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Brief paper

# Output-feedback anti-disturbance predictor-based control for discrete-time systems with time-varying input delays ☆

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ABSTRACT

This paper investigates the robust stabilization of discrete-time systems with time-varying input delays and model uncertainties by predictor-based anti-disturbance output-feedback control strategies. Here, a novel predictor-feedback control combined with an extended state observer is proposed. The objective is to counteract the negative effects of input delays while actively rejecting disturbance signals typically encountered in engineering practice, such as steps or harmonics. Differently from previous approaches, unknown but bounded time-varying delays are taken into consideration. Moreover, the complexity of the algorithm for control synthesis is notably reduced. Finally, an illustrative example from the literature is provided to show that better robust performance can be achieved with the proposed method.



## 1. Introduction

Time delays may degrade the control system or even jeopardize the stability if they are not taken into account in control synthesis (Fridman, 2014; Sipahi, Niculescu, Abdallah, Michiels, & Gu, 2011). The research of advanced control strategies with the objective to counteract the negative effect of time delays in the control loop has therefore received much attention in the last decades. Most of these methods are extensions of the classical Smith Predictor (Smith, 1959) and the Finite Spectrum Assignment (FSA) (Manitius & Olbrot, 1979). The Smith predictor takes advantage of prior knowledge of the plant model to remove the delayed components of the closed-loop dynamics using Internal Model Control approach, while predictor-feedback or FSA control incorporates in the control scheme a future prediction of the system state by the Artstein's reduction method (Artstein, 1982). These control strategies, known as delay compensation methods, allow designing the controller parameters using an equivalent delay-free model.

To overcome the difficulties in the implementation due to the infinite-dimensional nature of the FSA (Mondié & Michiels, 2003; Zhong, 2004), some adaptations were further developed. For instance, a truncated predictor feedback control was provided in Wei and Lin (2016), Yoon and Lin (2013), Zhou, Lin, and Duan (2012) and Zhou (2014) by ignoring the infinite-dimensional term in the control scheme. Other finite-dimensional delay compensation strategy was proposed in Besancon, Georges, and Benayache (2007) by modifying the classical Luenberger observer with the objective to obtain a future prediction of the state variable for systems with small input delays. This idea was extended in Najafi, Hosseinnia, Sheikholeslam, and Karimadini (2013) to cope with larger delays by using the cascade observer structure initiated in Germani, Manes, and Pepe (2002) by means of sequential sub-predictors (SSP), and further extended, among other works, to linear time-varying systems (Mazenc & Malisoff, 2017) and systems with input, state and output delays (Cacace & Germani, 2017).

Disturbance rejection has also been a major concern for control system design and performance optimization in many industrial applications (Liu & Gao, 2011). The reliability of the future prediction of the system state strongly depends on the accuracy of the system model. Therefore, predictor-feedback may lose effectiveness under model uncertainties and external disturbances (Karafyllis & Krstic, 2013a, 2013b; Li, Zhou, & Lin, 2014). A complete steady-state rejection for constant load disturbances of unknown amplitude and better attenuation for time-varying disturbances in the presence of delays were achieved by introducing disturbance information in the prediction scheme (Léchappé, Moulay, Plestan, Glumineau, & Chriette, 2015; Santos & Franklin, 2018) and by means a disturbance observer in Furtat, Fridman,

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and Fradkov (2017) and Sanz, García, and Albertos (2016) under the assumption of fully measurable plant state. Further extensions to output feedback were addressed via extended state observer in Sanz, Garcia, Fridman, and Albertos (2018).

For discrete-time systems, a disturbance rejection design was proposed by means of a extended state observer in Hao, Liu, and Zhou (2017), but no model uncertainties were considered. The presence of time-varying model mismatches for linear perturbed systems with input delays was considered in Liu, Hao, Li, Chen, and Wang (2017) by classical discrete-time predictor and in Hao, Liu, and Zhou (2019) by combining discrete-time SSP with ESO, where the control synthesis was addressed via LMI and Cone-Complementarity Linearization (CCL) approaches. Nevertheless, the proposed method (Hao et al., 2019) is restricted to time-constant delays.

In this work, a novel output predictor-feedback anti-disturbance control synthesis method is proposed for discrete-time systems. To this end, an iterative algorithm based on LMIs and CCL has been provided with the following improvements with respect to Hao et al. (2019): (i) The complexity of the control synthesis algorithm is independent of the size of delays, (ii) the proposed method can deal with time-varying delays and mismatched disturbances, and (iii) given any time-constant delay, a feasible solution for some controller and observer gains designed by standard procedures is guaranteed for the first CCL iteration.

The rest of the paper is organized as follows. The problem formulation and preliminaries are given in Section 2. The proposed strategy is presented in Section 3. The robust stability analysis and the controller design are tackled in Sections 4 and 5 respectively. Simulation results are provided in Section 6 and, finally, concluding remarks are drawn in Section 7.

## 2. Problem statement and preliminaries

Consider the following discrete-time uncertain system with unknown but bounded time-varying input delay $h_1 \leq d_k \leq h_2$ described by

$$\begin{aligned} x_{k+1} &= \left(A + \Delta_{A,k}\right) x_k + \left(B + \Delta_{B,k}\right) u_{k-d_k} \\ &+ \left(F + \Delta_{F,k}\right) f_k, \qquad y_k = C x_k \\ x_0 &= \theta,\ u_\kappa = \phi_u(\kappa), \qquad \kappa = -h_2, \ldots, -1. \end{aligned} \tag{1}$$

where $x_k \in \mathcal{R}^n$ is the system state, $u_k \in \mathcal{R}^m$ is the control input, $f_k \in \mathcal{R}^l$ represents an unmeasurable disturbance signal, $y_k \in \mathcal{R}^p$ is the controlled output, $h_1, h_2 > 0$ are known delay bounds, and $A, B, F, C$ are time-constant matrices of appropriate dimensions. The function $\phi_u(.) : \mathcal{N} \to \mathcal{R}^m$ and $\theta \in \mathcal{R}^n$ represents the initial conditions for the control input $u_k$ and system state $x_k$.

Time-varying model uncertainties are described as norm-bounded form as (Han & Gu, 2001):

$$\left(\Delta_{A,k}, \Delta_{B,k}, \Delta_{F,k}\right) = \lambda E \Delta_k \left(H_A, H_B, H_F\right) \tag{2}$$

where $\lambda \geq 0$ is a scalar that determines the size of uncertainties, $\Delta_k \in \mathcal{R}^{l_1 \times l_2}$ is an unknown time-varying matrix satisfying $\Delta_k^T \Delta_k \leq I, \forall k \geq 0$, and $E, H_A, H_B, H_F$ are time-constant matrices.

The objective is to design an output predictor-feedback control $u_k$ such that: (i) the closed-loop system (1) is robustly exponentially stable and (ii), the effect of the disturbance input $f_k$ is minimized in the controlled output $y_k$ to the greatest extent with a complete steady state rejection for some type of disturbances. The disturbance $f_k$ can be modelled by the following exogenous system (Hao et al., 2019):

$$\begin{aligned} \xi_{k+1} &= \Lambda \xi_k + M \delta_k, \\ f_k &= N \xi_k \end{aligned} \tag{3}$$

where the initial condition $\xi_0$ is assumed to be unknown, and matrices $\Lambda \in \mathcal{R}^{r \times r}$, $M \in \mathcal{R}^{r \times r_\delta}$, $N \in \mathcal{R}^{l \times r}$ determine the specific type of disturbance to be steady-state rejected in the controlled output $y_k$. Denoting $\rho_{sr}(.)$ the spectral radius of a given matrix, we assume that $\rho_{sr}(\Lambda) \leq 1$, which implies that $\xi_k$ is bounded $\forall k \geq 0$. The input $\delta_k \in \mathcal{R}^{r_\delta}$ represents an unknown and unmeasurable external disturbance signal, assumed to be energy-bounded and belong to $l_2[0, +\infty)$.

**Assumption 1.** The pair $(A, B)$ is stabilizable and the pair $(\mathcal{A}, \mathcal{C})$ is detectable, where $\mathcal{A} = \begin{bmatrix} A & FN \\ 0 & \Lambda \end{bmatrix}$ and $\mathcal{C} = \begin{bmatrix} C & 0_{l \times r} \end{bmatrix}$.

