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On a Ermakov–Kalitkin scheme based family of fourth order

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1 Introduction

This work addresses the resolution of nonlinear equations of the form

$$f(x) = 0, \text{ with } f : I \subset \mathbb{R} \to \mathbb{R},$$

by applying numerical iterative methods. To this days, Newton's method, one of the classic approaches, is still the most widely used way to find an approximated solution to this kind of problems arising in many and diverse branches of science. However, usually this method requires a "good and close enough" initial estimation to converge. Thus, finding complementary iterative methods that converge when Newton's does not work is a topic of interest.
Some of the aforementioned iterative methods have been designed aiming to enlarge the domain of convergence (see [6] and [7]). In general, an iterative method that takes the form

$$x_{k+1} = x_k - \beta_k \frac{f(x_k)}{f'(x_k)}, \tag{1}$$

where β_k is a sequence of real numbers is called a damped Newton's method (notice that taking $\beta_k = 1$ for every step k leads to Newton's method). In [5], V. V. Ermakov and N. N. Kalitkin designed a fairly efficient iterative method that follows the above expression, taking the sequence β_k as:

$$\beta_k = \frac{\|f(x_k)\|^2}{\|f(x_k)\|^2 + \|f(x_k - [f'(x_k)]^{-1} f(x_k))\|^2}, \quad k = 0, 1, ... \tag{2}$$

From this, D. A. Budzko with the second and third author introduced in [2] the scheme of two-steps iterative methods

$$\begin{aligned} y_k &= x_k - \alpha \frac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= y_k - \frac{f(x_k)^2}{b f(x_k)^2 + c f(y_k)^2} \frac{f(y_k)}{f'(x_k)}, \quad k = 0, 1, ..., \end{aligned} \tag{3}$$

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where α, b and c are given parameters. This multiparameter family take its first step (predictor) as a damped Newton's method, and the second step (corrector) resembles the Ermakov and Kalitkin formula 2 . It is proven that a method following this formula reaches third order of convergence whenever

$$\alpha \notin \{0,1\}, \quad b = \frac{1+\alpha^2}{2\alpha^2} \quad \text{and} \quad c = \frac{1+\alpha}{2(\alpha-1)\alpha^2}. \tag{4}$$

So, this provides a one-parameter family (depending on α) of two step iterative methods for solving nonlinear equations that has third order of convergence for every α different of 0 and 1. Additionally, this family provides bigger basins of attraction than Newthon's method (see [3]). In this work we present a new family of two-steps iterative method, also based on the Ermakov–Kalitkin formula, that reaches fourth order of convergence.

2 Methods

As we said in the previous section, in [2, Theorem 1] it is proved that every method of the form 3 has third order of convergence whenever the parameters are taken as in 4. However, this is the highest order of convergence that can be obtained following this scheme. To reach higher order while keeping the stability of the family, we modified 3 by adding an arbitrary weight function H in the corrector step:

$$\begin{aligned} y_k &= x_k - \alpha \frac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= y_k - \frac{f(x_k)^2}{bf(x_k)^2 + cf(y_k)^2} \frac{f(y_k)}{f'(x_k)} H\left(\frac{f(y_k)}{f(x_k)}\right), \quad k = 0, 1, \ldots \end{aligned} \tag{5}$$

The idea behind this change is that, when computing the error equation, the function H will be approximated by its Taylor series, and so, its Taylor series coefficients shall be taken into account. As the function H can be chosen, we may think of its Taylor coefficient as free parameters, and considering also the previous free parameters α, b and c, this provide us enough flexibility so we can find the relations that allows us to vanish as many terms as possible. This is the argument behind the prove of Theorem 1, which provides a way to find one-parameter families of fourth order by choosing a weigh function satisfying some specific relation.

3 Results

In this section we will state the main results and conclusions derived from the study. Due to the format of this document and for the sake of compactness, the proofs are omitted. We start with the theorem that crystallize the argument exposed above.

Theorem 1. *Let $\xi \in I$ be a simple zero of a differentiable function $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ in the open interval I, and let x_0 be an initial approximation close enough to ξ. Then, a two-step iterative method of the form*

$$\begin{aligned} y_k &= x_k - \alpha \frac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= y_k - \frac{f(x_k)^2}{bf(x_k)^2 + cf(y_k)^2} \frac{f(y_k)}{f'(x_k)} H\left(\frac{f(y_k)}{f(x_k)}\right), \end{aligned} \tag{6}$$

has 4th order of convergence whenever

$$\alpha = 1,\ b = h_0 = \frac{h_1}{2} \neq 0,$$

for every c and for every weight function $H(x)$ such that h_2 is finite, where $h_i = H^{(i)}(0)$. Moreover, the error equation of this method has the form

$$e_{k+1} = \Big((5 + \frac{c}{h_0} - \frac{h_2}{2h_0})c_2^3 - c_2c_3\Big)e_k^4 + O(e_k^5).$$

where $c_k = (1/k!)\frac{f^{(k)}(\xi)}{f'(\xi)}$ for $k = 1, 2, ...$ and $e_k = x_k - \xi$.

