Applied General Topology

Appl. Gen. Topol. 23, no. 1 (2022), 45-54
doi:10.4995/agt.2022.16128

Investigation of topological spaces using relators

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Communicated by H. Dutta

Abstract

In this paper, we define uniformities and topologies as relators and show the equivalences of these definitions with the classical ones. For this, we summarize the essential properties of relators, using their theory from earlier works of Á. Száz. Moreover, we prove implications between important topological properties of relators and disprove others. Finally, we show that our earlier analogous definition [G. Pataki, Investigation of proximal spaces using relators, Axioms 10, no. 3 (2021): 143.] for uniformly and proximally filtered property is equivalent to the topological one.
At the end of this paper, uniformities and topologies are defined in the same way. This will give us new possibilities to compare these and other topological structures.

2020 MSC: *54E15; 54A05; 54G15; 54G20; 54D10.*

Keywords: *(generalized) uniformities; (generalized) topologies; relators.*

1. Introduction

At the beginning of the 20th century some mathematicians tried to define abstract topological structures. The most relevant results due to Poincaré 1895, Fréchet 1906, Hausdorff 1914, and Kuratowski 1922.

Uniform spaces in terms of relations were introduced by Weil in 1937 [12].

Uniform and topological spaces formulated the recently usable form by Bourbaki in 1953 [1].

Received 26 August 2021 – Accepted 23 November 2021

After the works of Davis, Pervin, and Nakano [2], [8], and [4], in 1987 Száz [9] introduced the notion of relators and relator spaces in the following way.

Definition 1.1. A nonvoid family $\mathcal{R}$ of relations on a nonvoid set X is called a relator on X, and the ordered pair $(X, \mathcal{R})$ is called a relator space.

In the last decades, a few authors investigated the interpretation of well-known topological properties in terms of relators.

For more details, see, for instance, [7] and [6], but for the readers' convenience, we summarize the necessary notions and notations.

Remark 1.2. With the usual notations, the statement $\mathcal{R}$ is a relator on X means that

$$X \neq \varnothing, \qquad \varnothing \neq \mathcal{R} \subset \operatorname{Exp}(X^2),$$

where $\operatorname{Exp}(X)$ is the power set of X, and $X^2 = X \times X$.

If R is a relation on X, $x \in X$, and $A \subset X$, then the sets

$$R(x) = \{y \in X : (x,y) \in R\}, \qquad \text{and} \qquad R[A] = \bigcup_{x \in A} R(x)$$

are called the images of x and A under R, respectively.

2. Preliminary Concepts

Definition 2.1. If R and S are relations on X, then the composition of R and S can be defined, such that $(R \circ S)(x) = R[S(x)]$ for all $x \in X$.

Moreover, let $R^{-1} = \{(y,x) : (x,y) \in R\}$, $R^0 = \Delta_X = \{(x,x) : x \in X\}$ and $R^n = R \circ R^{n-1}$, for all $n = 1, 2, \ldots$. Finally, we say that R is

- reflexive if $R^0 \subset R$,
- symmetric if $R^{-1} \subset R$,
- transitive if $R^2 \subset R$.

Lemma 2.2. *If R is a relation on X, and $A, B \subset X$, then*

$$R[A] \subset B \iff R^{-1}[X \setminus B] \subset X \setminus A.$$

Definition 2.3. If $\mathcal{R}$ is a relator on X, then the relators

$$\mathcal{R}^* = \{S \subset X^2 : \exists R \in \mathcal{R} : R \subset S\},$$

and

$$\mathcal{R}^\wedge = \{S \subset X^2 : \forall x \in X : \exists R \in \mathcal{R} : R(x) \subset S(x)\},$$

are called the uniform and the topological refinements of $\mathcal{R}$, respectively. For more details, see [7].