**Lemma 1** (Seuret, Gouaisbaut, & Fridman, 2015)**.** *Given a symmetric positive definite matrix $Z$ and a time-constant delay $h > 0$, any sequence of discrete-time variables denoted by $u_k$, $k = 0, 1, 2, \ldots$, the following inequality holds:*

$$-h \sum_{i=k-h}^{k-1} \rho_i^T Z \rho_i \leq -\begin{bmatrix} \nu_{1,k}^T & \nu_{2,k}^T \end{bmatrix} \begin{bmatrix} Z & 0 \\ 0 & 3\alpha_h Z \end{bmatrix} \begin{bmatrix} \nu_{1,k} \\ \nu_{2,k} \end{bmatrix}$$

*where $\rho_i = u_{i+1} - u_i$, $\nu_{1,k} = u_k - u_{k-h}$, $\nu_{2,k} = u_k + u_{k-h} - \frac{2}{h+1} \sum_{i=0}^{h} u_{k-i}$ and $\alpha_h = \begin{cases} \frac{h-1}{h+1} & \text{if } h > 1 \\ 1 & \text{if } h = 1 \end{cases}$.*

**Lemma 2.** *Given any arbitrary discrete-time signal $u_k \in \mathcal{R}^m$ and time-varying delay $d_k$ satisfying $h_1 \leq d_k \leq h_2$, define*

$$\omega_k = \frac{2}{\tau} \left( u_{k-d_k} - \frac{1}{2} \left( u_{k-h_1} + u_{k-h_2} \right) \right), \tag{4}$$

*where $\tau = h_2 - h_1$. Then, the time-varying operator $\mathcal{D}_{h,k} : u \to \omega : \mathcal{R}^m \to \mathcal{R}^m$ renders $\omega_k = \frac{1}{\tau} \sum_{i=k-h_2}^{k-h_1-1} \phi_d(i) \rho_i$, where $\rho_i = u_{i+1} - u_i$ and*

$$\phi_d(i) = \begin{cases} 1 & \text{if } i < k - d_k - 1, \\ -1 & \text{otherwise}, \end{cases}$$

*Moreover, $\mathcal{D}_{h,k}$ satisfies $\| W \mathcal{D}_{h,k} W^{-1} \|_\infty \leq 1$ for any invertible matrix $W \in \mathcal{R}^m$, where $\|.\|_\infty$ represents the largest $\mathcal{L}_2$ induced norm of a general operator.*

**Proof.** The proof can be obtained by a straightforward adaptation of a similar result given in Li and Gao (2011, Lemma 2).

## 3. Proposed observer–predictor control scheme

This section is divided in two subsections: the first presents the proposed predictor-feedback control with an extended state observer, and the second obtains the equivalent closed-loop state-space model given in (14) based on a modified Artstein's state transformation to deal with time-varying delays.

### 3.1. Proposed control scheme

Consider the following predictor-feedback control scheme integrated by two components: the first one with $K$ corresponds to the predictor-feedback delay compensation control, and the second one with $K_d$ allows the steady-state rejection of the identified disturbance components:

$$u_k = K \left( A^{h_2} \hat{x}_k + A^\tau \Phi_{1,k} + \Phi_{2,k} \right) + K_d \hat{\xi}_k \tag{5}$$

where $\tau = h_2 - h_1$, and

$$\Phi_{1,k} \equiv \frac{1}{2} \sum_{i=0}^{h_1-1} A^{h_1-i-1} B \left( u_{k-h_1+i} + u_{k-h_2+i} \right), \tag{6}$$

$$\Phi_{2,k} \equiv \frac{1}{2}\sum_{i=0}^{\tau-1} A^{\tau-i-1}Bu_{k-\tau+i}$$

being $K \in \mathcal{R}^{m\times n}$ and $K_d \in \mathcal{R}^{m\times r}$ controller gains whose design is later discussed. $\hat{x}_k \in \mathcal{R}^n$, $\hat{\xi}_k \in \mathcal{R}^r$ in (5) are respectively the observer state and disturbance estimation defined below in (7) obtained from the available output system $y_k$ by the following ESO:

$$\begin{cases} \hat{x}_{k+1} = A\hat{x}_k + \frac{1}{2}Bu_{k-h_1} + \frac{1}{2}Bu_{k-h_2} + FN\hat{\xi}_k \\ +L\big(y_k - C\hat{x}_k\big) \\ \hat{\xi}_{k+1} = \Lambda\hat{\xi}_k + L_\xi\big(y_k - C\hat{x}_k\big), \end{cases} \tag{7}$$

where $\Lambda$ and $N$ are defined in (3). The observer gains $L \in \mathcal{R}^{n\times l}$, $L_\xi \in \mathcal{R}^{r\times l}$ must be designed to guarantee that $lim_{k\to\infty}\, x_k - \hat{x}_k = 0$ and $lim_{k\to\infty}\, \xi_k - \hat{\xi}_k = 0$.

### 3.2. Equivalent closed-loop representation

First, applying (2), (3) and the definition of $\omega_k$ in (4), let us reformulate system (1) as:

$$x_{k+1} = Ax_k + \frac{1}{2}Bu_{k-h_1} + \frac{1}{2}Bu_{k-h_2} + \frac{\tau}{2}B\omega_k + \lambda E w_{\Delta,k} + FN\xi_k \tag{8}$$

where $\omega_k$ is defined in (4), and $w_{\Delta,k} := \Delta_k y_{\Delta,k}$ with

$$y_{\Delta,k} = H_A x_k + \frac{1}{2}\sum_{g=1}^{2} H_B u_{k-h_g} + \frac{\tau}{2}H_B\omega_k + H_F N\xi_k \tag{9}$$

Let us propose the following new state transformation based on the Artstein's reduction method adapted for time-varying delays, where two delayed components for the lower and upper delay bounds $h_1$, $h_2$ are considered:

$$x_k^{(a)} = A^{h_2}x_k + A^\tau\Phi_{1,k} + \Phi_{2,k} \tag{10}$$

with $\Phi_{1,k}$, $\Phi_{2,k}$ defined in (6). The one-step ahead of $x_k^{(a)}$ defined in (10) renders $x_{k+1}^{(a)} = A^{h_2}x_{k+1} + A^\tau\Phi_{1,k+1} + \Phi_{2,k+1}$. Replacing $x_{k+1}$ from (8) into the latter expression yields:

$$\begin{aligned} x_{k+1}^{(a)} &= Ax_k^{(a)} + A^\tau\big(\Phi_{1,k+1} - A\Phi_{1,k}\big) + \big(\Phi_{2,k+1} - A\Phi_{2,k}\big) \\ &+ \frac{1}{2}A^{h_2}Bu_{k-h_1} + \frac{1}{2}A^{h_2}Bu_{k-h_2} + \frac{\tau}{2}A^{h_2}B\omega_k \\ &+ \lambda A^{h_2}Ew_{\Delta,k} + A^{h_2}FN\xi_k \end{aligned} \tag{11}$$

Taking into account that $\Phi_{1,k+1} - A\Phi_{1,k} = \frac{1}{2}B\big(u_k + u_{k-\tau}\big) - \frac{1}{2}A^{h_1}B\big(u_{k-h_1} + u_{k-h_2}\big)$, $\Phi_{2,k+1} - A\Phi_{2,k} = \frac{1}{2}Bu_k - \frac{1}{2}A^\tau Bu_{k-\tau}$ and the state transformation (10), we obtain from (11) an equivalent interconnected model for system (1) as:

$$\begin{aligned} x_{k+1}^{(a)} &= Ax_k^{(a)} + B_\tau u_k + \frac{\tau}{2}A^{h_2}B\omega_k \\ &+ \lambda A^{h_2}Ew_{\Delta,k} + A^{h_2}FN\xi_k \end{aligned} \tag{12}$$

where $B_\tau = \frac{1}{2}\big(A^\tau + I\big)B$.

