Symmetry, winding number, and topological charge of vortex solitons in discrete-symmetry media

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We determine the functional behavior near the discrete rotational symmetry axis of discrete vortices of the nonlinear Schrödinger equation. We show that these solutions present a central phase singularity whose charge is restricted by symmetry arguments. Consequently, we demonstrate that the existence of high-charged discrete vortices is related to the presence of other off-axis phase singularities, whose positions and charges are also restricted by symmetry arguments. To illustrate our theoretical results, we offer two numerical examples of high-charged discrete vortices in photonic crystal fibers showing hexagonal discrete rotational invariance.

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I. INTRODUCTION

Complex scalar solutions of wave equations can present dislocations similar to those found in crystals [1]. The essential mathematical property of these complex scalar functions in the point or line where a dislocation is localized is that its phase is increased or decreased in a multiple of 2π along a closed curve around it [1]. In these points or lines, also known as phase singularities, the amplitude of the function vanishes and its phase is undetermined. They play an important role in many branches of science, such as solid-state physics [2], Bose-Einstein condensates (BECs) [3], superfluidity [4], superconductivity [5], cosmology [6], molecular dynamics [7], nonlinear optics [8], etc. In the latter case, the study of such singularities is often enclosed in a separated branch called nonlinear singular optics [9].

An optical vortex is a complex scalar solution of a wave equation defined in a two-dimensional (2D) domain characterized by the presence of a phase singularity [8]. If the vortex has a circular symmetry there is only a singularity located at the rotational axis. Optical vortices with discrete symmetry, or discrete vortices (DVs), have been theoretically predicted to exist in inhomogeneous periodic media as solitonic solutions of a nonlinear wave equation, such as the nonlinear Schrödinger equation (NLSE). These periodic media include optically induced lattices [10], photonic crystal fibers [11], or Bessel lattices [12]. They have been experimentally observed in the former medium [13,14]. On the other hand, DVs have also been predicted to exist as solutions of NLSE in self-attracting BEC in the presence of periodic optical lattices [10,15]. Other solutions with discrete rotational symmetry and a complicated phase structure have been introduced in periodically modulated potentials, both in the framework of BEC and nonlinear optics [16–23]. Some of these solutions are characterized by the presence of more than one phase singularity [16–20]. Finally, other quasistationary solutions showing discrete symmetry in a homogeneous medium, known as necklace beams or soliton clusters, have been introduced, and some of them show a nontrivial phase structure [24–29]. Stationary solutions of this kind have also been obtained in an inhomogeneous media such as a photonic lattice [30–32].

In this paper, our aim is to study DVs with different possible configurations of singularities. We will analytically obtain the behavior of a DV near the symmetry axis to show that they always present a singularity in the rotational axis which can be completely characterized by discrete group theory arguments. Next, we will show that they can present more than one single singularity. The positions of these singularities are related according to the rules arising from discrete group theory arguments. We will also provide two numerical examples of DVs with more than one singularity to illustrate our results.

II. THEORY

To start with, let us introduce first some common definitions which we will be using in the text. If $\psi(\mathbf{x})$ is a complex scalar solution of a wave equation defined in a two-dimensional domain, $\mathbf{x}\in\mathbb{R}^2$, then the winding number γ of ψ along a closed curve Γ is given by the contour integral $\gamma=\frac{1}{2\pi}\oint_\Gamma \nabla\phi\cdot d\mathbf{l}$, where ϕ is the phase of the complex field $\psi=|\psi|e^{i\phi}$. Let $\mathbf{x}_0$ be the position of a phase singularity of ψ. The topological charge of the phase singularity located at $\mathbf{x}_0$ is the winding number of the complex field ψ for the smallest closed curve containing $\mathbf{x}_0$. That is, if Γ_ϵ is a family of closed curves containing $\mathbf{x}_0$ parametrized by ϵ such that $\lim_{\epsilon\to 0}\Gamma_\epsilon=\mathbf{x}_0$, then $v\equiv\lim_{\epsilon\to 0}\frac{1}{2\pi}\oint_{\Gamma_\epsilon}\nabla\phi\cdot d\mathbf{l}$. Additionally, one can define another quantity, the total angular momentum, as $\langle\psi|\mathcal{L}_z|\psi^*\rangle/\langle\psi|\psi^*\rangle$, where $\mathcal{L}_z=(\vec{r}\times\vec{\nabla})_z$.

