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Beyond the Hausdorff metric in digital topology

Laurence Boxer

Department of Computer and Information Sciences, Niagara University, USA; and
Department of Computer Science and Engineering, State University of New York at Buffalo, USA
(boxer@niagara.edu)

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Abstract

Two objects may be close in the Hausdorff metric, yet have very different geometric and topological properties. We examine other methods of comparing digital images such that objects close in each of these measures have some similar geometric or topological property. Such measures may be combined with the Hausdorff metric to yield a metric in which close images are similar with respect to multiple properties.

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1. Introduction

A key question in digital image processing is whether two digital images A and B represent the same object. If, after magnification or shrinking and translation, copies A' and B' of the respective images have been scaled to approximately the same size and are located in approximately the same position, a Hausdorff metric H may be employed: if $H(A', B')$ is small, then perhaps A and B represent the same object; if $H(A', B')$ is large, then A and B probably do not represent the same object. However, the Hausdorff metric is very crude as a measure of similarity. In this paper, we consider other comparisons of digital images.

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2. Preliminaries

Much of this section is quoted or paraphrased from [10].

We use $\mathbb{N}$ to indicate the set of natural numbers, $\mathbb{Z}$ for the set of integers, and $\mathbb{R}$ for the set of real numbers.

2.1. Adjacencies.

A digital image is a graph (X, κ), where X is a subset of $\mathbb{Z}^n$ for some positive integer n, and κ is an adjacency relation for the points of X. The c_u-adjacencies are commonly used. Let $x, y \in \mathbb{Z}^n$, $x \neq y$, where we consider these points as n-tuples of integers:

$$x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_n).$$

Let $u \in \mathbb{Z}$, $1 \leq u \leq n$. We say x and y are c_u-*adjacent* if

- There are at most u indices i for which $|x_i - y_i| = 1$.
- For all indices j such that $|x_j - y_j| \neq 1$ we have $x_j = y_j$.

Often, a c_u-adjacency is denoted by the number of points adjacent to a given point in $\mathbb{Z}^n$ using this adjacency. E.g.,

- In $\mathbb{Z}^1$, c_1-adjacency is 2-adjacency.
- In $\mathbb{Z}^2$, c_1-adjacency is 4-adjacency and c_2-adjacency is 8-adjacency.
- In $\mathbb{Z}^3$, c_1-adjacency is 6-adjacency, c_2-adjacency is 18-adjacency, and c_3-adjacency is 26-adjacency.

We write $x \leftrightarrow_\kappa x'$, or $x \leftrightarrow x'$ when κ is understood, to indicate that x and x' are κ-adjacent. Similarly, we write $x \leftrightarroweq_\kappa x'$, or $x \leftrightarroweq x'$ when κ is understood, to indicate that x and x' are κ-adjacent or equal.

A subset Y of a digital image (X, κ) is κ-*connected* [17], or *connected* when κ is understood, if for every pair of points $a, b \in Y$ there exists a sequence $\{y_i\}_{i=0}^m \subset Y$ such that $a = y_0$, $b = y_m$, and $y_i \leftrightarrow_\kappa y_{i+1}$ for $0 \leq i < m$.

2.2. Digitally continuous functions.

The following generalizes a definition of [17].

Definition 2.1 ([5])**.** Let (X, κ) and (Y, λ) be digital images. A single-valued function $f : X \to Y$ is (κ, λ)-continuous if for every κ-connected $A \subset X$ we have that $f(A)$ is a λ-connected subset of Y.

When the adjacency relations are understood, we will simply say that f is *continuous*. Continuity can be expressed in terms of adjacency of points:

Theorem 2.2 ([17, 5])**.** *A function $f : X \to Y$ is continuous if and only if $x \leftrightarrow x'$ in X implies $f(x) \leftrightarroweq f(x')$.*

See also [11, 12], where similar notions are referred to as *immersions*, *gradually varied operators*, and *gradually varied mappings*.