Let $e_k = x_k - \hat{x}_k$ and $\tilde{\xi}_k = \xi_k - \hat{\xi}_k$ be the observer errors. Then, (5) with (10) leads to $u_k = Kx_k^{(a)} - KA^{h_2}e_k + K_d\left(\xi_k - \tilde{\xi}_k\right)$, and the closed-loop system (12) can be written as:

$$\begin{aligned} x_{k+1}^{(a)} &= (A + BK)x_k^{(a)} - B_\tau KA^{h_2}e_k - B_\tau K_d\tilde{\xi}_k \\ &+ \frac{\tau}{2}A^{h_2}B\omega_k + \lambda A^{h_2}Ew_{\Delta,k} + \big(B_\tau K_d + A^{h_2}FN\big)\,\xi_k \end{aligned} \tag{13}$$

Given that $\xi_k$ in (3) is bounded, the last term with $\xi_k$ in (13) has been neglected in the subsequent stability analysis. Hence, the closed-loop system model with $\bar{x}_k = \left[\left(x_k^{(a)}\right)^T,\ e_k^T,\ \tilde{\xi}_k^T\right]^T$ is obtained from (3), (5), (7), (8) and (13) as:

$$\bar{x}_{k+1} = \big(\bar{A} + \bar{B}_\tau\bar{K}\big)\,\bar{x}_k + \frac{\tau}{2}\bar{B}\omega_k + \lambda\bar{E}w_{\Delta,k} + \bar{M}\delta_k \tag{14}$$

where

$$\bar{A} = \begin{bmatrix} A & 0 & 0 \\ 0 & A - LC & FN \\ 0 & -L_\xi C & \Lambda \end{bmatrix}, \quad \bar{K} = \begin{bmatrix} K & -KA^{h_2} & -K_d \end{bmatrix},$$

$$\bar{B}_\tau = \begin{bmatrix} B_\tau \\ 0 \\ 0 \end{bmatrix}, \bar{B} = \begin{bmatrix} A^{h_2}B \\ B \\ 0 \end{bmatrix}, \bar{E} = \begin{bmatrix} A^{h_2}E \\ E \\ 0 \end{bmatrix}, \bar{M} = \begin{bmatrix} 0 \\ 0 \\ M \end{bmatrix}$$

being $w_{\Delta,k}$, $\omega_k$ interconnected inputs $w_{\Delta,k} = \Delta_k y_{\Delta,k}$ and $\omega_k = \mathcal{D}_{h,k}\rho_k$ with $\rho_k = u_{k+1} - u_k$ and the time-varying operator $\mathcal{D}_{h,k}$ defined in Lemma 2.

**Corollary 1.** *In the absence of time-varying uncertainties, i.e., $\tau = 0$, $\lambda = 0$, the closed-loop system (14) is exponentially stable with decay rate $\beta$ regardless of $K_d$ if there exists $K$ and $\mathcal{L} = \begin{bmatrix} L^T & L_\xi^T \end{bmatrix}^T$ such that $A + BK$ and $\mathcal{A} - \mathcal{L}\mathcal{C}$ satisfy $\beta = max\,(\rho_{sr}(A + BK)$, $\rho_{sr}(\mathcal{A} - \mathcal{L}\mathcal{C}))$, where $\mathcal{A}$, $\mathcal{C}$ are defined in Assumption 1.*

**Proof.** Let $\tau = 0$ and $\lambda = 0$, Then, system (14) renders delay free. i.e., $\bar{x}_{k+1} = \big(\bar{A} + \bar{B}_\tau\bar{K}\big)\,\bar{x}_k + \bar{M}\delta_k$. The rest of the proof can easily be inferred noting that $B_\tau = B$ ($\tau = 0$) and the triangular form of $\big(\bar{A} + \bar{B}_\tau\bar{K}\big) = \begin{bmatrix} A + BK & \Xi \\ 0 & \mathcal{A} - \mathcal{L}\mathcal{C} \end{bmatrix}$, with $\Xi = \begin{bmatrix} -BKA^{h2} & -BK_d \end{bmatrix}$. ■

Under Assumption 1 and in the absence of uncertainties, Corollary 1 shows that the controller gains $K$ and $K_d$ can be designed for any time-constant delay in two steps: (i) find $K$ and $\mathcal{L}$ such that the eigenvalues of $(A + BK)$ and $(\mathcal{A} - \mathcal{L}\mathcal{C})$ are placed for a desired closed-loop dynamics, and (ii) find $K_d$ with the objective of achieving a steady-state rejection of the identified disturbance component $f_k$ in the controlled output $y_k$ by solving the resulting equality constraints depending of the type of disturbance. For step signal: solve $K_d$ s.t $G_{yd}(0) = 0$, where $G_{yd}(z)$ is the $z$-transfer function of the nominal closed-loop system from the disturbance input $f_k$ to the controlled output $y_k$. For harmonic disturbance components of frequency $w_f$, solve $K_d$ s.t $|G_{yd}(e^{iw_f})| = 0$, $|G_{yd}(e^{-iw_f})| = 0$.

To cope with time-varying model uncertainties and delay mismatches, next section addresses the exponential stability analysis and control synthesis via LKF approach and LMI/CCL framework:

## 4. Robust stability analysis

This section presents a sufficient condition for robust exponential stability with guaranteed decay rate $\beta$ and $H_\infty$ disturbance rejection $\gamma$ from the unknown disturbance $\delta_k \in l_2[0, +\infty)$ to the controlled output $y_k$ for the closed-loop system (14).

**Theorem 1.** *Given delay bounds $0 < h_1 \le h_2$, if there exist symmetric matrices $P \in \mathcal{R}^{\bar{n}} > 0$, $S_1, S_2 \in \mathcal{R}^n > 0$, $Q_1, Q_2, Z_1, Z_2, W \in \mathcal{R}^m > 0$ with $\bar{n} = 2n + r + 3m$, a matrix $T \in \mathcal{R}^{(5n+r+3m)\times n}$ and a scalar $\varepsilon > 0$ such that the LMI given below is true, the closed-loop system (14) is robustly exponentially stable with decay rate $\beta$, $H_\infty$ disturbance attenuation $\gamma$ and guaranteed level of robustness $\lambda$:*

$$\Xi + \mathcal{T}\Pi_5 + (\mathcal{T}\Pi_5)^T < 0 \tag{15}$$

where

$$\Xi=\begin{bmatrix}\Xi_1 & \Xi_2 & \bar{\mathcal{A}}^{*T}P & \Xi_5^T\bar{Z} & \Xi_5^T W & \varepsilon\bar{\mathcal{H}}_A^T & \bar{\mathcal{C}}^T\\ (*) & \Xi_3 & \Xi_4^T P & 0 & 0 & \varepsilon\Xi_6^T & 0\\ (*) & (*) & -P & 0 & 0 & 0 & 0\\ (*) & (*) & (*) & -\bar{Z} & 0 & 0 & 0\\ (*) & (*) & (*) & (*) & -W & 0 & 0\\ (*) & (*) & (*) & (*) & (*) & -\varepsilon I_{l_2} & 0\\ (*) & (*) & (*) & (*) & (*) & 0 & -I_l\end{bmatrix}<0, \tag{16}$$

$$\Xi_1=-\beta^2\Pi_1^TP\Pi_1+\Pi_2^T(Q_1+Q_2)\Pi_2+\Pi_3+\Pi_4,$$

$$\Xi_2=\begin{bmatrix}0_{\bar{n}_1\times\bar{n}_3}\\ \sum_{g=1}^2\beta^{2(h_g-1)}\left(Z_g-3\alpha_{(h_g-1)}Z_g\right)\mathcal{V}_g\\ 6\mathcal{Z}_1\mathcal{V}_1\\ 6\mathcal{Z}_2\mathcal{V}_2\\ 0_{3n\times\bar{n}_3}\end{bmatrix},$$