Vortices with circular symmetry can be written as $\psi(r,\theta)=g(r)e^{il\theta}$, where (r,θ) are the polar coordinates. They present well-defined angular momentum since $\mathcal{R}_\alpha\psi=e^{il\alpha}\psi$, where $\mathcal{R}_\alpha=e^{i\mathcal{L}_z\alpha}$ is a continuous rotation of angle $\alpha\in\mathbb{R}$ and $\mathcal{L}_z=i\frac{\partial}{\partial\theta}$ is the generator of the $O(2)$ rotation group. Therefore, vortices are eigenfunctions of the angular-momentum operator, satisfying $\mathcal{L}_z\psi = l\psi$. Note that the winding number and the topological charge of a singularity are phase-related concepts while the angular momentum is a symmetry-related concept. These circularly symmetric vortices present a single singularity at the origin and fulfill $l = v = \gamma$, provided that γ is calculated in a closed curve that surrounds the phase singularity. Also, they satisfy $g(r) \sim ar^{|l|} + O(r^{|l|+2})$ when $r \to 0$ [8].

For a DV, the winding number γ and the topological charge v, i.e., the phase-related concepts, are also well defined. However angular momentum is not. Nevertheless, another symmetry-related concept, the angular pseudomomentum m, has been defined for discrete-symmetry media [33]. It has been also demonstrated that this quantity is conserved during propagation [33]. By construction, the angular pseudomomentum m completely defines the representation of the $\mathcal{C}_n$ group to which ψ belongs. It has been also shown that $m = v = \gamma$ for a DV with a single singularity [34]. We discuss the relationship of these three quantities for all types of DVs. This relationship will permit us to study the number of phase singularities associated to a DV.

Let us show next how the behavior of a DV near the symmetry axis can be analytically determined in terms of the angular pseudomomentum m. This will permit us to establish that there is always a phase singularity of charge m located at the symmetry axis. To do this, let the function $\psi(r,\theta)$ be a stationary solution of the 2D nonlinear eigenvalue equation,

$$[L_0 + L_{NL}(|\psi|)]\psi = \mu\psi, \tag{1}$$

with $L_0 = -\nabla_t^2 + V(\mathbf{x})$ and $L_{NL}(|\psi|)$ as the invariant operators under the $\mathcal{C}_n$ point-symmetry group formed by discrete rotations of order n around the rotation axis. It has been proven in Refs. [33,35] that if ψ is a self-consistent solution of Eq. (1) satisfying the symmetry condition $|\psi(r,\theta+2\pi/n)|^2 = |\psi(r,\theta)|^2$, it can be expressed as $\psi_{m,\alpha}(r,\theta) = e^{im\theta}u_{m,\alpha}(r,\theta)$, where $u_{m,\alpha}(r,\theta) = u_{m,\alpha}(r,\theta+\frac{2\pi}{n})$, m is the angular pseudomomentum, and α is a band index. The function ψ is said to be a *symmetric stationary solution* of the 2D nonlinear eigenvalue Eq. (1) if it satisfies the condition $|\psi(r,\theta+2\pi/n)|^2 = |\psi(r,\theta)|^2$.

It has been also demonstrated that the angular pseudomomentum presents a cutoff related with the order n of discrete rotational symmetry of the medium. Particularly, it has been shown that $|m| \leq \frac{n}{2}$ for even n and $|m| \leq \frac{n-1}{2}$ for odd n [33,34]. Under these conditions it can be proven that the solutions of Eq. (1) with $|m| = 1, \ldots, \frac{n}{2}-1$ for even n and $|m| = 1, \ldots, \leq \frac{n-1}{2}$ for odd n cannot be real. It is easy to see that the operator $L = [L_0 + L_{NL}(|\psi|)]$ is Hermitian. Therefore its eigenvalues are real numbers. Since L satisfies $L = L^*$, solutions with $m = 0$ or $m = \frac{n}{2}$ for even n are real (up to a global phase), since they belong to one-dimensional irreducible representations. On the other hand, solutions with angular pseudomomentum different from $m = 0$ or $m = \frac{n}{2}$ for even n correspond to complex solutions belonging to two-dimensional representations of the $\mathcal{C}_n$ group. Therefore, DVs are characterized by $|m| = 1, \ldots, \frac{n}{2}-1$ for even n and $|m| = 1, \ldots, \frac{n-1}{2}$ for odd n.