2.3. Pseudometrics and metrics.

Definition 2.3 ([13])**.** Let X be a nonempty set. Let $d : X^2 \to [0, \infty)$ be a function such that for all $x, y, z \in X$,

- $d(x, y) \geq 0$;

- $d(x, x) = 0$;
- $d(x, y) = d(y, x)$; and
- $d(x, z) \leq d(x, y) + d(y, z)$.

Then d is a *pseudometric* for X. If, further, $d(x, y) = 0$ implies $x = y$ then d is a metric for X.

Pseudometrics that can be applied to pairs (A, B) of nonempty subsets of a digital image X include the absolute values of the differences in their

- deviations from convexity. Several such deviations are discussed in [19, 4], for each of which it was shown that two objects can be "close" in the Hausdorff metric yet quite different with respect to the deviation from convexity. These can be adapted to digital images with respect to digital convexity as defined in [7].
- Euler characteristics. I.e., the function

$$s_\chi(A, B) = |\chi(A) - \chi(B)|,$$

where $\chi(X)$ is the Euler characteristic of (X, κ), is a pseudometric for digital images in $\mathbb{Z}^n$. An improper definition of the Euler characteristic for digital images was given in [15]. An appropriate definition is given in [8].
- Lusternik-Schnirelman category $cat_\kappa(X)$ [1]. I.e., the function

$$s_{LS,\kappa}(A, B) = |cat_\kappa(A) - cat_\kappa(B)|,$$

where $cat_\kappa(X)$ is the Lusternik-Schnirelman category of (X, κ), is a pseudometric for digital images in $\mathbb{Z}^n$.
- diameters. This is discussed below.

The following is easily verified and extends an assertion of [4].

Lemma 2.4. *Let $\Delta_i : X^2 \to [0, \infty)$ be a pseudometric, $1 \leq i \leq n$. Then $D = \sum_{i=1}^n \Delta_i : X^2 \to [0, \infty)$ is a pseudometric. Further, if at least one of the Δ_i is a metric, then D is a metric.*

Here we mention metrics we use in this paper for $\mathbb{R}^n$ or $\mathbb{Z}^n$. Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$.

- Let $p \geq 1$. The ℓ_p *metric* for $\mathbb{R}^n$ is given by

$$d_p(x, y) = \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{1/p}.$$

The special case $p = 1$ gives the *Manhattan* or *city block* metric $d_1 : (\mathbb{R}^n)^2 \to [0, \infty)$, given by

$$d_1(x, y) = \sum_{i=1}^n |x_i - y_i|.$$

The special case $p = 2$ gives the *Euclidean metric* $d_2 : (\mathbb{R}^n)^2 \to [0, \infty)$, given by

$$d_2(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}.$$

- The *shortest path metric* [14]: Let (X, κ) be a connected digital image. For $x, y \in X$, let

$$d_\kappa(x, y) = \min\{n \mid \text{there is a } \kappa\text{-path of length } n \text{ in } X \text{ from } x \text{ to } y.\}$$

- The *Hausdorff metric based on a metric d* [16]: Let $d : X^2 \to [0, \infty)$ be a metric where $X \subset \mathbb{R}^n$. The Hausdorff metric for nonempty bounded and closed subsets A and B of X (hence, in the case $X \subset \mathbb{Z}^n$, finite subsets of X) based on d is

$$H(A, B) = \min \left\{ \begin{array}{l} \varepsilon > 0 \mid \forall (a, b) \in A \times B, \ \exists \ (a', b') \in A \times B \\ \text{such that } \varepsilon \geq d(a, b') \text{ and } \varepsilon \geq d(a', b) \end{array} \right\}.$$

We make the following modification of the Hausdorff metric based on d_κ as presented in [20].