From this, in Theorem 2 we introduce a new one-parameter family of iterative methods which we refer as the **EK** family. We built it by taking $H(x) = 1 + 2x$ in Theorem 1, the simplest function that satisfy the conditions. Also, notice that as a formal/aesthetic rearrangement, the $1/b$ term in the second step of (6) is absorbed by the weight function H, and by putting $\beta = c/b$ we get the statement:

Theorem 2. *Let $\xi \in I$ be a simple zero of a differentiable function $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ in the open interval I, and let x_0 be an initial approximation close to ξ. Then, the two step iterative method*

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= y_k - \frac{f(x_k)^2}{f(x_k)^2 + \beta f(y_k)^2}\frac{f(y_k)}{f'(x_k)}\left(1 + 2\frac{f(y_k)}{f(x_k)}\right), \end{aligned} \tag{\textbf{EK}}$$

has 4th order of convergence for every parameter β. Moreover, the error equation for this method has the form

$$e_{k+1} = \Big((5 + \beta)c_2^3 - c_2c_3\Big)e_k^4 + O(e_k^5),$$

where $c_k = (1/k!)\frac{f^{(k)}(\xi)}{f'(\xi)}$ for $k = 1, 2, ...$ and $e_k = x_k - \xi$.

The **EK** family is the main contribution of this work. Now we will include a summary of an introductory study of its complex dynamical behaviour depending on the parameter β (basic concepts and definitions from holomorphic dynamical systems can be found in [1] and [4]). In order to do this, we construct a rational operator associated with the family, on a generic two-degree nonlinear polynomial, and we analyze the stability and convergence of the corresponding fixed and critical points.

We start by considering $f(z) = (z - r)(z - s)$ a generic quadratic polynomial with roots $r, s \in \mathbb{R}$. Applying a method of the **EK** family to this function, we get the following rational function which depends on the roots of f and the parameter β.

$$Q_f(z, \beta) = \frac{\frac{(r-z)^4(s-z)^4\left(r^2-6z(r+s)+4rs+s^2+6z^2\right)}{\beta(r-z)^4(s-z)^4+(r-z)^2(s-z)^2(r+s-2z)^4} + rs - z^2}{r + s - 2z}.$$

We take the Möbius transformation $M_f(z) = \dfrac{z - s}{z - r}$ and its inverse $M_f^{-1}(z) = \dfrac{rz - s}{z - 1}$. Thus, the rational operator is constructed as:

$$R(z, \beta) = M_f \circ Q_f(\cdot, \beta) \circ M_f^{-1}(z) = \frac{z^4(\beta + z(z + 4) + 5)}{z((\beta + 5)z + 4) + 1}.$$

Through this process, we have constructed a rational operator $R(z,\beta)$ that is independent of the original roots of f, while is topologically conjugate to $Q_f(z,\beta)$. This last means that both share the same dynamical properties. So, this allow us to study the complex behaviour of the **EK** family applied to any quadratic polynomial by just studying the $R(z,\beta)$ operator. Notice that M_f produces a biholomorphic transformation of the Riemann sphere that satisfy $M_f(\infty)=1$, $M_f(r)=\infty$ and $M_f(s)=0$. Thus, roughly speaking, we can think on the dynamical planes of $R(z,\beta)$ as a "nice rearrangement" of the dynamical planes of $Q_f(z,\beta)$, where the relevant points s, r and ∞ are now located in their images through M_f.
The result below sums up the fixed points of $R(z,\beta)$ and its stability.