We leave the study of the solutions with $m = 0$ or $m = \frac{n}{2}$ for future research. For the rest, i.e., for DVs, we will obtain next the mathematical behavior near the symmetry axis. Particularly, we will show that if the function ψ is a symmetric stationary solution of Eq. (1) and satisfies the following mathematical conditions:

(1) $|\psi(r,\theta)| \overset{r\to 0}{\approx} |\psi_0| + \delta\psi(r,\theta)$, where $\psi_0 \in \mathbb{R}$ and $\delta\psi \ll \psi_0$,

(2) $L_{NL}(|\psi|) \overset{r\to 0}{\approx} L_{NL}(|\psi_0|) + \delta L_{NL}$, where $L_{NL}(|\psi_0|) \in \mathbb{R}$ and $\delta L_{NL} \ll L_{NL}(|\psi_0|)$, and

(3) $V(r,\theta) \overset{r\to 0}{\approx} V_0 + \delta V(r,\theta)$, where $V_0 \in \mathbb{R}$ and $\delta V \ll V_0$, then

$$\psi(r,\theta) \overset{r\to 0}{\approx} ar^m e^{im\theta} + \mathcal{O}(r^{m+1}),$$

where $a \in \mathbb{R}$.

From a physical point of view, these mathematical conditions establish that the amplitude, the nonlinear operator, and the potential V have a smooth nonsingular behavior near the symmetry axis. Therefore, the previous conditions are easily satisfied by DVs in discrete-symmetry media such as the systems mentioned above. We will prove next that the mathematical behavior of DVs near the symmetry axis depends only on the angular pseudomomentum.

To demonstrate the above behavior of the function $\psi(r,\theta)$, we need to express it as $\psi_{m,\alpha}(r,\theta) = e^{im\theta}u_{m,\alpha}(r,\theta)$. The functions $u_{m,\alpha}(r,\theta)$ satisfy the following differential equation:

$$\left[-\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{m^2}{r^2} - i\frac{2m}{r^2}\frac{\partial}{\partial\theta} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]u_{m,\alpha} + [V(r,\theta) + L_{NL}(|\psi|)]u_{m,\alpha} = \mu_{m,\alpha}u_{m,\alpha}. \tag{2}$$

Because of the periodic behavior of the wave function $u_{m,\alpha}(r,\theta)$ and the potential $V(r,\theta)$, we can expand them in Fourier series in the angular variable as

$$u_{m,\alpha}(r,\theta) = \sum_k e^{ikn\theta}u^k_{m,\alpha}(r), \quad V(r,\theta) = \sum_{k'} e^{ik'n\theta}V_{k'}(r),$$

and after performing the angular integrals we get the following set of differential equations for the angular Fourier components $u^{\bar{k}}_{m,\alpha}(r)$:

$$\left[-\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \frac{(m+\bar{k}n)^2}{r^2}\right]u^{\bar{k}}_{m,\alpha} + \sum_k V_{\bar{k}-k}(r)u^k_{m,\alpha} + \sum_k L^{k-\bar{k}}_{NL}(|\psi|)u^{\bar{k}}_{m,\alpha} = \mu_{m,\alpha}u^{\bar{k}}_{m,\alpha}. \tag{3}$$

Assumptions (1)–(3) for the limit $r \to 0$ allow us to write $V(r,\theta) \approx V_0$, $|\psi(r,\theta)| \approx \psi_0$, and $L_{NL}(|\psi|) \approx L_{NL}(|\psi_0|)$, where V_0, ψ_0, and $L_{NL}(|\psi_0|)$ are real numbers. Therefore, in this limit, we obtain

$$V_{\bar{k}-k}(r) = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{i(\bar{k}-k)n\theta}V(r,\theta) \overset{r\to 0}{\approx} \delta_{\bar{k},k}V_0 + \delta V_{\bar{k}-k}(r)$$

and

$$L_{NL}^{k-\bar{k}}(|\psi|) = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{-i(\bar{k}-k)n\theta} L_{NL}(|\psi|)$$
$$\overset{r\to 0}{\approx} \delta_{\bar{k},k} L_{NL}(|\psi_0|) + \delta L_{NL}^{k-\bar{k}}.$$

Then, the set of differential equations [Eq. (3)] for the angular Fourier components behave for $r\to 0$ as

$$\left[-\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \frac{(m+\bar{k}n)^2}{r^2}\right]u_{m,\alpha}^{\bar{k}}$$
$$\overset{r\to 0}{\approx} [\mu_{m,\alpha} - V_0 - L_{NL}(|\psi_0|)]u_{m,\alpha}^{\bar{k}}$$
$$= K_{m,\alpha}u_{m,\alpha}^{\bar{k}}, \tag{4}$$

which is the same equation (but in the linear case) obtained for circularly symmetric vortices of the form $\psi(r,\theta)=u(r)e^{il\theta}$ carrying angular momentum $l=m+\bar{k}n$. As discussed in [8], the behavior of a solution of this type close to the origin is given by $u(r) \overset{r\to 0}{\approx} Ar^{|l|}+O(r^{|l|+1})$. Therefore, the solution of the previous equation behaves in the vortex core region as

$$u_{m,\alpha}^{\bar{k}}(r) \overset{r\to 0}{\approx} ar^{|m+\bar{k}n|} + O(r^{|m+\bar{k}n|+1}). \tag{5}$$