Moreover, for all $n = -1, 0, 1, 2, \ldots$, we define

$$\mathcal{R}^n = \{R^n : R \in \mathcal{R}\}.$$

Remark 2.4. $*$ and $\wedge$ are really refinements as we defined in [7], that is, they are self-increasing in the sense that

$$\mathcal{R} \subset \mathcal{S}^* \iff \mathcal{R}^* \subset \mathcal{S}^* \qquad \text{and} \qquad \mathcal{R} \subset \mathcal{S}^\wedge \iff \mathcal{R}^\wedge \subset \mathcal{S}^\wedge,$$

or equivalently, they are expansive, increasing, and idempotent, in the sense that

$$\mathcal{R} \subset \mathcal{R}^*, \ \mathcal{R} \subset \mathcal{S} \implies \mathcal{R}^* \subset \mathcal{S}^*, \ \mathcal{R}^{**} = \mathcal{R}^*$$

and

$$\mathcal{R} \subset \mathcal{R}^\wedge, \ \mathcal{R} \subset \mathcal{S} \implies \mathcal{R}^\wedge \subset \mathcal{S}^\wedge, \ \mathcal{R}^{\wedge\wedge} = \mathcal{R}^\wedge,$$

for all $\mathcal{R}$ and $\mathcal{S}$ relators on X.

Moreover, $\wedge$ is $*$-dominating, $*$-invariant, $*$-absorbing and $*$-compatible, that is if $\mathcal{R}$ is a relator on X, then

$$\mathcal{R}^* \subset \mathcal{R}^\wedge, \qquad \mathcal{R}^\wedge = \mathcal{R}^{\wedge *}, \qquad \mathcal{R}^\wedge = \mathcal{R}^{*\wedge}, \qquad \mathcal{R}^{\wedge *} = \mathcal{R}^{*\wedge}.$$

For all $n = -1, 0, 1, 2, \dots$ the mapping $\mathcal{R} \mapsto \mathcal{R}^n$ of relators on X is increasing.

Finally, $*$ is inversion-compatible, that is, for all $\mathcal{R}$ relators on X

$$\mathcal{R}^{*-1} = \mathcal{R}^{-1*}.$$

And we have that for all $\mathcal{R}$ relators on X

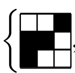
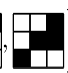
$$\mathcal{R}^{2*} = \mathcal{R}^{*2*}.$$

The following examples show, that the analog assertions are not true for $\wedge$.

Example 2.5. Let $X = \{1, 2, 3\}$, and

$$\mathcal{R} = \{\Delta_X \cup \{(1,2)\}, \Delta_X \cup \{(3,2)\}\}$$

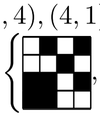
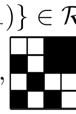
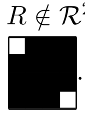
is an elementwise reflexive and transitive relator on X. Now, $\mathcal{R}^{\wedge -1} \not\subset \mathcal{R}^{-1\wedge}$, since $\Delta_X \in \mathcal{R}^{\wedge -1}$ however $\Delta_X \notin \mathcal{R}^{-1\wedge}$. Moreover, if $\mathcal{S} = \mathcal{R}^{-1}$, then $\Delta_X \in \mathcal{S}^{-1\wedge} \setminus \mathcal{S}^{\wedge -1}$.

Note, that $\mathcal{R} = \Big\{$ ,  $\Big\}$.

Example 2.6. Let $X = \{1, 2, 3, 4\}$, and

$$\mathcal{R} = \{\Delta_X \cup \{(1,2),(4,2),(2,1),(2,4)\}, \Delta_X \cup \{(1,3),(4,3),(3,1),(3,4)\}\}$$

is an elementwise reflexive and symmetric relator on X. Now, $\mathcal{R}^{\wedge 2} \not\subset \mathcal{R}^{2\wedge}$, since $R = X^2 \setminus \{(1,4),(4,1)\} \in \mathcal{R}^{\wedge 2}$ however $R \notin \mathcal{R}^{2\wedge}$.

Note, that $\mathcal{R} = \Big\{$ ,  $\Big\}$, and $R =$ .

Definition 2.7. If $\mathcal{R}$ is a relator on X, then for any $x \in X$ and $A \subset X$, we write:

$$x \in \operatorname{int}_{\mathcal{R}}(A) \text{ if } R(x) \subset A \text{ for some } R \in \mathcal{R},$$

and

$$A \in \mathcal{T}_{\mathcal{R}} \text{ if } A \subset \operatorname{int}_{\mathcal{R}}(A).$$

The relation $\operatorname{int}_{\mathcal{R}}$ is called the topological interior, and the elements of $\mathcal{T}_{\mathcal{R}}$ are called the topologically open subsets induced by $\mathcal{R}$ on X.

