is a zero-set, by hypothesis.

$(2) \Rightarrow (3)$ Let $\{f_1, f_2, ..., f_n, ...\}$ be a countable subset of $C(X)$. Clearly, $\bigcap_{i=1}^{\infty} P_{f_i} = \bigcap_{i=1}^{\infty} O_{\mathrm{int}_X Z(f_i)} = O_{\bigcup_{i=1}^{\infty} \mathrm{int}_X Z(f_i)}$. By hypothesis, $\mathrm{cl}_X(\bigcup_{i=1}^{\infty} \mathrm{int}_X Z(f_i)) = Z(f)$, for some $f \in C(X)$. Hence $O_{\bigcup_{i=1}^{\infty} \mathrm{int}_X Z(f_i)} = M_{\bigcup_{i=1}^{\infty} \mathrm{int}_X Z(f_i)} = M_{Z(f)} = M_f$, i.e., $\bigcap_{i=1}^{\infty} P_{f_i} = M_f$.

$(3) \Rightarrow (4)$ The proof is similar to the proof of $(2) \Rightarrow (3)$ of Theorem 5.1, step by step.

$(4) \Rightarrow (5)$ Trivial.

$(5) \Rightarrow (1)$ Consider $\mathrm{cl}_X(X \setminus Z(f))$ as a support. By hypothesis, $\mathrm{Ann}(f) = M_g$ for some $g \in C(X)$. Thus $M_{X \setminus Z(f)} = M_g = M_{Z(g)}$. This implies $\mathrm{cl}_X(X \setminus Z(f)) = \mathrm{cl}_X(Z(g)) = Z(g)$. $\qquad \square$

A point $p \in X$ is said to be an almost P-point if $f \in M_p$, $\mathrm{int}_X Z(f) \neq \varnothing$, and X is called an almost P-space if every point of X is an almost P-point. It is easy to see that a space X is an almost P-space if and only if every zero-set in X is regular-closed. The reader is referred to [1], [7], [11] and [13], for more details and properties of almost P -spaces.

Definition 5.3. A space X is called a C-almost P-space (briefly CAP-space) if the closure of the interior of every zero-set in X is a zero-set.

Example 5.4.

(1) Clearly every almost P-space is a CAP-space.
(2) Every CZ-space is a CAP-space. For, let $Z \in Z[X]$. Then $\mathrm{cl}_X(X \setminus Z) = X \setminus \mathrm{int}_X Z = Z(f)$, for some $f \in C(X)$. Thus $\mathrm{cl}_X \mathrm{int}_X(Z) = \mathrm{cl}_X(X \setminus Z(f)$. As X is CZ, $\mathrm{cl}_X(X \setminus Z(f))$ is a zero-set, hence $\mathrm{cl}_X \mathrm{int}_X Z$ is a zero-set. This implies every perfectly normal space (hence a metric space) is a CAP-space. Thus $\mathbb{R}$ with usual topology is a CAP-space which is not an almost P-space.
(3) Clearly every O_Z-space (i.e., a space which in every regular-closed subset is a zero-set) is a CAP-space.

Lemma 5.5. *A space X is a CAP-space if and only if every basic z^o-ideal of $C(X)$ is a basic z-ideal.*

Proof. $\Rightarrow$ Let P_f be a basic z^o-ideal. By hypothesis, there exists $g \in C(X)$ such that $\mathrm{cl}_X(\mathrm{int}_X Z(f)) = Z(g)$. Thus we have,

$$P_f = O_{\mathrm{int}_X Z(f)} = M_{\mathrm{int}_X Z(f)} = M_{\mathrm{cl}_X(\mathrm{int}_X(Z(f)))} = M_{Z(g)} = M_g.$$

$\Leftarrow$ Let $Z(f) \in Z[X]$. There exists $g \in C(X)$ such that $P_f = M_g$. Since $P_f = O_{\mathrm{int}_X Z(f)}$ and $M_g = M_{Z(g)}$. The equality $P_f = M_g$ implies $\mathrm{cl}_X(\mathrm{int}_X Z(f)) = Z(g)$. So X is a CAP-space. $\qquad\square$