We now analyzed which are the dominant terms in the limit $r\to 0$ in the Fourier expansion of $\psi_{m,\alpha}(r,\theta)$,

$$\psi_{m,\alpha}(r,\theta) = e^{im\theta}\left[\sum_k e^{ikn\theta}u_{m,\alpha}^k(r)\right]. \tag{6}$$

In the limit $r\to 0$ the dominant term in the expression above will be given by that which minimizes the exponent of the leading term in expansion (5). The behavior of $\eta(k)=|m+kn|$ has to be then analyzed for different m and k values in order to find the dominant contribution, i.e., that given by the lowest value of $\eta(k)$.

Let us first consider that according to the cutoff theorem, $|m|\leq\frac{n}{2}$ for n even and $|m|\leq\frac{n-1}{2}$ for n odd. Besides, as stated above, for DVs, $m\neq 0$ and $m\neq\frac{n}{2}$, and then $|m|=1,\ldots,\frac{n}{2}-1$ for even n or $|m|=1,\ldots,\frac{n-1}{2}$ for odd n. Therefore, it is easy to see that $|kn|>|m|$ for $|k|=1,2,\ldots$ and, of course, $|kn|<|m|$ for $k=0$. With this in mind, let us see that $\eta(0)<\eta(k)$ for all possible values of m and k.

Let us first consider that $m>0$ and $k>0$. In this case, we have $\eta(k)=|m+kn|=m+kn$ for $k=0,1,\ldots$, since both quantities are positive. Hence, this gives $\eta(0)<\eta(k)$ for $k=1,2,\ldots$. Next, let us consider that $m>0$ and $k<0$. For $|k|=1,2,\ldots$, we have $\eta(k)=|m+kn|=|k|n-m$ since $|kn|>|m|$. Then, this gives $\eta(k)<\eta(k+1)$. On the other hand, for $k=0$, we have $\eta(0)=m$. Finally, let us show that $m<|k|n-m$. For DVs it is always satisfied that $|m|<\frac{n}{2}$ in all cases. Then, we obtain $2m<|k|n$ for $k\neq 0$ and consequently $m<|k|n-m$. Then, this gives $\eta(0)<\eta(k)$.

Now, let us consider that $m<0$ and $k\geq 0$. Then, we have $\eta(0)=|m|$ and $\eta(k)=|m+kn|=kn-|m|$, for $k=1,2,\ldots$ since $kn>|m|$ and then $kn-|m|$ is always a positive value. Then, this gives $\eta(k)<\eta(k+1)$ for $k=1,2,\ldots$. On the other hand, since for DV we have $2|m|<kn$ for $k=1,2,\ldots$, we obtain $|m|<kn-|m|$. Therefore, this gives $\eta(0)<\eta(k)$ for $k=1,2,\ldots$. Finally, if $m<0$ and $k\leq 0$ then we have $\eta(0)=|m|$ and $\eta(k)=|m+kn|=|kn|+|m|$, for $k=1,2,\ldots$ since both quantities are negative. Then, obviously $\eta(0)<\eta(k)$ is satisfied for $|k|=1,2,\ldots$.

In conclusion, for all the possible values of m that a DV can present, the dominant contribution to the wave function arises from the term given by $k=0$ and, consequently, according to Eqs. (5) and (6) the wave function behaves like

$$\psi_{m,\alpha}(r,\theta) \propto e^{im\theta}r^{|m|}$$

in the limit $r\to 0$.

Therefore, it can be easily proven that there always exists a phase singularity of charge m located at point $\mathbf{x}_r$ where the rotation axis intersects the 2D plane, i.e., the topological charge of this singularity is $v=m$.