Theorem 2.8.

$$\operatorname{int} : \operatorname{Exp}(\operatorname{Exp}(X^2)) \setminus \{\varnothing\} \to \operatorname{Exp}(\operatorname{Exp}(X) \times X)$$

is a $\wedge$–increasing set-valued function for relators on X in the sense that

$$\mathcal{S} \subset \mathcal{R}^{\wedge} \iff \operatorname{int}_{\mathcal{S}} \subset \operatorname{int}_{\mathcal{R}}$$

for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X.

Moreover, it follows that if $\mathcal{R}$ is a relator on X, then $\mathcal{R}^{\wedge}$ is the largest relator on X such that

$$\operatorname{int}_{\mathcal{R}} = \operatorname{int}_{\mathcal{R}^{\wedge}} .$$

Theorem 2.9.

$$\mathcal{T} : \operatorname{Exp}(\operatorname{Exp}(X^2)) \setminus \{\varnothing\} \to \operatorname{Exp}(\operatorname{Exp}(X))$$

is an increasing set-valued function for relators on X in the sense that

$$\mathcal{S} \subset \mathcal{R} \implies \mathcal{T}_{\mathcal{S}} \subset \mathcal{T}_{\mathcal{R}}$$

for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X.

Moreover, if $\mathcal{R}$ is a relator on X, then

$$\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\wedge}}.$$

3. A new form of generalized topologies

Definition 3.1. Let $\mathcal{R}$ be a relator on X, and let $\square \in \{*, \wedge\}$ be a refinement for relators on X. We define the followings.

- $\mathcal{R}$ is $\square$-reflexive, if $\mathcal{R} \subset \mathcal{R}^{0\square}$;
- $\mathcal{R}$ is $\square$-symmetric, if $\mathcal{R} \subset \mathcal{R}^{-1\square}$;
- $\mathcal{R}$ is $\square$-transitive, if $\mathcal{R} \subset \mathcal{R}^{2\square}$;
- $\mathcal{R}$ is $\square$-fine, if $\mathcal{R} = \mathcal{R}^{\square}$.

For instance, we say that $\mathcal{R}$ is uniformly symmetric or topologically transitive instead of $*$-symmetric or $\wedge$-transitive, respectively.

Following Weil, we say that the relator $\mathcal{R}$ on X is a generalized uniformity on X, and the relator space $(X, \mathcal{R})$ is a generalized uniform space if it is

- uniformly reflexive;
- uniformly symmetric;
- uniformly transitive;
- uniformly fine.

Moreover, we say that the relator $\mathcal{R}$ on X, is a generalized topology on X, and the relator space $(X, \mathcal{R})$ is a generalized topological space if it is

- topologically reflexive;
- topologically transitive;
- topologically fine.

Definition 3.2. If X is an arbitrary set, and $\mathcal{T} \subset \operatorname{Exp}(X)$ satisfies the following axioms, then we say that it is a generalized set-topology on X.

(1) $\mathcal{T}$ is closed under arbitrary union, that is $\bigcup \mathcal{A} \in \mathcal{T}$ for any $\mathcal{A} \subset \mathcal{T}$;
(2) $X \in \mathcal{T}$.

The following relations were investigated by Davis, Pervin, and Száz.

Definition 3.3. Let A be a subset of X. Then, the relation

$$D_A = A^2 \cup (X \setminus A) \times X$$

is called the Davis–Pervin relation on X generated by A.

Some parts of the following theorem were proved in [10] and [11].

Proposition 3.4. *If $\mathcal{R}$ is a relator on X, and*

$$\psi : \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(\mathcal{P}(X^2)), \qquad \psi(\mathcal{T}) = \{D_A : A \in \mathcal{T}\},$$

then

(1) $\mathcal{T}_\mathcal{R}$ is a generalized set-topology for all $\mathcal{R}$ relators on X;
(2) $\psi(\mathcal{T})^\wedge$ is a generalized topology for all $\varnothing \neq \mathcal{T} \subset \mathcal{P}(X)$;
(3) If $\mathcal{T}$ is a generalized set-topology on X, then $\mathcal{T} = \mathcal{T}_{\psi(\mathcal{T})}$;
(4) If $\mathcal{R}$ is a generalized topology on X, then $\mathcal{R} = \psi(\mathcal{T}_\mathcal{R})^\wedge$.

Proof. (1) If $\mathcal{A} \subset \mathcal{T}_\mathcal{R}$ and $x \in \bigcup \mathcal{A}$, then there exists an $A \in \mathcal{A}$ such that $x \in A$. It follows that $R(x) \subset A \subset \bigcup \mathcal{A}$ for some $R \in \mathcal{R}$, since $A \in \mathcal{T}_\mathcal{R}$. Therefore, $\bigcup \mathcal{A} \in \mathcal{T}_\mathcal{R}$.