Proposition 5.6. *The following statements hold.*

(1) *Every CZ-space is a cozero complemented space.*
(2) *Every support in X is a zero-set if and only if X is a cozero complemented space and a CAP-space.*

Proof. (1) X is a CZ-space. Thus every support is a zero-set, so by Proposition 5.2, for each $f \in C(X)$ there exists $g \in C(X)$ such that $\mathrm{Ann}(f) = M_g$. As $\mathrm{Ann}(f)$ is a z^o-ideal, this implies M_g is a z^o-ideal and hence equals with P_g. Now this follows from Theorem 5.1.

(2) First assume every support in X is a zero-set and $Z \in Z[X]$. Then $\mathrm{cl}_X(X \setminus Z) = Z(f)$ for some $f \in C(X)$. This implies $\mathrm{cl}_X(\mathrm{int}_X(Z)) = \mathrm{cl}_X(X \setminus Z(f))$, i.e., X is a cozero complemented space, by Proposition 1.5 in [4]. Again by hypothesis, $\mathrm{cl}_X(X \setminus Z(f))$ is a zero-set. Thus X is a CAP-space. Conversely, let $\mathrm{cl}_X(X \setminus Z)$ be a support. By hypothesis, $\mathrm{cl}_X(X \setminus Z) = \mathrm{cl}_X(\mathrm{int}_X(Z(g)))$, for some $g \in C(X)$. As X is a CAP-space, $\mathrm{cl}_X(\mathrm{int}_X(Z(g)))$ is a zero-set. Thus $\mathrm{cl}_X(X \setminus Z)$ is a zero-set. $\qquad\square$

Part 2 of the above result shows that if X is a CAP-space which is not a cozero complemented space or a cozero complemented space which is not a CAP-space, then there is a non zero-set support in X. Thus X is not a CZ-space. To see examples, first, consider the space X as the one point compactification of an uncountable discrete space. Then X is an almost P-space, since any non-empty G_δ of it contains an isolated point of the space, hence is a CAP-space. But this is not a cozero complemented space and hence is not a CZ-space, see example 3.3 in [4]. Next, consider the space $\Lambda = \beta\mathbb{R} \setminus (\beta N \setminus N)$ in [8, 6P.5]. We have $\beta\Lambda = \beta\mathbb{R}$. Thus $\beta\Lambda$ is a cozero complemented and hence Λ is a cozero complemented space. However, we know that it is not a CZ. On the other hand, by [8, 6p.5], Λ is pseudocompact, so if $\beta\Lambda = \upsilon\Lambda$ is CZ, then we must have Λ is CZ, by Proposition 4.8, which is not true. Thus $\beta\Lambda = \beta\mathbb{R}$ is not a CZ-space while we know that $\mathbb{R}$ is a CZ-space.

Henriksen and Woods in [10] showed that for an uncountable discrete space D, the Stone-Čhech compactification βD of D is cozero complemented but $\beta D \times \beta D$ is not, and so by part (1) of Proposition 5.6, $\beta D \times \beta D$ is not a CZ-space.

Definition 5.7. A subspace X of a space T is called CRZ (resp., CZ)-extended in T if for each regular-closed zero-set (resp., zero-set) $Z \in Z[X]$, $\mathrm{cl}_T Z$ is a zero-set in T.

Example 5.8.

(1) Let X be a C^*-embedded in T. If X is C-embedded in T, then it is CZ-extended in T. For, $\mathrm{cl}_T Z(f) = Z(f^o)$, for $f \in C(X)$ and f^o is the continuous extension of f to T.

(2) Every pseudocomact space is a CZ-extended in βX. For, if X is pseudocompact, then $\beta X = \upsilon X$ and for each $Z \in Z[X]$, $\mathrm{cl}_{\beta X} Z = \mathrm{cl}_{\upsilon X} Z = Z^\upsilon = Z^\beta$.