Definition 2.5. Let $X \subset \mathbb{Z}^n$, $\varnothing \neq A \subset X$, $\varnothing \neq B \subset X$. Let κ be an adjacency on X. Then

$$H_{(X,\kappa)}(A, B) = \min \left\{ \begin{array}{l} \varepsilon \geq 0 \mid \forall (a, b) \in A \times B, \ \exists (a', b') \in A \times B \\ \text{such that there are } \kappa\text{-paths in } X \text{ of length} \leq \varepsilon \\ \text{from } a \text{ to } b' \text{ and from } b \text{ to } a' \end{array} \right\}.$$

In the version of the Hausdorff metric based on d_κ in [20], $X = \mathbb{Z}^n$. We show below that we can get very different results for the more general situation $\varnothing \neq X \subset \mathbb{Z}^n$.

We use the notations H_d for the Hausdorff metric based on the metric d, H_p for the Hausdorff metric based on the ℓ_p metric d_p (i.e., $H_p = H_{d_p}$), and $H_{(X,\kappa)}$ for the Hausdorff metric based on d_κ for subsets of X (i.e., $H_\kappa = H_{d_\kappa}$).

Another metric from classical topology that is easily adapted to digital topology is Borsuk's *metric of continuity* [2, 3] based on a metric d which is typically, but not necessarily, the Euclidean metric. For digital images (X, κ) and (Y, κ) in $\mathbb{Z}^n$, define the metric of continuity $\delta_d(X, Y)$ as the greatest lower bound of numbers $t > 0$ such that there are κ-continuous $f : X \to Y$ and $g : Y \to X$ with

$$d(x, f(x)) \leq t \text{ for all } x \in X \text{ and } d(y, g(y)) \leq t \text{ for all } y \in Y.$$

Proposition 2.6. *Given finite digital images (X, κ) and (Y, κ) in $\mathbb{Z}^n$ and a metric d for $\mathbb{Z}^n$, $H_d(X, Y) \leq \delta_d(X, Y)$.*

Proof. This is largely the argument of the analogous assertion in [2]. Let $u = H_d(X, Y)$. Since X and Y are finite, without loss of generality, there exists $x_0 \in X$ such that $u = \min\{y \in Y \mid d(x_0, y)\}$. Then for all κ-continuous $f : X \to Y$, $d(x_0, f(x_0)) \geq u$. Therefore, $\delta_d(X, Y) \geq u$. □

An example for which the inequality of Proposition 2.6 is strict is given in the following.

Theorem 2.7. *Let* $X = \{(x,y) \in \mathbb{Z}^2 \mid |x| = n \text{ or } |y| = n\}$*. Let* $Y = X \setminus \{(n,n)\}$*. Then, using the Manhattan metric for* d *and* $\kappa = c_1$*, we have* $H_1(X,Y) = 1$ *and* $\delta_d(X,Y) \geq 2n-1$*.*

Proof. It is clear that $H_1(X,Y) = 1$.

Notice there is an isomorphism $F : (Y, c_1)$ to a subset of $(\mathbb{Z}, c_1)$. Let $f : X \to Y$ be c_1-continuous. By [6], there is a pair of antipodal points $P, -P \in X$ such that $|F \circ f(P) - F \circ f(-P)| \leq 1$. Since F is an isomorphism, we must have $f(P) \leftrightarrows_{c_1} f(-P)$. We will show that either $d(P, f(P)) \geq 2n-1$ or $d(-P, f(-P)) \geq 2n-1$, as follows.

If $P = (n,u)$ then $-P = (-n,-u)$. Then:

- If $f(P) = (n,-n)$ then $f(-P) \in \{(n-1,-n),(n,-n),(n,-n+1)\}$, so $d(-P, f(-P)) \geq 2n-1$.
- Note $(n,n) \notin Y$ so $f(P)$ cannot equal (n,n).
- If $f(P) = (n,v)$ for $|v| < n$ then $f(-P) \in \{(n,v-1),(n,v),(n,v+1)\}$, so $d(-P, f(-P)) \geq 2n$.