$X \in \mathcal{T}_\mathcal{R}$ means only that the triviality $R(x) \subset X$ for all $x \in X$ and $R \in \mathcal{R}$. Note that $\mathcal{R} \neq \varnothing$.

(2) $\psi(\mathcal{T})^\wedge$ is obviously reflexive and topologically fine. For proving topologically transitivity of $\psi(\mathcal{T})^\wedge$, that is $\psi(\mathcal{T})^\wedge \subset \psi(\mathcal{T})^{\wedge 2 \wedge}$ or equivalently $\psi(\mathcal{T}) \subset \psi(\mathcal{T})^{\wedge 2 \wedge}$ let $R \in \psi(\mathcal{T})$ be arbitrary. By the definition of the Davis–Pervin relations, we have that $R^2 = R$ and by using the expansivity of $\wedge$ the proof is complete.

(3) At first, we prove that $\mathcal{T} \subset \mathcal{T}_{\psi(\mathcal{T})}$ for all $\mathcal{T} \subset \mathcal{P}(X)$. If $A \in \mathcal{T}$, then $D_A \in \psi(\mathcal{T})$. Since $D_A[A] = A$, we have that $A \in \mathcal{T}_{\psi(\mathcal{T})}$.

On the other hand, let $A \in \mathcal{T}_{\psi(\mathcal{T})}$. If $A = X$, then $A \in \mathcal{T}$, since $\mathcal{T}$ is a generalized set-topology.

If $A \neq X$, and $x \in A$, then there exists a $U_x \in \mathcal{T}$ such that $x \in U_x$ and $U_x = D_{U_x}(x) \subset A$.

$$A = \bigcup_{x \in A} \{x\} \subset \bigcup_{x \in A} U_x \subset \bigcup_{x \in A} A = A$$

implies that $A \in \mathcal{T}$ since $\mathcal{T}$ is closed under arbitrary union.

(4) Let $R \in \psi(\mathcal{T}_\mathcal{R})^\wedge$ and $x \in X$. There exists an $A \in \mathcal{T}_\mathcal{R}$ such that $D_A \in \psi(\mathcal{T}_\mathcal{R})$ and $D_A(x) \subset R(x)$. If $x \in A$ then $S(x) \subset A$ for some $S \in \mathcal{R}$ since A is topologically open, and $D_A(x) = A$ implies $S(x) \subset R(x)$. If $x \notin A$, then $D_A(x) = X$ therefore obviously $S(x) \subset X = R(x)$ for

some $S \in \mathcal{R}$. We proved that $\psi(\mathcal{T}_\mathcal{R})^\wedge \subset \mathcal{R}^\wedge$ for an arbitrary $\mathcal{R}$ relator on X, that is

$$\psi(\mathcal{T}_\mathcal{R})^\wedge \subset \mathcal{R}$$

if $\mathcal{R}$ is topologically fine.

For the converse inclusion let $R \in \mathcal{R}$, $x \in X$ and $A = \operatorname{int}_\mathcal{R}(R(x))$. By the definition of interior, for all $y \in A$ there exists an $S \in \mathcal{R}$ such that $S(y) \subset R(x)$. Topological transitivity of $\mathcal{R}$ implies that there exists a $Q \in \mathcal{R}$ such that $Q[Q(y)] = Q^2(y) \subset S(y) \subset R(x)$, that is $Q(y) \subset A$.

It follows that $A \in \mathcal{T}_\mathcal{R}$, and then $D_A \in \psi(\mathcal{T}_\mathcal{R})$. Since $x \in A$ and $\mathcal{R}$ is reflexive we have that $D_A(x) = A \subset R(x)$. We proved that $R \in \psi(\mathcal{T}_\mathcal{R})^\wedge$ for an arbitrary $R \in \mathcal{R}$.

□

Several papers including [10] used $\{R \circ S : R, S \subset \mathcal{R}\}$ instead of our $\mathcal{R}^2$. Because of the definition of uniformities, we need our one, but later in Theorem 4.8 we will see that in some cases, these definitions are equivalent.