Proposition 3. *The fixed points of the rational operator $R(z,\beta)$ are the roots of the equation $R(z,\beta)=z$. Then, for each value of β it has the next fixed points on the Riemann's sphere:*

- *$z=0$ and $z=\infty$ are superattractors for every β.*
- *The pair **FP1** $=\frac{1}{4}\left(-\sqrt{-4\beta-7}\pm\sqrt{-4\beta+10\sqrt{-4\beta-7}+2}-5\right)$. Its stability is described by Figure 1.*
- *The pair **FP2** $=\frac{1}{4}\left(\sqrt{-4\beta-7}\pm\sqrt{-4\beta-10\sqrt{-4\beta-7}+2}-5\right)$. Its stability is described by Figure 2.*
- *$z=1$ is a fixed point if and only if $\beta\neq-10$. Moreover, the point is an attractor whenever $\beta=v+iw$ is taken such that (v,w) belongs to the disc $(v+21)^2+w^2\leq 133$, and it is superattractor if and only if $\beta=-16$.*

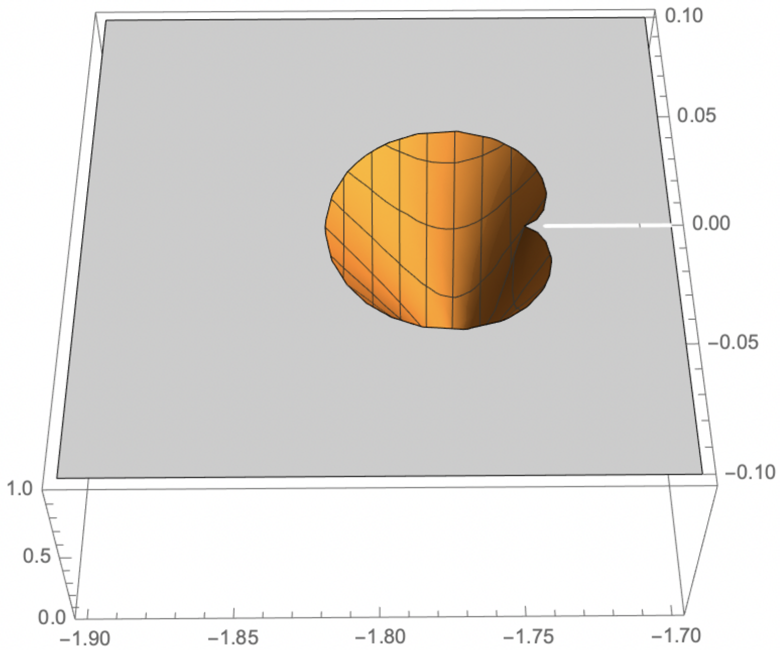

Figure 1: Stability surface of the pair **FP1** (in color, the complex area where the pair is repulsive, being attracting in the rest).

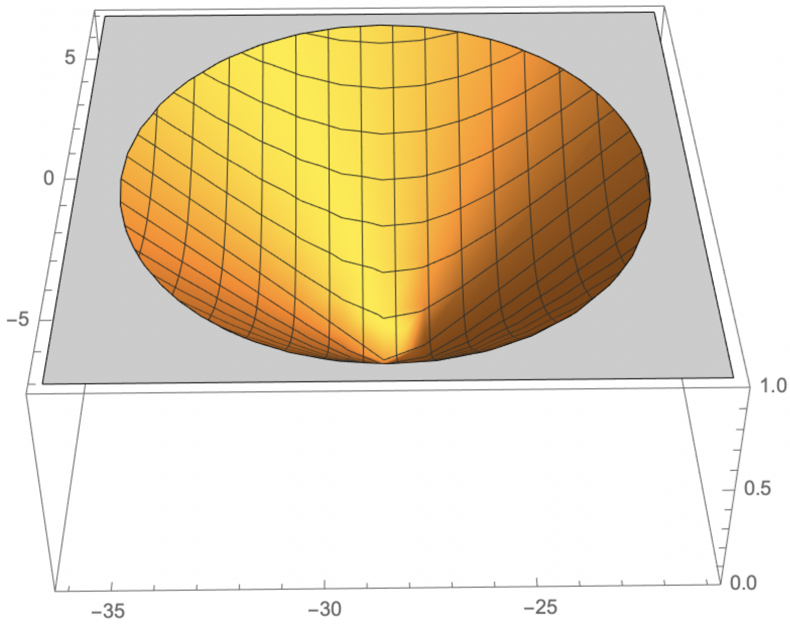

Figure 2: Stability surface of the pair **FP2** (in color, the complex area where the pair is repulsive, being attracting in the rest).

By the classic Fatou–Julia theorem, it is known that every basins of attraction of a rational function must contain, at least, one critical point. Thus, by locating the critical points and computing their orbits, we can find if there is some basin of attraction other than the generated by $z = 0$ and $z = \infty$.