$$\bar{n}_1=2n+r,\qquad \bar{n}_2=3m+3n,\qquad \bar{n}_3=3m+l_1+l,$$

$$\begin{aligned}&\Xi_3=\Xi_{31}+\Xi_{32},\\ &\Xi_{31}=diag\,\big(-\beta^{2(h_1-1)}Q_1,\ -\beta^{2(h_2-1)}Q_2,\\ &\quad -W,\ -\varepsilon I_{l_1},\ -\gamma^2 I_l\big),\\ &\Xi_{32}=diag\big(-\bar{\mathcal{Z}}_1,\ -\bar{\mathcal{Z}}_2,\ 0_{m+l_1+l}\big),\\ &\Xi_4=\begin{bmatrix}\bar{\mathcal{B}}_1 & \bar{\mathcal{B}}_2 & \bar{\mathcal{B}}_3 & \bar{\mathcal{E}} & \bar{\mathcal{M}}\end{bmatrix},\\ &\Xi_5=\bar{\mathcal{K}}-\Pi_2,\\ &\Xi_6=\begin{bmatrix}\tfrac12 v_1H_B & \tfrac12 v_2H_B & \tfrac{\tau}{2}H_B & 0_{l_2\times l_1} & 0\end{bmatrix},\\ &\bar{\mathcal{A}}^*=\bar{\mathcal{A}}+\bar{\mathcal{B}}\bar{\mathcal{K}},\qquad \bar{\mathcal{K}}=\begin{bmatrix}\bar{K} & 0_{m\times\bar{n}_2}\end{bmatrix},\end{aligned}\tag{17}$$

$$\bar{\mathcal{A}}=\begin{bmatrix}\bar{A} & 0 & 0 & 0 & 0_{\bar{n}_1\times 3n}\\ 0 & 0_m & 0 & 0 & 0_{n\times 3n}\\ 0 & (1-v_1)I_m & h_1I_m & 0 & 0_{m\times 3n}\\ 0 & (1-v_2)I_m & 0 & h_2I_m & 0_{m\times 3n}\end{bmatrix},$$

$$\bar{\mathcal{B}}=\begin{bmatrix}\bar{B}_\tau^T & I_m & 0_{m\times 2m}\end{bmatrix}^T,$$

$$\bar{\mathcal{B}}_1=\begin{bmatrix}0_{m\times(\bar{n}_1+m)} & -v_1I_m & 0_m\end{bmatrix}^T,$$

$$\bar{\mathcal{B}}_2=\begin{bmatrix}0_{m\times(\bar{n}_1+m)} & 0_m & -v_2I_m\end{bmatrix}^T,$$

$$\bar{\mathcal{B}}_3=\begin{bmatrix}\tfrac{\tau}{2}\bar{B}^T & 0_{m\times\bar{n}_2}\end{bmatrix}^T,$$

$$\bar{\mathcal{E}}=\begin{bmatrix}\lambda\bar{E}^T & 0_{l_1\times\bar{n}_2}\end{bmatrix}^T,\ \bar{\mathcal{M}}=\begin{bmatrix}\bar{M}^T & 0_{l\times\bar{n}_2}\end{bmatrix}^T,$$

$$\Pi_1=\begin{bmatrix}I_{\bar{n}_1} & 0 & 0 & 0 & 0_{\bar{n}_1\times 3n}\\ 0 & I_m & 0 & 0 & 0_{m\times 3n}\\ 0 & -I_m & h_1I_m & 0 & 0_{m\times 3n}\\ 0 & -I_m & 0 & h_2I_m & 0_{m\times 3n}\end{bmatrix},$$

$$\Pi_2=\begin{bmatrix}0_{m\times\bar{n}_1} & I_m & 0_m & 0_m & 0_{m\times 3n}\end{bmatrix},$$

$$\Pi_3=\begin{bmatrix}0_{\bar{n}_1} & 0 & 0 & 0 & 0\\ (*) & -\bar{\mathcal{Z}}_1-\bar{\mathcal{Z}}_2 & 6\mathcal{Z}_1 & 6\mathcal{Z}_2 & 0\\ (*) & (*) & -12\mathcal{Z}_1 & 0 & 0\\ (*) & (*) & (*) & -12\mathcal{Z}_2 & 0\\ (*) & (*) & (*) & (*) & 0_{3n}\end{bmatrix},$$

$$\Pi_4=\begin{bmatrix}0_{\bar{n}_1} & 0 & 0 & 0 & 0 & 0\\ (*) & \pi_{41} & 0 & \pi_{42} & \pi_{43} & 0\\ (*) & (*) & 0_{2m} & 0 & 0 & 0\\ (*) & (*) & (*) & -S_1 & 0 & 0\\ (*) & (*) & (*) & (*) & -S_2 & 0\\ (*) & (*) & (*) & (*) & (*) & 0_n\end{bmatrix},$$

$$\pi_{41}=\bar{S}-\frac14\sum_{g=1}^{2}\left(B^TS_gB\right),$$

$$\pi_{42}=v_3B^TS_1,\qquad \pi_{43}=\frac12B^TS_2,$$

$$\bar{S}=\frac14(h_2-1)\sum_{j=1}^{h_2-1}\beta^{-2j}\phi_{1,j+1}^TS_1\phi_{1,j+1}$$

$$+\frac{\tilde{\tau}}{4}\sum_{j=1}^{\tau-1}\beta^{-2j}\phi_{2,j+1}^TS_2\phi_{2,j+1},\qquad \tilde{\tau}=\begin{cases}\tau-1 & if\ \tau>0\\ 0 & if\ \tau=0,\end{cases}$$

$$\Pi_5=\begin{bmatrix}\tilde{\Pi}_5 & 0\end{bmatrix},\ \tilde{\Pi}_5=\begin{bmatrix}-I_n & 0_{n\times(n+r+3m)} & A^\tau & I_n & A^{h_2}\end{bmatrix},$$

$$\mathcal{T}=\begin{bmatrix}T^T & 0\end{bmatrix}^T,$$

$$\bar{\mathcal{H}}_A=\begin{bmatrix}0_{l_2\times\bar{n}_1} & \mathcal{H}_b & 0_{l_2\times(2m+2n)} & H_A\end{bmatrix},$$

$$\mathcal{H}_b=\frac12(1-v_1)H_B+\frac12(1-v_2)H_B,$$

$$\bar{\mathcal{C}}=\begin{bmatrix}0_{p\times(\bar{n}_1+3m+2n)} & C\end{bmatrix},$$

$$\mathcal{V}_1=\begin{bmatrix}I_m & 0_m & 0_{m\times(m+l_1+l)}\end{bmatrix},$$

$$\mathcal{V}_2=\begin{bmatrix}0_m & I_m & 0_{m\times(m+l_1+l)}\end{bmatrix},$$

$$\bar{\mathcal{Z}}_g=\beta^{2(h_g-1)}\left(3\alpha_{(h_g-1)}Z_g+Z_g\right),$$

$$\mathcal{Z}_g=\beta^{2(h_g-1)}\left(\alpha_{(h_g-1)}Z_g\right),\qquad g=1,2,$$

$$\bar{Z}=\sum_{g=1}^{2}\sum_{j=0}^{h_g-2}\beta^{2j}(h_g-1)Z_g,$$

$$\phi_{1,j}=\begin{cases}\frac12A^{j-1}B & if\ j\le\tau\\ \frac12\left(A^{j-1}+A^{j-\tau-1}\right)B & if\ j>\tau\ and\ j\le h_2-\tau\\ \frac12A^{j-\tau-1}B & if\ j>h_2-\tau\\ 0 & otherwise,\end{cases}$$

$$\phi_{2,j}=\frac12A^{j-1}B,$$

$$v_g=\begin{cases}1 & if\ h_g>1, g=1,2\\ 0 & otherwise,\end{cases}\qquad v_3=\begin{cases}1 & if\ \tau=0\\ \frac12 & otherwise\end{cases}$$

**Proof.** See the Appendix

**Remark 1.** The novel term $V_{4,k}$ has been defined in the LKF (A.1) to deal with all delayed terms coming from $\Phi_{1,k}$ and $\Phi_{2,k}$ respectively with single decision variables ($S_1$ and $S_2$ respectively). As a result, the size of LMI (15) and the number of decision variables keep independent of the size of delays.

**Corollary 2.** *A feasible solution for LMI (15) is always guaranteed for sufficiently small values of $\lambda,\tau$ and $\gamma^{-1}$ for any decay rate $\beta\ge max\,(\rho_{sr}(A+BK),\rho_{sr}(\mathcal{A}-\mathcal{L}\mathcal{C}))$.*

**Proof.** The existence of a feasible solution for LMI (15) can be deduced by choosing $\lambda=0$, $\tau=0$ and $\gamma^{-1}=0$. After removing their respective rows and columns from (15), the condition $\bar{A}^TX\bar{A}-\beta^2X<0$ for some $X>0$ is obtained, which is true for any $\beta\ge max\,(\rho_{sr}(A+BK),\rho_{sr}(\mathcal{A}-\mathcal{L}\mathcal{C}))$. ■

## 5. Robust control synthesis

This section provides a CCL-based algorithm based on Theorem 1 for control and observer design in the presence of time-varying delays and model mismatches.