(3) Consider the infinite P-space X (e.g., the discrete space $\mathbb{N}$). Then every zero-set $Z \in Z[X]$ is open and hence $\mathrm{cl}_{\beta \mathbb{X}} Z$ is clopen in βX. Thus it is a zero-set in βX. This says that X is a CZ-extended in βX. However X need not be C-embedded in βX.

(4) Trivially every CZ-extended in a space containing it, is a CRZ-extended. But the space $\Sigma = \mathbb{N} \cup \{\sigma\}$ in [8, 4M] is CRZ-extended in $\beta \Sigma = \beta \mathbb{N}$ which is not a CZ-extended in $\beta \mathbb{N}$, see [8, 6E].

(5) Trivially every CZ-extended X in a space containing it, is Z-embedded. However a Z-embedded need not be a CZ-extended. For example, Σ is Z-embedded in $\beta \Sigma$, but is not CZ-extended in it.

Recall from [10], let X and T be two completely regular spaces and X be a subspace of T. X is said to be $Z^\#$-embedded in T if for each $f \in C(X)$, there exists a $g \in C(T)$ such that $\mathrm{cl}_X(\mathrm{int}_X(Z(f))) = \mathrm{cl}_T(\mathrm{int}_T(Z(g))) \cap X$. The following lemma is proved in [10].

Lemma 5.9. *If X is a subspace of T that is either open or dense, then the following are equivalent.*

(1) X *is $Z^\#$-embedded in T.*

(2) *If $Z \in Z[X]$, then there is a $Z^t \in Z[T]$ such that $\mathrm{int}_X Z = (\mathrm{int}_T Z^t) \cap X$.*

Lemma 5.10. *Suppose that X is dense or open as well as being $Z^\#$-embedded in a space T.*

(1) *If T is CAP, then so is X.*

(2) *If every support in T is a zero-set, then every support in X is a zero-set.*

Proof. (1) First assume X is dense in T. We show X is a CAP-space. Let $Z \in Z[X]$. By hypothesis and Lemma 5.9, there is $Z^t \in Z[T]$ such that $\mathrm{int}_X Z = \mathrm{int}_T Z^t \cap X$. Thus $\mathrm{cl}_X(\mathrm{int}_X Z) = \mathrm{cl}_X(\mathrm{int}_T Z^t \cap X) = \mathrm{cl}_T(\mathrm{int}_T Z^t \cap X) \cap X = \mathrm{cl}_T(\mathrm{int}_T Z^t) \cap X$. By hypothesis, there exists $Z(f) \in Z[T]$ such that $\mathrm{cl}_T(\mathrm{int}_T Z^t) = Z(f)$. Hence $\mathrm{cl}_X(\mathrm{int}_X Z) = Z(f) \cap X$ which is a zero-set in X. If X is open in T, then for each $p \in \mathrm{cl}_T(\mathrm{int}_T Z^t) \cap X$ and U open in X containing p, we have U is open in T and so $U \cap \mathrm{int}_X Z = U \cap \mathrm{int}_T(Z^t) \cap X \neq \varnothing$. Hence $\mathrm{cl}_X(\mathrm{int}_X Z) = \mathrm{cl}_T(\mathrm{int}_T Z^t) \cap X$. So we are done.

(2) Every support in T is a zero-set, so by Proposition 5.6, T is a cozero complemented space and a CAP-space. Thus X is a cozero complemented space, by Lemma 2.5 in [10]. Also, by part 1, X is CAP. Now Proposition 5.6 implies every support in X is a zero-set. $\qquad\square$

Since every open (dense) z-embedded subset is $Z^{\#}$-embedded, and every cozero (resp., C^*-embedded)-subset is z-embedded in any space containing it, the above lemma implies next result.