Theorem 3.5. *If $\mathcal{R}$ is a relator on X, then $\mathcal{R} \mapsto \mathcal{T}_\mathcal{R}$ is a bijection of the set of generalized topologies on X onto the set of generalized set-topologies on X.*

Proof. The injectivity and surjectivity are followed by Proposition 3.4 (4) and (3), respectively. □

4. A new form of topologies

Definition 4.1. Let $\mathcal{A}$ be a family of sets, or equivalently $\mathcal{A} \subset \operatorname{Exp}(X)$ for some set X. We call

$$\Phi(\mathcal{A}) = \left\{ \bigcap \mathcal{B} : \varnothing \neq \mathcal{B} \subset \mathcal{A}, \text{ and } \mathcal{B} \text{ is finite} \right\}$$

the filtered family of sets generated by $\mathcal{A}$.

Moreover, we say that $\mathcal{A}$ is filtered if $\Phi(\mathcal{A}) = \mathcal{A}$.

Remark 4.2. Since Φ is a refinement for relators on X, we write $\mathcal{R}^\Phi$ instead of $\Phi(\mathcal{R})$, if $\mathcal{R}$ is a relator on X. Note also that $\mathcal{R}$ is filtered iff $\mathcal{R}^\Phi \subset \mathcal{R}$.

Moreover, note that Φ is an inversion compatible refinement for relators on X, that is, if $\mathcal{R}$ is a relator on X, then $\mathcal{R}^{-1\Phi} = \mathcal{R}^{\Phi-1}$.

Finally, if $\mathcal{R}$ is finite, then $\mathcal{R}^{\Phi *} = \{\bigcap \mathcal{R}\}^*$.

Remark 4.3. By [6] we know that if $\mathcal{R}$ is a relator on X, then $\mathcal{R}^{*\Phi} = \mathcal{R}^{\Phi *}$, $\mathcal{R}^{2\Phi} \subset \mathcal{R}^{\Phi 2 *}$. Now, we can state a similar assertion for topological refinement.

Lemma 4.4. *If $\mathcal{R}$ is a relator on X, then $\mathcal{R}^{\wedge\Phi} \subset \mathcal{R}^{\Phi\wedge}$.*

Proof. It is easy to see that

$$\exists \varnothing \neq \mathcal{S} \subset \mathcal{R}^\wedge \text{ finite } : \bigcap \mathcal{S} = R \implies$$
$$\implies \forall x \in X : \exists \varnothing \neq \mathcal{Q} \subset \mathcal{R} \text{ finite } : \bigcap \mathcal{Q}(x) \subset R(x).$$

□

Proposition 4.5. *If $\mathcal{R}$ is a relator on X and $\square \in \{*, \wedge\}$, then the following assertions are equivalent.*

(1) $\mathcal{R}^{\Phi} \subset \mathcal{R}^{\square}$;
(2) *there exists an $\mathcal{S}$ relator on X, such that* $\mathcal{S}^{\Phi} \subset \mathcal{S}^{\square} = \mathcal{R}^{\square}$.

Proof. We need only to prove the (2) $\Longrightarrow$ (1) implication. For this, we note that if (2) is true, then

$$\mathcal{R}^{\Phi} \subset \mathcal{R}^{\square\Phi} = \mathcal{S}^{\square\Phi} \subset \mathcal{S}^{\Phi\square} \subset \mathcal{S}^{\square\square} = \mathcal{S}^{\square} = \mathcal{R}^{\square}.$$

□

Because of the above Proposition, the following definition seems unduly overcomplicated, but from [6] we know, that this is reasonable.

Definition 4.6. If $\square$ is a refinement for relators on X, then we say that the $\mathcal{R}$ relator on X is $\square$-filtered if there exists an $\mathcal{S}$ relator on X such that $\mathcal{S}^{\Phi} \subset \mathcal{S}^{\square} = \mathcal{R}^{\square}$.

We use the uniformly filtered and topologically filtered notions instead of $*$-filtered and $\wedge$-filtered.

Note that by Proposition 4.5 we have that $\mathcal{R}$ is a uniformly (topologically) filtered relator on X iff $\mathcal{R}^{\Phi} \subset \mathcal{R}^{*}$ ($\mathcal{R}^{\Phi} \subset \mathcal{R}^{\wedge}$), which can be found in the definition of uniformities by Weil.

Definition 4.7. If the generalized uniformity/generalized topology $\mathcal{R}$ on X is also uniformly/topologically filtered, then we say that $\mathcal{R}$ is a uniformity/topology on X, and $(X, \mathcal{R})$ is a uniform space/topological space.