The cases $P = (-n,u)$, $P = (w,n)$, and $P = (w,-n)$ are similar. Thus $\delta_d(X,Y) \geq 2n-1$. □

We say the *diameter* of a nonempty bounded set $A \subset \mathbb{R}^n$ with respect to a metric d is

$$diam_d(A) = \max\{d(a,b) \mid a,b \in A\}.$$

We will use the notations $diam_p$ for $diam_{d_p}$, and $diam_\kappa$ for $diam_{d_\kappa}$.

We define a function s_d for pairs of nonempty bounded sets in $\mathbb{R}^n$ by

$$s_d(A,B) = |diam_d(A) - diam_d(B)|.$$

We use notations s_p for s_{d_p}, and s_κ for s_{d_κ}.

The following is easily verified.

Lemma 2.8. *The function* s_d *is a pseudometric.*

3. Comparing (pseudo)metrics on digital images

In this section, we compare the use of some of the (pseudo)metrics discussed above.

Theorem 3.1. *Let* A *and* B *be nonempty, bounded subsets of* $\mathbb{R}^n$*. Let* H_p *be the Hausdorff metric based on the* ℓ_p *metric* d_p *and suppose* $H_p(A,B) \leq m$*. Then* $s_p(A,B) \leq 2m$*.*

Proof. There exist $a, a' \in A$ such that $d_p(a,a') = diam_p(A)$. There exist $b, b' \in B$ such that $d_p(a,b) \leq m$ and $d_p(a',b') \leq m$. So

$$\begin{aligned} diam_p(A) = d_p(a,a') &\leq d_p(a,b) + d_p(b,b') + d_p(b',a') \leq m + diam_p(B) + m \\ &= diam_p(B) + 2m. \end{aligned}$$

Similarly, $diam_p(B) \leq diam_p(A) + 2m$. The assertion follows. □

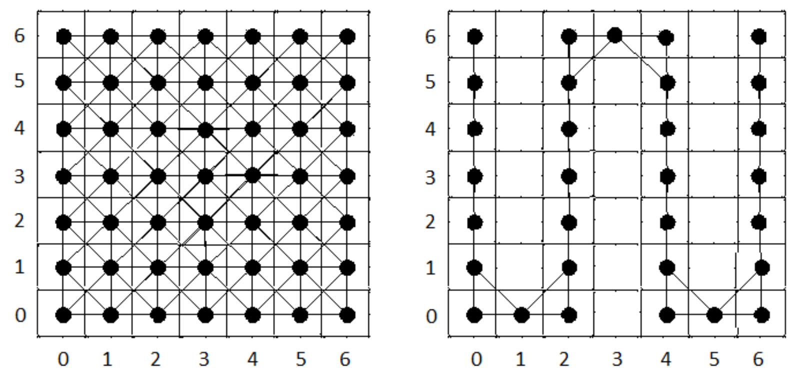

FIGURE 1. Left: $Q = [0, n]_{\mathbb{Z}}^2$ (here, $n = 6$).
Right:
$S = Q \setminus \left[(\bigcup_{k \in \mathbb{Z}} \{4k+1\} \times [1, n]_{\mathbb{Z}}) \cup (\bigcup_{k \in \mathbb{Z}} \{4k+3\} \times [0, n-1]_{\mathbb{Z}}) \right]$
(here, $n = 6$).
Q and S are within 1 with respect to the Hausdorff metric based on the Manhattan metric; however, they differ considerably with respect to diameter in the shortest path metric.

By contrast, we have the following.

Example 3.2. Let $n \in \mathbb{N}$ such that n is even. Let $Q = [0, n]_{\mathbb{Z}}^2$. Let

$$S = Q \setminus \left[(\bigcup_{k \in \mathbb{Z}} \{4k+1\} \times [1, n]_{\mathbb{Z}}) \cup (\bigcup_{k \in \mathbb{Z}} \{4k+3\} \times [0, n-1]_{\mathbb{Z}}) \right].$$

(See Figure 1.) Then $s_1(Q, S) = 0$, but while $diam_{c_1}(Q) = 2n$, we have $diam_{c_1}(S) = n + n(1 + n/2)$. Thus $s_{c_1}(Q, S) = n^2/2$.