Corollary 5.11. *The following statements hold.*

(1) *Every open (dense) z-embedded subspace of a CAP-space is CAP.*
(2) *Every cozero-set in a CAP-space is CAP.*
(3) *Every open (dense) C^*-embedded in a CAP-space is a CAP.*

Recall that if every open cover of a space X contains a countable subfamily whose union is dense in X, then X is called weakly Lindelöf space. Every Lindelöf space and every ccc-space is weakly Lindelöf space, while an uncountable discrete space is not a weakly Lindelöf space.

Corollary 5.12. *If $S \cap W$ is a weakly Lindelöf space where S is a dense subspace and W is an open subspace of a CAP-space T, then $S \cap W$ is CAP.*

Proof. Similar to the proof of Theorem 2.6 in [10], we have $\mathrm{cl}_W(S \cap W) = \mathrm{cl}_T(S \cap W) \cap W = \mathrm{cl}_T W \cap W = W$. Thus $S \cap W$ is dense in W. So, since $S \cap W$ is weakly Lindelöf, so is W. We have W is open in T, hence this is $Z^{\#}$-embedded, by Lemma 2.4 of [10]. Thus W is CAP, by Lemma 5.10. But $S \cap W$ is dense in W, so by Lemma 2.4 in [10], $S \cap W$ is $Z^{\#}$-embedded in W. Thus by Lemma 5.10, we are done. $\square$

Theorem 5.13. *Let X be dense and $Z^{\#}$-embedded in a space T. Then the following statements hold.*

(1) *T is a CAP-space if and only if X is a CAP-space and CRZ-extended in T.*
(2) *Every support in T is a zero-set if and only if every support in X is a zero-set and X is CRZ-extended in T.*

Proof. (1)$\Rightarrow$ Part 1 of Lemma 5.10 shows X is a CAP-space. Now, assume $Z(f) \in Z[X]$ be a regular-closed zero-set. Then $Z(f) = \mathrm{cl}_X(\mathrm{int}_X(Z(f)))$ and by hypothesis, there exists $f^t \in C(T)$ such that $\mathrm{int}_X Z(f) = \mathrm{int}_T Z(f^t) \cap X$. T is CAP, hence there is a $g \in C(T)$ such that $\mathrm{cl}_T \mathrm{int}_T Z(f^t) = Z(g)$. Thus we have,

$$\mathrm{cl}_T Z(f) = cl_T(\mathrm{cl}_X(\mathrm{int}_X(Z(f)))) = cl_T(\mathrm{cl}_X(\mathrm{int}_T(Z(f^t)) \cap X))$$

$$= cl_T(\mathrm{int}_T(Z(f^t)) \cap X) = \mathrm{cl}_T(\mathrm{int}_T(Z(f^t))) = Z(g).$$

This completes the proof.

$\Leftarrow$ Let $Z(f^t) \in Z[T]$. Then $Z(f^t) \cap X \in Z[X]$. X is a CAP-space, hence $\mathrm{cl}_X(\mathrm{int}_X(Z(f^t) \cap X)) = Z(g)$ for some $g \in C(X)$. Thus $Z(g)$ is a regular-closed set. On the other hand, we have $\mathrm{int}_X(Z(f^t) \cap X) = \mathrm{int}_T Z(f^t) \cap X$. To see it, $\mathrm{int}_T Z(f^t) \cap X$ is open in X and contained in $Z(f^t) \cap X$. Thus it is contained in $\mathrm{int}_X(Z(f^t) \cap X)$. Now, suppose $p \in \mathrm{int}_X(Z(f^t) \cap X)$. Then $p \in X$ and there is an open subset U of T such that $p \in U \cap X \subseteq Z(f^t)$. This implies

$p \in U \subseteq \mathrm{cl}_T U = \mathrm{cl}_T(U \cap X) \subseteq Z(f^t)$, i.e., $p \in \mathrm{int}_T Z(f^t) \cap X$. It is easy to see that;

$$\mathrm{cl}_T(\mathrm{int}_T(Z(f^t))) = \mathrm{cl}_T(\mathrm{int}_T(Z(f^t)) \cap X) = \mathrm{cl}_T(\mathrm{int}_X(Z(f^t) \cap X)) =$$
$$\mathrm{cl}_T(\mathrm{cl}_X(\mathrm{int}_X(Z(f^t) \cap X)) = \mathrm{cl}_T(Z(g)).$$

As X is CRZ-extended in T, $\mathrm{cl}_T Z(g)$ is a zero-set in T, so $\mathrm{cl}_T(\mathrm{int}_T(Z(f^t)))$ is a zero-set in T, i.e., T is a CAP-space.