Let us denote $\tilde{P}=P^{-1}$, $\tilde{\bar{Z}}=\bar{Z}^{-1}$, $\tilde{W}=W^{-1}$ and $\tilde{\varepsilon}=\varepsilon^{-1}$. Hence, pre- and post multiplying (15) by $diag\left(I,\tilde{P},\tilde{\bar{Z}},\tilde{W},\tilde{\varepsilon}I_{l_2},I_l\right)$, we obtain the matrix inequality $\tilde{\Xi}+\mathcal{T}\Pi_5+(\mathcal{T}\Pi_5)^T<0$, where

$$\tilde{\Xi}=\begin{bmatrix}\Xi_1 & \Xi_2 & \bar{\mathcal{A}}^{*T} & \Xi_5^T & \Xi_5^T & \bar{\mathcal{H}}_A^T & \bar{\mathcal{C}}^T\\ (*) & \tilde{\Xi}_3 & \tilde{\Xi}_4^T & 0 & 0 & \tilde{\Xi}_6^T & 0\\ (*) & (*) & -\tilde{P} & 0 & 0 & 0 & 0\\ (*) & (*) & (*) & -\tilde{\bar{Z}} & 0 & 0 & 0\\ (*) & (*) & (*) & (*) & -\tilde{W} & 0 & 0\\ (*) & (*) & (*) & (*) & (*) & -\tilde{\varepsilon}I_{l_2} & 0\\ (*) & (*) & (*) & (*) & (*) & 0 & -I_l\end{bmatrix} \quad (18)$$

and

$$\tilde{\Xi}_3=\tilde{\Xi}_{31}+\Xi_{32}, \quad (19)$$
$$\tilde{\Xi}_{31}=diag\left(-\beta^{2h_1}Q_1,\ -\beta^{2h_2}Q_2,\ -\tilde{W},\ -I_{l_1},\ -\gamma^2 I_l\right),$$
$$\tilde{\Xi}_4=\begin{bmatrix}\bar{\mathcal{B}}_1 & \bar{\mathcal{B}}_2 & \tau\bar{\mathcal{B}}_3\tilde{W} & \tilde{\varepsilon}\lambda\bar{\mathcal{E}} & \bar{\mathcal{M}}\end{bmatrix},$$
$$\tilde{\Xi}_6=\begin{bmatrix}\frac{1}{2}H_B & \frac{1}{2}H_B & \frac{\tau}{2}H_B\tilde{W} & 0_{l_2\times l_1} & 0\end{bmatrix}$$

Now, let us introduce the LMI conditions to relax the equality constraints $P\tilde{P}=I$ and $\bar{Z}\tilde{\bar{Z}}=I$ for the CCL algorithm:

$$\begin{bmatrix}P & I_{\bar{n}}\\ I_{\bar{n}} & \tilde{P}\end{bmatrix}\geq 0,\qquad \begin{bmatrix}\bar{Z} & I_m\\ I_m & \tilde{\bar{Z}}\end{bmatrix}\geq 0, \quad (20)$$

together with the objective function to minimize:

$$\min(\mathrm{trace}\left(P\tilde{P}+\tilde{P}P+\bar{Z}\tilde{\bar{Z}}+\tilde{\bar{Z}}\bar{Z}\right)$$

### 5.1. CCL algorithm description

The detailed CCL-algorithm is described as follows:

- Step (i): Given $K$ and $\mathcal{L}$ satisfying the conditions given in Corollary 2, find a feasible solution for LMIs (15) choosing small values for $\lambda$, $\tau$ and $\tilde{\gamma}=\gamma^{-1}$. Then, set $\lambda^{(0)}=\lambda$, $\tau^{(0)}=\tau$, $\tilde{\gamma}^{(0)}=\tilde{\gamma}$, $P^{(0)}=P$, $\bar{Z}^{(0)}=\bar{Z}$. Set $q:=1$.
- Step (ii): Set $P^{(q)}:=P^{(q-1)}$, $\tilde{P}^{(q)}:=\left(P^{(q-1)}\right)^{-1}$, $\bar{Z}^{(q)}:=\bar{Z}^{(q-1)}$, $\tilde{\bar{Z}}^{(q)}:=\left(\bar{Z}^{(q-1)}\right)^{-1}$ and go to Step (iii).
- Step (iii): Solve the LMI's (18) and (20) subject to

  $$\min(\mathrm{trace}\left(P\tilde{P}^{(q)}+\tilde{P}P^{(q)}+\bar{Z}\tilde{\bar{Z}}^{(q)}+\tilde{\bar{Z}}\bar{Z}^{(q)}\right)$$

  setting $\lambda=\lambda^{(q-1)}+\mathcal{I}_\lambda$, $\tau=\tau^{(q-1)}+\mathcal{I}_\tau$, $\tilde{\gamma}=\tilde{\gamma}^{(q-1)}+\mathcal{I}_{\tilde{\gamma}}$, being $\mathcal{I}_\lambda$, $\mathcal{I}_\tau$, $\mathcal{I}_{\tilde{\gamma}}$ an incremental value for each iteration. Matrices $P, Q_1, Q_2, Z_1, Z_2, S_1, S_2, \tilde{W}, K$ and $\mathcal{L}$ are defined in this step as LMI decision variables.
- Step (iv): If a feasible solution is found, set $\lambda^{(q)}=\lambda$, $\tau^{(q)}=\tau$, $\tilde{\gamma}^{(q)}=\tilde{\gamma}$ and go to step (v). Otherwise, set $\mathcal{I}_\lambda=\mathcal{I}_\lambda/n_\lambda$, $\mathcal{I}_\tau=\mathcal{I}_\tau/n_\tau$, $\mathcal{I}_{\tilde{\gamma}}=\mathcal{I}_{\tilde{\gamma}}/n_{\tilde{\gamma}}$ for some step reduction factors $n_\lambda, n_\tau, n_{\tilde{\gamma}}>1$ and go to Step (iii).
- Step (v): If the maximum number of iterations is still not reached, and $\tau^{(q)}<h_2-h_1$ or $\lambda^{(q)}<\bar{\lambda}$ or $\tilde{\gamma}^{(q)}<\bar{\gamma}^{-1}$ with $\bar{\lambda},\bar{\gamma}$ prescribed levels of robustness and $H_\infty$ disturbance rejection, set $q:=q+1$, $P^{(q)}:=P^{(q-1)}$, $\tilde{P}^{(q)}:=\left(P^{(q-1)}\right)^{-1}$, $\bar{Z}^{(q)}:=\bar{Z}^{(q-1)}$, $\tilde{\bar{Z}}^{(q)}:=\left(\bar{Z}^{(q-1)}\right)^{-1}$ and go to Step (iii). Otherwise, stop and exit.

**Remark 2.** It is worthwhile mentioning that a feasible solution for the first iteration in the CCL iterative loop is always guaranteed regardless of the delay values $h_1, h_2$ by solving LMI (15) choosing a sufficiently small value for $\lambda$, $\tau$, $\gamma^{-1}$ and controller and observer gains $K$ and $\mathcal{L}$ satisfying the conditions given in Corollary 2 for a given decay rate $\beta$. Hence, differently from Hao et al. (2019), the controller and observer gains can be designed by standard techniques (i.e, pole placement) in the particular case of time-constant delays and the absence of model uncertainties.

**Remark 3.** The size of LMI conditions (18) and (20) for CCL is $4\bar{n}+3n+5m+l_1+l_2+2l$ and the number of decision variables (NoV) is $\bar{n}(\bar{n}+1)+n(n+1)+3m(m+1)+m(n+r)+p(n+r)+(5n+3m+r)n+1$ with $\bar{n}=2n+r+3m$, which are both independent on delay. Hence, differently from Hao et al. (2019), the CPU time for the proposed algorithm is not influenced by the size of delays. The LKF definition (A.1) including $u_{k-1}$ in the augmented states has been crucial to avoid the presence of cross-product terms between $\bar{\mathcal{K}}$ and other decision variables in matrix inequalities (18) without being necessary to resort to applying multiple Schur Complements.