Moreover, if $\mathcal{T}$ is a generalized set-topology on X, such that $\mathcal{T}$ is filtered, then we say that $\mathcal{T}$ is a set-topology on X.

Several papers including [7] investigated the following properties.

Theorem 4.8. *If $\mathcal{R}$ is topologically fine and topologically reflexive (or topologically filtered), then the following assertions are equivalent.*

(1) *$\mathcal{R}$ is topologically transitive;*
(2) $\forall x \in X, R \in \mathcal{R} : \exists P, Q \in \mathcal{R} : P[Q(x)] \subset R(x)$;
(3) $\forall x \in X, R \in \mathcal{R} : x \in \operatorname{int}_{\mathcal{R}}(\operatorname{int}_{\mathcal{R}}(R(x)))$;
(4) $\forall A \subset X : \operatorname{int}_{\mathcal{R}}(A) \in \mathcal{T}_{\mathcal{R}}$;
(5) $\forall x \in X, R \in \mathcal{R} : \operatorname{int}_{\mathcal{R}}(R(x)) \in \mathcal{T}_{\mathcal{R}}$.

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) are quite obvious (without extra conditions). Note only for proving (3) $\Longrightarrow$ (4), that if $x \in \operatorname{int}_{\mathcal{R}}(A)$, that is there exists an $R \in \mathcal{R}$ such that $R(x) \subset A$, then by (3) $x \in \operatorname{int}_{\mathcal{R}}(\operatorname{int}_{\mathcal{R}}(R(x))) \subset \operatorname{int}_{\mathcal{R}}(\operatorname{int}_{\mathcal{R}}(A))$ and it follows that there exists an $S \in \mathcal{R}$ such that $S(x) \subset \operatorname{int}_{\mathcal{R}}(A)$.

Therefore we prove only the (5) $\Longrightarrow$ (1) implication.

For this, let $R \in \mathcal{R}$ and $x \in X$ be fixed and use (5). We have $x \in \operatorname{int}_{\mathcal{R}}(R(x)) \in \mathcal{T}_{\mathcal{R}}$, that is there exists a $Q \in \mathcal{R}$ such that $Q(x) \subset \operatorname{int}_{\mathcal{R}}(R(x))$.

For such a Q, and for all $y \in Q(x)$ there exists a $P_y \in \mathcal{R}$ such that $P_y(y) \subset R(x)$. Now define the S relation on X by the following.

$$S(y) = \begin{cases} Q(y), & \text{if } y = x, \\ P_y(y), & \text{if } y \in Q(x) \setminus \{x\}, \\ X, & \text{else.} \end{cases}$$

Note that because of $\mathcal{R}$ is topologically fine, we have that $S \in \mathcal{R}^\wedge = \mathcal{R}$.

If $\mathcal{R}$ is topologically reflexive, then $\operatorname{int}_{\mathcal{R}}(R(x)) \subset R(x)$, therefore

$$S^2(x) = S[Q(x)] = \bigcup_{y \in Q(x) \setminus \{x\}} P_y(y) \cup Q(x) \subset R(x).$$

If $\mathcal{R}$ is topologically filtered, then write $Q \cap R$ in place of Q. Note that $Q \cap R \in \mathcal{R}^\Phi \subset \mathcal{R}^\wedge = \mathcal{R}$. In this case

$$S^2(x) = S[Q(x)] \subset \bigcup_{y \in Q(x) \setminus \{x\}} P_y(y) \cup Q(x) \subset R(x).$$

□

Proposition 4.9. *Let $\mathcal{R}$ be a relator on X.*

(1) If $\mathcal{R}$ is topologically filtered, then $\mathcal{T}_{\mathcal{R}}$ is filtered.

(2) If $\mathcal{R}$ is topologically reflexive and topologically transitive, moreover $\mathcal{T}_{\mathcal{R}}$ is filtered, then $\mathcal{R}$ is topologically filtered.