$(2) \Rightarrow$ Part 2 of Lemma 5.10 implies every support in X is a zero-set. On the other hand, T is CAP, by Proposition 5.6. Thus by Part (1), for any regular-closed zero-set Z in X, $\mathrm{cl}_T Z$ is a zero-set in T, i.e., X is CRZ-extended in T.

$\Leftarrow$ Every support in X is a zero-set, hence X is a cozero complemented space and a CAP-space, by Proposition 5.6. This implies T is a cozero complemented space, by Theorem 2.8 in [10]. Moreover, T is a CAP, by part (1). Now, again by using Proposition 5.6, we have every support in T is a zero-set. $\qquad \square$

It is easy to see that if X is CZ-extended in T, then X is $Z^{\#}$-embedded in T. So we conclude the following result from the above theorem.

Corollary 5.14. *The following statements hold.*

    (1) *If X is weakly Lindelöf and dense in T, then T is CAP if and only if X is CAP and CRZ-extended in T.*

    (2) *If X is CZ-extended in T, then X is CAP if and only if T is a CAP-space.*

    (3) *βX is a CAP-space if and only if X is a CAP and CRZ-extended in βX.*

    (4) *If X is a CZ-extended in βX, then βX is a CAP-space if and only if X is so.*

    (5) *Every support in βX is a zero-set if and only if every support in X is a zero-set and X is CRZ-extended in βX.*

    (6) *If X is a CZ-extended in βX, then every support in βX is a zero-set if and only if every support in X is a zero-set.*

6. Directions and some questions

CZ – space. As we have shown a space X is CZ if and only if the set of basic z-ideals is closed under countable intersection. This shows that this class of spaces is important. So we may have focus on the spaces $\mathrm{Max}(C(X))$ and $\mathrm{Spec}(C(X))$, with Zariski topology, whenever X is a CZ-space. On the other hand, since the set $L = \{M_f : f \in C(X)\}$ is a lattice with two operations: $M_f \vee M_g = M_{f^2+g^2}$ and $M_f \wedge M_g = M_f \cap M_g = M_{fg}$. So, we have X is a CZ-space if and only if for every countable subset S of L, $\wedge S \in L$. This can help us to investigate more properties of CZ-spaces by the lattice properties of L.

We have seen that if X is a pseudocompact and CZ, then βX is a CZ-space. As an example, we have seen that $\mathbb{R}$ is a CZ-space, but $\beta \mathbb{R}$ is not a CZ. However, this is a remainder question as follows:

Question 6.1. *When is βX a CZ-space?*

We also have some other questions as follows:

Question 6.1. Is every CZ hereditarily CZ (i.e., all subspaces are CZ)?

Question 6.2. Is the product of two (infinite) CZ-spaces a CZ?

Question 6.3. Suppose $X \times Y$ is a CZ-space. Must X or Y be CZ?

CAP $-$ space. We observed that the class of CAP-spaces behaves like cozero-complemented spaces. So we may investigate other properties of CAP-spaces. It also is important to know the relation between CAP and cozero-complemented spaces. We know that X is cozero-complemented if and only if $\mathrm{Min}\,(C(X))$ is compact. So this is a motivation to ask the following questions:

Question 6.4. What is $\mathrm{Min}\,(C(X))$, when X is a CZ?

Question 6.5. What is $\mathrm{Max}\,(C(X))$, when X is a CZ?

We also have some questions as follows:

Question 6.6. Characterize the CAP spaces which are hereditarily CAP.

Question 6.7. Is the product of two (infinite) CAP-spaces a CAP?

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