Proof. It is easy to see that both Q and S have diagonally opposed points that are maximally distant in the d_1 metric. Therefore, $diam_1(S) = diam_1(Q) = 2n$, so $s_1(Q, S) = 0$.

Diagonally opposed points of Q are maximally separated with respect to d_{c_1}, so $diam_{c_1}(Q) = 2n$. Maximally separated points of S with respect to d_{c_1} are

$$\begin{array}{ll} (0, n) \text{ and } (n, n) & \text{if } n = 4k + 2 \text{ for some } k \in \mathbb{Z}; \\ (0, n) \text{ and } (n, 0) & \text{if } n = 4k \text{ for some } k \in \mathbb{Z}. \end{array}$$

In either case, the unique shortest c_1-path between maximally separated points requires n horizontal steps. The number of vertical steps is computed as follows. There are $1 + n/2$ vertical line segments that must be traversed, each of length n, so the number of vertical steps is $n(1 + n/2)$. Thus the number of steps between maximally separated members of S is $diam_{c_1}(S) = n + n(1 + n/2)$.

Hence for $\kappa = c_1$ we have $s_\kappa(Q, S) = |n + n(1 + n/2) - 2n| = n^2/2$. □

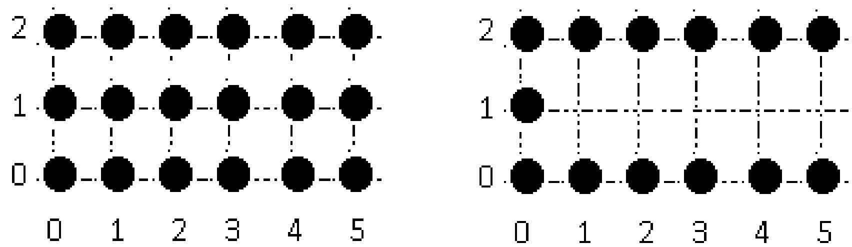

FIGURE 2. Digital images A (left) and B (right) for Example 3.3, using $n = 5$. Using the shortest path metric and either $\kappa = c_1$ or $\kappa = c_2$, maximally distant points in A are $(0, 0)$ and $(n, 2)$, and maximally distant points in B are $(n, 0)$ and $(n, 2)$.

We do not get an analog of Theorem 3.1 by using the Hausdorff metric based on an adjacency κ instead of H_p. This is shown in the following example.

Example 3.3. Let $A = [0, n]_{\mathbb{Z}} \times [0, 2]_{\mathbb{Z}}$. Let $B = A \setminus ([1, n]_{\mathbb{Z}} \times \{1\})$. (See Figure 2.) Then $H_1(A, B) = 1$. However, we have the following.

- For $\kappa = c_1$, $diam_\kappa(A) = n+2$ and $diam_\kappa(B) = 2n+2$, so $s_\kappa(A, B) = n$.
- For $\kappa = c_2$, $diam_\kappa(A) = n$ and $diam_\kappa(B) = 2n$, so $s_\kappa(A, B) = n$.

Theorem 3.4. *Let A and B be finite, nonempty c_u-connected subsets of a c_u-connected subset X of $\mathbb{Z}^n$, where $1 \leq u \leq n$. Suppose we have $H_{(X,c_u)}(A, B) \leq m$ for some $m \in \mathbb{N}$. Then $H_p(A, B) \leq mu^{1/p}$.*

Proof. By hypothesis, given $x \in A$ and $y \in B$, there exist $x' \in A$, $y' \in B$, and c_u-paths P from x to y' and Q from y to x' in X such that each of P and Q has length of at most m. Since each c_u-adjacency corresponds to a Euclidean distance of at most $u^{1/p}$, it follows that $d_p(x, y') \leq mu^{1/p}$ and $d_p(y, x') \leq mu^{1/p}$. It follows that $H_p(X, Y) \leq mu^{1/p}$. □

We do not get a converse for Theorem 3.4, as the following shows.