Proof. (1) If $A \in \Phi(\mathcal{T}_{\mathcal{R}})$, then there exists a nonvoid finite subset $\mathcal{B}$ of $\mathcal{T}_{\mathcal{R}}$ such that $A = \bigcap \mathcal{B}$. Let $x \in A$ be an arbitrary fixed point. For all $B \in \mathcal{B}$ we have that $x \in B \in \mathcal{T}_{\mathcal{R}}$, therefore there exists $S_B \in \mathcal{R}$ such that $S_B(x) \subset B$. Now, with $R = \bigcap_{B \in \mathcal{B}} S_B \in \mathcal{R}^\Phi \subset \mathcal{R}^\wedge$ we can see that $R(x) \subset \bigcap \mathcal{B} = A$, that is $A \in \mathcal{T}_{\mathcal{R}^\wedge} = \mathcal{T}_{\mathcal{R}}$.

(2) If $R \in \mathcal{R}^\Phi$, then let $\mathcal{S} \subset \mathcal{R}$ nonvoid and finite such that $R = \bigcap \mathcal{S}$. Let $x \in X$ be an arbitrary fixed point. We need to show that there exists a $P \in \mathcal{R}$ such that $P(x) \subset R(x)$.

Topologically transitivity of $\mathcal{R}$, the filtered property of $\mathcal{T}_{\mathcal{R}}$ and Theorem 4.8 give that $U = \bigcap_{S \in \mathcal{S}} \operatorname{int}_{\mathcal{R}}(S(x)) \in \mathcal{T}_{\mathcal{R}}$. It is easy to see, that $x \in \operatorname{int}_{\mathcal{R}}(S(x))$ for all $S \in \mathcal{R}$, that is $x \in U$, therefore there exists a $P \in \mathcal{R}$ such that $P(x) \subset U$.

On the other hand, since $\mathcal{R}$ is topologically reflexive, we have that $\operatorname{int}_{\mathcal{R}}(S(x)) \subset S(x)$, and it follows that $U \subset \left(\bigcap \mathcal{S}\right)(x) = R(x)$.

□

It gives the following.

Theorem 4.10. *If $\mathcal{R}$ is a relator on X, then $\mathcal{R} \mapsto \mathcal{T}_{\mathcal{R}}$ is a bijection of the set of topologies on X onto the set of set-topologies on X.*

Proof. Proposition 4.9 yields that the range of the bijection in Theorem 3.5 restricted to the set of topologies on X is the set of set-topologies on X. □

5. A new form of S_0-topologies

Following notations of [3], $(X, \mathcal{R})$ is called quasi-uniformities, iff $\mathcal{R}$ is a uniformly reflexive, uniformly transitive, uniformly filtered, and uniformly fine relator on X.

By Definition 4.7, we have that $(X, \mathcal{R})$ is a topology, iff $\mathcal{R}$ is a topologically reflexive, topologically transitive, topologically filtered and topologically fine relator on X.

Uniformities have the symmetric property. Let us see topologies with this.

Definition 5.1. If $\mathcal{T}$ is a (generalized) set-topology on X, such that for all $x, y \in X$

$$(\exists U \in \mathcal{T} : x \in U \subset X \setminus \{y\}) \implies (\exists U \in \mathcal{T} : y \in U \subset X \setminus \{x\}),$$

then we say that $\mathcal{T}$ is a (generalized) S_0-set-topology on X.

Lemma 5.2. *If $\mathcal{R}$ is a relator on X, then $\bigcap \mathcal{R} \subset \bigcap \mathcal{R}^{\wedge}$.*

Proof. On the contrary, assume that there exist an $(x, y) \in \bigcap \mathcal{R}$ and an $R \in \mathcal{R}^{\wedge}$ such that $(x, y) \notin R$. In this case, $y \notin R(x)$ that is $R(x) \subset X \setminus \{y\}$. It follows that there exists an $S \in \mathcal{R}$ such that $S(x) \subset R(x)$, and this is a contradiction because $y \notin S(x)$ means $y \notin \left(\bigcap \mathcal{R}\right)(x)$. ☐

Note that by the above Lemma, we have that $\bigcap \mathcal{R} = \bigcap \mathcal{R}^{\wedge}$ for all $\mathcal{R}$ relators on X.

Proposition 5.3. *If $\mathcal{R}$ is a generalized topology on X, then the following assertions are equivalent.*

- *(1) $\bigcap \mathcal{R} \subset \bigcap \mathcal{R}^{-1}$;*
- *(2) $\bigcap \mathcal{R} \supset \bigcap \mathcal{R}^{-1}$;*
- *(3) $\bigcap \mathcal{R} = \bigcap \mathcal{R}^{-1}$;*
- *(4) $\mathcal{R}$ is topologically symmetric;*
- *(5) $\mathcal{T}_{\mathcal{R}}$ is a generalized S_0-set-topology on X.*

Proof. (1) $\iff$ (2) $\iff$ (3) is quite obvious since $\bigcap \mathcal{R}^{-1} = \left(\bigcap \mathcal{R}\right)^{-1}$.