## 6. Simulation results

First example provides comparative results with Hao et al. (2019) in order to illustrate the achieved improvements in the sense of robustness and closed-loop performance. Second example gives simulation results with a mismatched case (She, Fang, Ohyama, Hashimoto, & Wu, 2008).

### 6.1. Example 1 (matched case)

Consider the open-loop unstable plant model studied in Hao et al. (2019) with sampling period $T_s=0.1$. The discrete-time model is obtained as (1) with matrices $A=\begin{bmatrix}0.9505 & 0.1149\\ -1.0339 & 1.2952\end{bmatrix}$, $B=F=\begin{bmatrix}0.0055\\ 0.1149\end{bmatrix}$, and $C=\begin{bmatrix}1 & 0\end{bmatrix}$. In this example, Case 3 in Hao et al. (2019) will be proposed for comparison since time-varying model uncertainties and sinusoidal-like disturbance signals are considered. For a fair comparison, the same simulation conditions will be taken into account: (i) time-varying uncertainties (2) with $E=[0\ 0.03]^T$, $H_A=[0.1\ 0.2]$, $H_B=0.05$, $\Delta_k=sin(k)$, (ii) disturbance composed of a step and a sinusoidal signal of frequency $w_c=0.5$ but unknown amplitude and phase, and (iii) an additional perturbation in the exogenous system $\delta_k=sin(k)/(1+k)$. Disturbances are therefore described as Hao et al. (2019) by (3) with $\Lambda=\begin{bmatrix}0.9988 & 0.05 & 0\\ -0.05 & 0.9988 & 0\\ 0 & 0 & 1\end{bmatrix}$, $M=\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $N=\begin{bmatrix}1 & 0 & 1\end{bmatrix}$. In this case, a stabilizing controller is obtained after 6 CCL iterations for a time-constant delay $d_k\equiv 6$ with guaranteed exponential decay rate performance $\beta=0.98$. The designed controller and observer parameters are $K=[8.5926\quad -3.947]$, $K_d=[-2.3830\quad -0.5639\quad -2.4540]$, $L=[0.95955\quad 2.6947]^T$ and $L_\xi=[0.48501\quad 0.7032\quad 0.55686]^T$. where the CCL parameters have been chosen as $\mathcal{I}_\lambda=0.2$, $\mathcal{I}_\tau=0$, $\mathcal{I}_{\tilde{\gamma}}=0$, $n_\lambda=n_\tau=n_{\tilde{\gamma}}=0.5$, and the initial $K$ and $\mathcal{L}$ for the first iteration have been designed by pole placement with controller poles in $\{0.92,\ 0.93\}$ and observer poles in $\{0.87,\ 0.88,\ 0.89,\ 0.90,\ 0.91\}$. Simulation results are given in Fig. 1, where it can be appreciated a slight improvement in terms of closed-loop dynamic performance.

With the proposed method, longer delays are allowed up to a time-constant delay $d_k\equiv 16$ with a guaranteed level of robustness $\lambda=0.1$ and exponential decay rate $\beta=0.9896$ (simulation results are given in Fig. 2). The designed controller and observer gains after 3 CCL iterations for (5) and (7) are $K=[8.3633\quad -3.8756]$, $K_d=[14.5315\quad 1.28002\quad 14.2773]$, $L=[0.76209\quad 2.0754]^T$ and $L_\xi=[0.4183\quad 0.70198\quad 0.50556]^T$, where the same initial values for $K$ and $\mathcal{L}$ have been set, and the CCL parameters were chosen as $\mathcal{I}_\lambda=0.05$, $\mathcal{I}_\tau=0$, $\mathcal{I}_{\tilde{\gamma}}=0$, $n_\lambda=n_\tau=n_{\tilde{\gamma}}=0.5$. It is noteworthy that, in view of Table 1 in Hao et al. (2019), the computational time becomes prohibitive for $d_k=16$. In contrast, the proposed method (as discussed in Remark 3) allows finding a stabilizing controller with the same computational effort (around 130$s$) due to the fact that

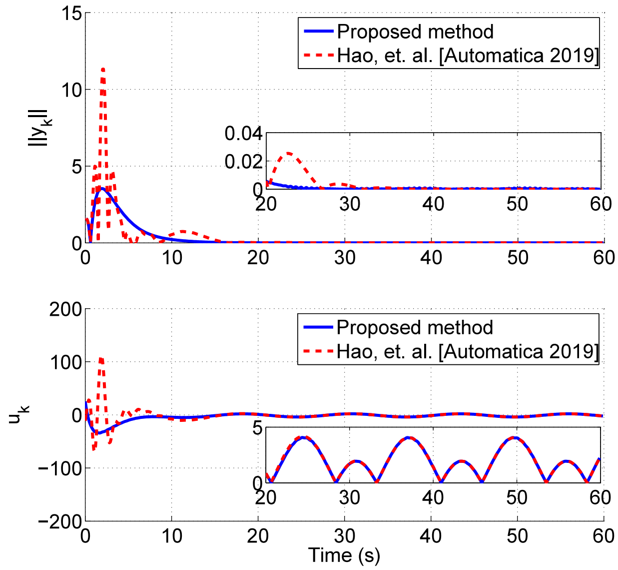


**Fig. 1.** Control results for $d_k \equiv 6$ (comparative analysis).

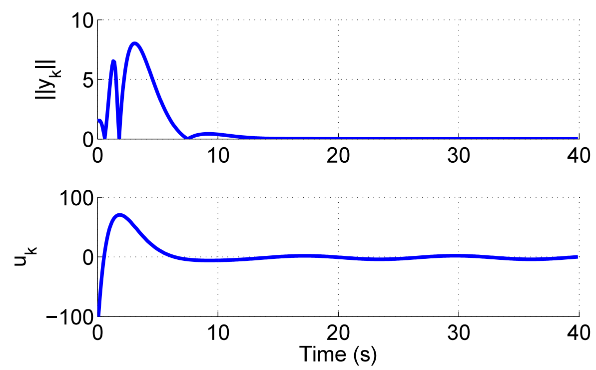


**Fig. 2.** Control results for $d_k \equiv 16$.

the number of decision variables and size of LMIs are the same in both cases: 165 and 57 respectively.

Now, consider a time-varying delay $5 \leq d_k \leq 6$. Applying the proposed algorithm, a stabilizing controller (5)–(7) is designed with the controller and observer parameters $K = [6.1852 \quad -3.7406]$, $K_d = [-2.6429 \quad -0.50889 \quad -2.7002]$, $L = [0.92691 \quad 2.8183]^T$ and $L_\xi = [0.13407 \quad 0.23022 \quad 0.1789]^T$. Simulation results are given in Fig. 3 for $d_k = h_1 + \tau|cos(k)|$, where it can also be appreciated that the disturbance signal is effectively steady-state rejected. The CCL algorithm was first executed to achieve $\lambda = 0$, $\tau = 1$ with $\mathcal{I}_\lambda = 0$, $\mathcal{I}_\tau = 0.1$, $\mathcal{I}_{\tilde{\gamma}} = 0$, $n_\lambda = n_\tau = n_{\tilde{\gamma}} = 0.5$ and initial $K, \mathcal{L}$ with the same pole location as in the previous cases. A second CCL execution was performed to achieve $\lambda = 1$, $\tau = 1$ starting from the obtained $K, \mathcal{L}$ of the first execution and CCL parameters $\mathcal{I}_\lambda = 0.05$, $\mathcal{I}_\tau = 0$, $\mathcal{I}_{\tilde{\gamma}} = 0$, $n_\lambda = n_\tau = n_{\tilde{\gamma}} = 0.5$. Moreover, an exponential decay rate $\beta = 0.9989$ is ensured for any time-varying delay pattern.