Proposition 4. *The critical points of the rational operator $R(z, \beta)$ are the roots of the equation $R'(z, \beta) = 0$. Then, for each value of β it has the next critical points on the Riemann's sphere:*

- $z = 0$ *and* $z = \infty$ *(the roots).*
- *The pair* ***CR1*** $= \frac{1}{4}\left(\pm\sqrt{2}\sqrt{-\frac{\beta^4+15\beta^3+80\beta^2-3\left(|\beta+5|\sqrt{-\beta(2\beta^2+13\beta+20)}-50\right)\beta-20|\beta+5|\sqrt{-\beta(2\beta^2+13\beta+20)}}{(\beta+5)^3}} - \frac{\sqrt{-\beta(2\beta^2+13\beta+20)}}{|\beta+5|} - \frac{3\beta}{\beta+5} - \frac{20}{\beta+5}\right)$;
- *The pair* ***CR2*** $= \frac{1}{4}\left(\pm\sqrt{2}\sqrt{-\frac{\beta^4+15\beta^3+80\beta^2+3\left(\sqrt{(\beta+5)^2}\sqrt{-\beta(2\beta^2+13\beta+20)}+50\right)\beta+20\sqrt{(\beta+5)^2}\sqrt{-\beta(2\beta^2+13\beta+20)}}{(\beta+5)^3}} + \frac{\sqrt{-\beta(2\beta^2+13\beta+20)}}{|\beta+5|} - \frac{3\beta}{\beta+5} - \frac{20}{\beta+5}\right)$;

A parameter space associated to a critical point is defined as a mesh in the complex plane where each point corresponds to a different value of of the parameter β. If a value of β is represented in red it means that for this value of the parameter the critical point is being attracted by a root ($z = 0$ or $z = \infty$). Otherwise, the pixel is colored black. In Figures 3 and 4 we included the parameter spaces related to the pairs **CR1** and **CR2** respectively. They show that as long as we avoid the β values from the black regions, we get nice iterative methods where the only basins of attraction are the associated to the roots.

4 Conclusions

In this paper we have introduced a result to generate families of iterative methods of fourth order that follows the Ermakov–Kalitkin scheme, allowing to choose freely a weight function as long as it satisfies the hypotheses. From this, we designed the **EK** family by taking the simplest possible

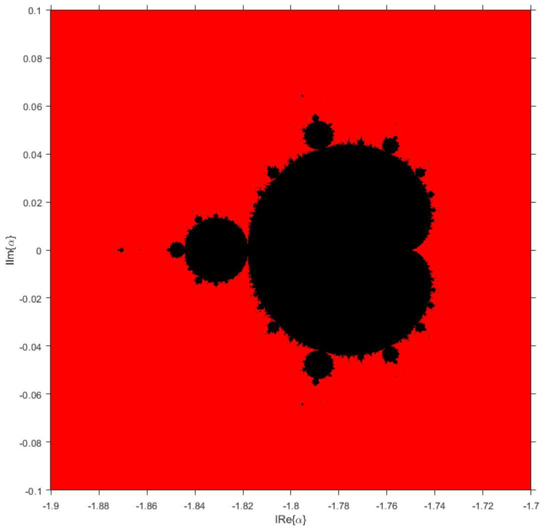

Figure 3: Parameter space associated to the pair **CR1**.

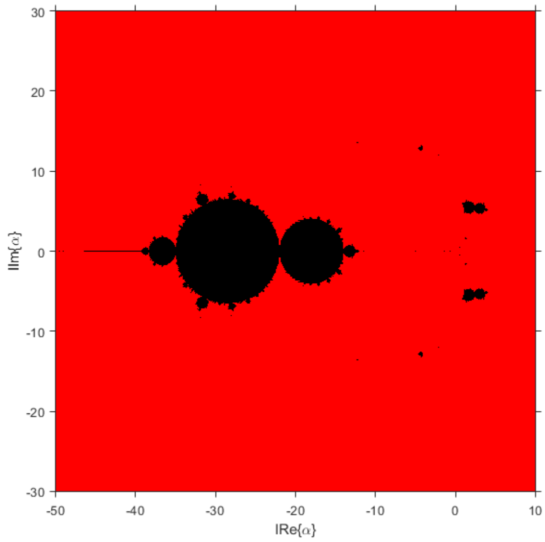

Figure 4: Parameter space associated to the pair **CR2**.

weight function satisfying the conditions, and included a summary of its behaviour on quadratic polynomials depending on the free parameter. The dynamical study shows that by avoiding some values of the parameter we can get nice iterative methods of fourth order where the only attracting behaviour belongs to the roots. Future lines of work will involve closing the search for the best value of the parameter and an in-depth study of the good behavior of this methods applied to other non-linear functions.

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