- (4) $\implies$ (2): By Lemma 5.2 and (4), we have that $\bigcap \mathcal{R}^{-1} \subset \bigcap \mathcal{R}^{-1\wedge} \subset \bigcap \mathcal{R}$.
- (1) $\implies$ (4): By (1) $\bigcap \mathcal{R} \subset \bigcap \mathcal{R}^{-1} \subset R^{-1}$ for all $R \in \mathcal{R}$, that is $\mathcal{R}^{-1} \subset \{\bigcap \mathcal{R}\}^{*} = \mathcal{R}^{\wedge-1\wedge-1}$ since [5] Definition 3.1. (3) and 4.1., Remark 4.2. and Theorem 5.3. It follows that $\mathcal{R} \subset \mathcal{R}^{\wedge-1\wedge} = \mathcal{R}^{-1\wedge}$ because of $\mathcal{R}$ is topologically fine.
- (3) $\implies$ (5): Let $x, y \in X$ be fixed, and assume that there exists a $U \in \mathcal{T}_{\mathcal{R}}$ such that $x \in U \subset X \setminus \{y\}$. Since $U \in \mathcal{T}_{\mathcal{R}}$ hence $y \notin R(x)$ for some $R \in \mathcal{R}$, and hence $y \notin \left(\bigcap \mathcal{R}\right)(x) = \left(\bigcap \mathcal{R}^{-1}\right)(x)$. In this case, $x \notin \left(\bigcap \mathcal{R}\right)(y)$, that is there exists an $R \in \mathcal{R}$ such that $x \notin R(y)$. By Theorem 4.8 (5), we have that $\operatorname{int}_{\mathcal{R}}(R(y)) \in \mathcal{T}_{\mathcal{R}}$. It is easy to see that $y \in \operatorname{int}_{\mathcal{R}}(R(y))$. Because of the topologically reflexivity of $\mathcal{R}$ it is also easy to see that $x \notin \operatorname{int}_{\mathcal{R}}(R(y))$ that is $\operatorname{int}_{\mathcal{R}}(R(y)) \subset X \setminus \{x\}$.

(5) $\Longrightarrow$ (1): Let $(x,y) \in \bigcap \mathcal{R}$ be arbitrary. If the U topologically open subset contains x, then there exists an $R \in \mathcal{R}$ such that $y \in (\bigcap \mathcal{R})(x) \subset R(x) \subset U$. Now, by using (5), we have that for all $U \in \mathcal{T}_{\mathcal{R}}$ $y \in U \implies x \in U$.

If $R \in \mathcal{R}$, then $y \in \operatorname{int}_{\mathcal{R}}(R(y)) \in \mathcal{T}_{\mathcal{R}}$ and hence $x \in \operatorname{int}_{\mathcal{R}}(R(y)) \subset R(y)$ since $\mathcal{R}$ is topologically reflexive. It follows that $(y,x) \in \bigcap \mathcal{R}$ that is $(x,y) \in \bigcap \mathcal{R}^{-1}$. It holds for an arbitrary $(x,y) \in \bigcap \mathcal{R}$, therefore (1) is true.

□

The previous Proposition shows the appropriateness of the following.

Definition 5.4. If $\mathcal{R}$ is a topologically symmetric (generalized) topology on X, then we say that $\mathcal{R}$ is a (generalized) S_0-topology on X, and the ordered pair $(X, \mathcal{R})$ is called a (generalized) S_0-topological space.

The previous Proposition gives the following.

Theorem 5.5. *If $\mathcal{R}$ is a relator on X, then $\mathcal{R} \mapsto \mathcal{T}_{\mathcal{R}}$ is a bijection of the set of (generalized) S_0-topologies on X onto the set of (generalized) S_0-set-topologies on X.*

Proof. Proposition 5.3 yields that the range of the bijection in Theorem 3.5 restricted to the set of generalized S_0-topologies on X is the set of generalized S_0-set-topologies on X.

On the other hand, Proposition 5.3 yields that the range of the bijection in Theorem 4.10 restricted to the set of S_0-topologies on X is the set of S_0-set-topologies on X. □

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