### 6.2. Example 2 (mismatched case)

Consider the speed control of a rotational control system borrowed from She et al. (2008) consisting of two DC motors. After discretizing with ZOH and sampling period $T_s = 10$ ms, system (1) is obtained with state-space matrices

$$A = \begin{bmatrix} -0.101 & 0.975 & -117.919 \\ 0.275 & 0.787 & 44.753 \\ 0.004 & -0.006 & -0.246 \end{bmatrix},$$

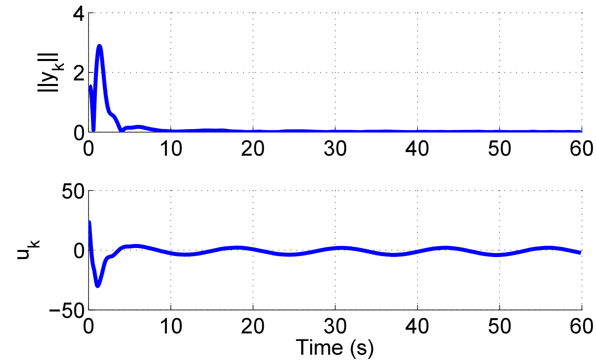


**Fig. 3.** Control results for $5 \leq d_k \leq 6$.

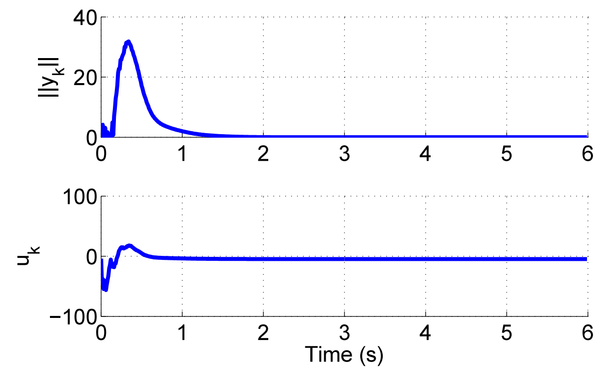


**Fig. 4.** Control results for $8 \leq d_k \leq 12$ (Case mismatched).

$B = \begin{bmatrix} 0.147 & 0.030 & 0.001 \end{bmatrix}^T$, $F = \begin{bmatrix} -6.756 & 2.225 & 0.037 \end{bmatrix}^T$ and $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$. The disturbance signal is assumed to be a constant load ($\Lambda = 1$, $M = 1$, $N = 0$ in (3)), and time-varying delays $h_1 \leq d_k \leq h_2$ are assumed with $h_1 = 8$ and $h_2 = 12$. Applying the proposed CCL algorithm with $\mathcal{I}_\lambda = 0$, $\mathcal{I}_\tau = 0.1$, $\mathcal{I}_{\tilde{\gamma}} = 0$, $n_\lambda = n_\tau = n_{\tilde{\gamma}} = 0.5$, and the initial $K$ and $\mathcal{L}$ for the first iteration designed by pole placement with controller poles in $\{0.91, \ 0.92, \ 0.93\}$ and observer poles in $\{0.87, \ 0.88, \ 0.89, \ 0.90\}$, the controller and observer gains given below with exponential decay rate $\beta = 0.9828$ are obtained after 12 iterations as $K = [-4.5829 \quad 4.5175 \quad 611.7487]$, $L = [-0.5465 \quad 0.1589 \quad 0.0052]^T$ and $L_\xi = 9.3477 \cdot 10^{-4}$ and $K_d = -1.3402$. Simulation results are depicted in Fig. 4, where the output system is shown to be stable with a complete steady-state disturbance rejection.

## 7. Conclusions and perspectives

This paper has presented an output-feedback anti-disturbance predictor-feedback control with an extended state observer, which differently from previous approaches, time-varying delays can be considered. Moreover, a CCL-based algorithm has been provided where the complexity (number of decision variables and size of LMIs) is independent of the time-delay magnitude. Finally, comparative simulation results have been provided to show the achieved improvements of the proposed method in the sense of better robust performance against time-varying delays and model uncertainties. Future extension of this work could be aimed at including other performance criteria in control synthesis, such as input saturation or regional pole constraints.

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## Appendix. Proof of Theorem 1

Let $\bar{\eta}_k = [\bar{x}_k^T \;\; u_{k-1}^T \;\; \mu_{1,k}^T \;\; \mu_{2,k}^T \;\; \Phi_{1,k}^T \;\; \Phi_{2,k}^T \;\; x_k]^T$, $\bar{\eta}_k^* = \left[\bar{x}_k^T \;\; u_{k-1}^T \;\; \mu_{1,k}^{*T} \;\; \mu_{2,k}^{*T}\right]^T$, where $\mu_{g,k} = \frac{1}{h_g}\sum_{i=1}^{h_g} u_{k-i}$ with g = 1, 2, and $\mu_{g,k}^* = \sum_{i=2}^{h_g} u_{k-i}$. Noting that $\Phi_{k+1}(h_g) - A\Phi_k(h_g) = \frac{1}{2}A^{-h_g}Bu_k - \frac{1}{2}Bu_{k-h_g}$ and $\mu_{g,k+1}^* = h_g\mu_{g,k} - u_{k-h_g}$, we can write $\bar{\eta}_{k+1}^* = \bar{\mathcal{A}}\bar{\eta}_k + \bar{\mathcal{B}}_1 u_{k-h_1} + \bar{\mathcal{B}}_2 u_{k-h_2} + \bar{\mathcal{B}}_\tau \omega_k + \bar{\mathcal{E}} w_{\Delta,k} + \bar{\mathcal{M}}\delta_k$, or similarly in compact form as $\bar{\eta}_{k+1}^* = \bar{\mathcal{A}}\bar{\eta}_k + \Xi_4\bar{\omega}_k$, where $\bar{\omega}_k = \left[u_{k-h_1}^T \;\; u_{k-h_2}^T \;\; \omega_k^T \;\; w_{\Delta,k}^T \;\; \delta_k^T\right]^T$. Now, consider the LKF: $V_k = V_{1,k} + V_{2,k} + V_{3,k} + V_{4,k}$, where

$$V_{1,k} = \bar{\eta}_k^{*T} P \bar{\eta}_k^*, \tag{A.1}$$

$$V_{2,k} = \sum_{g=1}^{2}\sum_{i=2}^{h_g} \beta^{2(i-2)} u_{k-i}^T Q_g u_{k-i},$$

$$V_{3,k} = \sum_{g=1}^{2}(h_g - 1)\left(\sum_{i=2}^{h_g}\sum_{j=2}^{i} \beta^{2(i-2)} \rho_{k-j}^T Z_g \rho_{k-j}\right),$$

$$V_{4,k} = (h_2 - 1)\left(\sum_{i=2}^{h_2}\sum_{j=2}^{i} \beta^{2(j-i-1)} u_{k-j}^T \phi_{1,i}^T S_1 \phi_{1,i} u_{k-j}\right)$$
$$+\, \tilde{\tau}\left(\sum_{i=2}^{\tau}\sum_{j=2}^{i} \beta^{2(j-i-1)} u_{k-j}^T \phi_{2,i}^T S_2 \phi_{2,i} u_{k-j}\right)$$

where $\rho_k = u_{k+1} - u_k$ and $\phi_{1,i}$, $\phi_{2,i}$ are defined in Theorem 1. The system is stable with decay rate $\beta$ if there exists $V_k > 0$ such that $V_{k+1} - \beta^2 V_k < 0$. Then, defining the forward difference $\Delta_\beta V_{l,k} = V_{l,k+1} - \beta^2 V_{l,k}$, $l = 1,2,3,4$, taking into account that $\mu_{g,k}^* = h_g\mu_{g,k} - u_{k-1}$, $g = 1,2$ (and therefore $\bar{\eta}_k^* = \Pi_1\bar{\eta}_k$) and noting that $u_{k-1} = \Pi_2\bar{\eta}_k$, we have that:

$$\Delta_\beta V_{1,k} = \bar{\eta}_k^T \bar{\mathcal{A}}^T P \bar{\mathcal{A}} \bar{\eta}_k \tag{A.2}$$
$$+\, 2\bar{\eta}_k^T \bar{\mathcal{A}}^T P \Xi_4 \bar{\omega}_k + \bar{\omega}_k^T \Xi_4^T P \Xi_4 \bar{\omega}_k - \beta^2 \bar{\eta}_k^T \Pi_1^T P \Pi_1 \bar{\eta}_k,$$