Example 3.5. Let $B = [0, n]_{\mathbb{Z}} \times [0, 2]_{\mathbb{Z}} \setminus ([1, n]_{\mathbb{Z}} \times \{1\})$ as in Example 3.3. (See Figure 2.) Let $C = [0, n]_{\mathbb{Z}} \times \{0\} \subset B$. Then $H_1(B, C) = H_2(B, C) = 2$. However, $H_{(B,c_1)}(B, C) = n + 2$ and $H_{(B,c_2)}(B, C) = n + 1$.

Proof. It is easy to see that $H_1(B, C) = H_2(B, C) = 2$.

Since $C \subset B$, finding a Hausdorff distance between B and C comes down to considering a furthest point of B from C. With respect to $\kappa = c_1$ and also with respect to $\kappa = c_2$, the furthest point of B from C in the shortest path metric is $b = (n, 2)$ and its closest point of C is $c = (0, 0)$. Since $d_{c_1}(b, c) = n + 2$ and $d_{c_2}(b, c) = n + 1$, the assertion follows. □

Roughly, it appears that the great differences found in Examples 3.3 and 3.5, between measures based in ℓ_p metrics and measures based on the shortest path metric, are due to significant deviations from convexity. If we consider $H_{(X,c_i)}(A,B)$ for a set X such as a digital cube, we may find H_p and $H_{(X,c_p)}$ are more alike, as we see below.

Proposition 3.6. *Let $A \neq \varnothing \neq B$, $A \cup B \subset J = [0,m]_{\mathbb{Z}}^2$. Then $H_1(A,B) = H_{(J,c_1)}(A,B)$.*

Proof. Let $n = H_1(A,B)$. Let $x \in A$. Then there exists $y \in B$ such that $d_1(x,y) \leq n$. By definition of d_1, it follows that there is a c_1-path in J of length at most n from x to y. Similarly, given u in B, there is a c_1-path in J of length at most n from u to a point $v \in A$. Therefore, $H_{(J,c_1)}(A,B) \leq n = H_1(A,B)$.

Now let $n = H_{(J,c_1)}(A,B)$. Then given $x \in A$, there is a c_1-path in J of length at most n from x to some $y \in B$. Similarly, given $u \in B$, there is a c_1-path in J of length at most n from u to some $v \in A$. Since every c_1 adjacency represents a d_1 distance of 1, it follows that $d_1(x,y) \leq n$ and $d_1(u,v) \leq n$. Thus $H_1(A,B) \leq n = H_{(J,c_1)}(A,B)$. The assertion follows. □

Using the observation that a c_u-adjacency in $\mathbb{Z}^r$, $1 \leq u \leq r$, represents a d_p distance between the adjacent points that is between 1 and $u^{1/p}$, we can generalize the argument used to prove Proposition 3.6, getting the following.

Theorem 3.7. *Let $A \neq \varnothing \neq B$, $A \cup B \subset J = [0,m]_{\mathbb{Z}}^v$. Then for $1 \leq u \leq v$, $H_{(J,c_u)}(A,B) \leq u^{1/p} \cdot H_{(J,c_u)}(A,B)$.*

4. Further remarks

The Hausdorff metric is often used to compare objects A and B. It is easy to compute efficiently [18, 9] and gives a good indication of how well each of its arguments approximates the other with respect to location.

However, two objects may be close in the Hausdorff metric and yet have very different geometric or topological properties. Lemma 2.4 tells us that by adding other pseudometrics or metrics, such as those we have discussed, to the Hausdorff metric, we can get another metric in which closeness is more likely to validate the parameters as digital images representing the same physical object.

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