$$\Delta_\beta V_{2,k} = u_{k-1}^T(Q_1 + Q_2)u_{k-1} - \sum_{g=1}^{2}\beta^{2(h_g-1)} u_{k-h_g}^T Q_g u_{k-h_g}$$
$$= \bar{\eta}_k^T \Pi_2^T (Q_1 + Q_2)\Pi_2 \bar{\eta}_k - \bar{\omega}_k^T \mathcal{V}^T \bar{Q} \mathcal{V} \bar{\omega}_k, \quad \mathcal{V}^T = \left[\mathcal{V}_1^T \;\; \mathcal{V}_2^T\right],$$
$$\bar{Q} = diag\left(\beta^{2(h_1-1)}Q_1,\; \beta^{2(h_2-1)}Q_2\right),$$

$$\Delta_\beta V_{3,k} = \rho_{k-1}^T \bar{Z} \rho_{k-1} - \sum_{g=1}^{2}\sum_{j=2}^{h_g}(h_g - 1)\beta^{2(h_g-1)} \rho_{k-j}^T Z_g \rho_{k-j}$$
$$= \bar{\eta}_k^T \Xi_5^T \bar{Z} \Xi_5 \bar{\eta}_k - \sum_{g=1}^{2}(h_g - 1)\sum_{j=2}^{h_1} \rho_{k-j}^T Z_g \rho_{k-j},$$

$$\Delta_\beta V_{4,k} = (h_2 - 1)\sum_{j=2}^{h_2} \beta^{-2(j-1)} u_{k-1}^T \phi_{1,j}^T S_1 \phi_{1,j} u_{k-1}$$
$$-\, (h_2 - 1)\sum_{j=2}^{h_1} u_{k-j}^T \phi_{1,j}^T S_1 \phi_{1,j} u_{k-j}$$
$$+\, \tilde{\tau}\sum_{j=2}^{\tau} \beta^{-2(j-1)} u_{k-1}^T \phi_{2,j}^T S_2 \phi_{2,j} u_{k-1}$$
$$-\, \tilde{\tau}\sum_{j=2}^{\tau} u_{k-j}^T \phi_{2,j}^T S_2 \phi_{2,j} u_{k-j}$$

Applying Lemma 1, from the rightmost part of $\Delta_\beta V_{3,k}$ and taking into account that $\sum_{j=2}^{h_g} \rho_{k-j} = u_{k-1} - u_{k-h_g}$, $g = 1,2$, we obtain:

$$-\sum_{g=1}^{2}\left((h_g - 1)\sum_{j=2}^{h_1} \rho_{k-j}^T Z_g \rho_{k-j}\right)$$
$$\le \begin{bmatrix}\bar{\eta}_k \\ \bar{\omega}_k\end{bmatrix}^T \begin{bmatrix}\Pi_3 & \Xi_2 \\ (*) & \Xi_{32}\end{bmatrix} \begin{bmatrix}\bar{\eta}_k \\ \bar{\omega}_k\end{bmatrix} \tag{A.3}$$

where $\Pi_3$, $\Xi_2$, and $\Xi_{32}$ are defined in (16). Also, by Jensen's inequality (Briat, 2011), from the rightmost part of $\Delta_\beta V_{4,k}$ and taking into account the definition of $\Phi_k(.)$ in (6), we have that:

$$-\,(h_2 - 1)\left(\sum_{j=2}^{h_2}\left(\phi_{1,j}u_{k-j}\right)^T S_1 \left(\phi_{1,j}u_{k-j}\right)\right) \tag{A.4}$$
$$\le -\left(\sum_{j=2}^{h_2}\phi_{1,j}u_{k-j}\right)^T S_1 \left(\sum_{j=2}^{h_2}\phi_{1,j}u_{k-j}\right) \tag{A.5}$$
$$= -\left(\Phi_{1,k} - \upsilon_3 B\right)^T S_1 \left(\Phi_{1,k} - \upsilon_3 B\right),$$

$$-\,\tilde{\tau}\left(\sum_{j=2}^{\tau}\left(\phi_{2,j}u_{k-j}\right)^T S_2 \left(\phi_{2,j}u_{k-j}\right)\right)$$
$$\le -\left(\sum_{j=2}^{\tau}\phi_{2,j}u_{k-j}\right)^T S_2 \left(\sum_{j=2}^{\tau}\phi_{2,j}u_{k-j}\right)$$
$$= -\left(\Phi_{2,k} - \frac{1}{2}B\right)^T S_2 \left(\Phi_{2,k} - \frac{1}{2}B\right)$$

Taking into account (A.4), we have that $\Delta_\beta V_{4,k} \le \bar{\eta}_k^T \Pi_4 \bar{\eta}_k$, where $\Pi_4$ are defined in Theorem 1. Applying Schur complement, we have that (15) is equivalent to $\Omega + T\tilde{\Pi}_5 + \left(T\tilde{\Pi}_5\right)^T < 0$, where

$$\Omega = \begin{bmatrix}\Xi_1 & \Xi_2 \\ (*) & \Xi_3\end{bmatrix} + \begin{bmatrix}\bar{\mathcal{A}}^T \\ \Xi_4^T\end{bmatrix} P \begin{bmatrix}\bar{\mathcal{A}} & \Xi_4\end{bmatrix} + \begin{bmatrix}\Xi_5^T \\ \Xi_6^T\end{bmatrix} \bar{Z} \begin{bmatrix}\Xi_5 & \Xi_6\end{bmatrix}$$
$$\begin{bmatrix}\Xi_5^T \\ \Xi_6^T\end{bmatrix} W \begin{bmatrix}\Xi_5 & \Xi_6\end{bmatrix} + \begin{bmatrix}\mathcal{H}_A^T \\ \Xi_6^T\end{bmatrix} \begin{bmatrix}\mathcal{H}_A & \Xi_6\end{bmatrix} + \begin{bmatrix}\mathcal{C}^T \\ 0\end{bmatrix} \begin{bmatrix}\mathcal{C} & 0\end{bmatrix} \tag{A.6}$$

Taking into account from (10) that $\tilde{\Pi}_5\bar{\eta}_k = 0$ with $\tilde{\Pi}_5$ defined in (16), the augmented vector $\bar{\eta}_k$ above defined, and applying Finlser's theorem, we have that $\bar{\eta}_k^T\left(\Omega + T\tilde{\Pi}_5 + \left(T\tilde{\Pi}_5\right)^T\right)\bar{\eta}_k < 0$ is equivalent to $\Delta_\beta V_k = \sum_{i=1}^4 \Delta_\beta V_{i,k}$. From (A.2) and the inequalities (A.3), (A.4), it can be deduced that $\Delta_\beta V_k + J_k \le \bar{\psi}_k^T \Omega \bar{\psi}_k < 0$ with $\bar{\psi}_k = \left[\bar{\eta}_k^T,\; \bar{\omega}_k^T\right]^T$ and $\bar{\eta}_k, \bar{\omega}_k$ above defined, where:

$$J_k = \rho_{k-1}^T W \rho_{k-1} - \omega_k^T W \omega_k + \varepsilon y_{\Delta,k}^T y_{\Delta,k} - \varepsilon w_{\Delta,k}^T w_{\Delta,k}$$
$$+\, y_k^T y_k - \gamma^2 \delta_k^T \delta_k \tag{A.7}$$

and $\rho_{k-1} = u_k - u_{k-1} = \left(\bar{\mathcal{K}} - \Pi_2\right)\bar{\eta}_k = \Xi_5\bar{\eta}_k$, $y_{\Delta,k} = H_A x_k = \bar{\mathcal{H}}_A\bar{\eta}_k + \Xi_6\bar{\omega}_k$, $y_k = Cx_k = \bar{\mathcal{C}}\bar{\eta}_k$. Hence, the fulfilment of (15) implies that $\Delta_\beta V_k < 0$, which is a sufficient condition for exponential stability with decay rate $\beta$. Also, the fulfilment of (15) under zero initial conditions implies that the $l_2$ induced norm from $\delta_k$ to $y_k$ is not greater than $\gamma$, while the condition $\|W\mathcal{D}_{h,k}W^{-1}\|_\infty \le 1$ (Lemma 2) and $w_{\Delta,k}^T w_{\Delta,k} \le y_{\Delta,k}^T y_{\Delta,k}$ are satisfied $\forall k \ge 0$. Hence, the robust exponential stability with $H_\infty$ disturbance rejection level $\gamma$ of the closed-loop system formed by (1) and the proposed control scheme can be ensured if the inequality (15) is true. ■

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