Once this has been established, let us go into the relationship among m, the winding number, and the topological charge of a singularity in depth. The winding number of the symmetric stationary solution ψ can be calculated using any closed curve. Let us consider a closed curve Γ that surrounds point $\mathbf{x}_r$ and let us assume that the winding number of the symmetric stationary solution ψ is such that $\gamma\neq m$. Then (1) there always exists a phase singularity of charge m located *on axis* and (2) there must exist a number of *off-axis* singularities inside the closed curve Γ fulfilling the condition

$$\gamma = m + \sum_{j=1}^{V} v_j, \tag{7}$$

where v_j is the topological charge of the jth phase singularity and V is the total number of singularities. (3) Additionally, there could exist a number of off-axis phase singularity pairs with opposite charges (vortex-antivortex pairs) inside the closed curve Γ.

This result is obtained after splitting the path integral in the definition of γ, in $V+1$ path integrals, all of them related with the topological charge of each singularity. The integral around the rotational axis will offer a topological charge equal to m. The rest must be related with the existence of V off-axis singularities or vortex-antivortex pairs that contribute with a null net charge to the integral.

Alternatively, let us consider a closed curve Γ that surrounds point $\mathbf{x}_r$ for which the winding number of the symmetric stationary solution ψ is such that $\gamma=m$. Then, using similar arguments, either (1) there exists *only* one phase singularity of charge m located on axis or (2) there exist one phase singularity of charge m located on axis and a number of off-axis phase singularity pairs with opposite charges (vortex-antivortex pairs) inside the closed curve Γ. Hence, we have proven that it can be stated that a DV presents a phase singularity of topological charge m located at the rotational axis and a number of off-axis phase singularities verifying Eq. (7).

Let us show now that the position of these off-axis phase singularities of ψ is always symmetric with respect to the rotation axis. Let us assume that a phase singularity is located at point $\mathbf{x}_0$ different from point $\mathbf{x}_r$, where the symmetry axis intersects the transverse plane. Then, the modulus of the

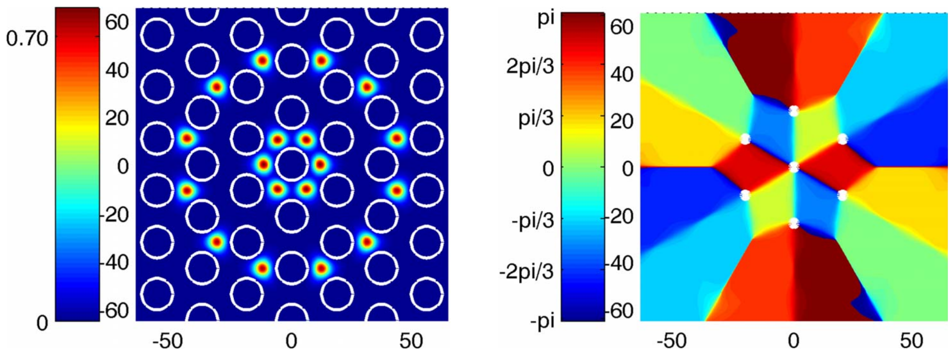

FIG. 1. (Color online) (a) Amplitude and (b) phase of a DV with angular pseudomomentum $m=-1$, $\mu=-0.08$, and more than one singularity. White circles in (a) represent the air holes in the photonic crystal fiber. White circles in (b) represent the positions of the singularities.

function at this point vanishes, $|\psi(\mathbf{x}_0)|=0$ and the argument is not defined. Besides, as stated above, the function ψ of a DV must belong to one of the representations of the group [11,33]. Then, if one transforms ψ according to the rotational elements of the group, i.e., the transformations $C_n^t=e^{it(2\pi/n)}$, with $t=1,\ldots,n-1$, then the transformed function must be

$$\psi_t = e^{itm(2\pi/n)}\psi. \tag{8}$$

Then the transformed function and the function itself differ only in a fixed phase, i.e., $|\psi_t|=|\psi|$ and $\angle\psi_t=\angle\psi+\alpha_t$, where $\alpha_t=tm\frac{2\pi}{n}$.

Let us apply the rotational elements of the group C_n^t, with $t=1,\ldots,n-1$, to the function at point $\mathbf{x}_0$. The transformed function is the function ψ evaluated at a set of rotated points $\mathbf{x}_t$ given by $(r_t,\theta_t)=(r_0,\theta_0+t\frac{2\pi}{n})$, with $t=1,\ldots,n-1$, where we have used polar coordinates and where (r_0,θ_0) are the polar coordinates of point $\mathbf{x}_0$. According to Eq. (8), if $|\psi(\mathbf{x}_0)|=0$ then $|\psi_t|=0$ for $t=1,\ldots,n-1$. Consequently, the modulus of the function vanishes in this set of points. Let us show that there is a phase singularity located in each of these points.

To do so, let us consider points $\mathbf{x}_0^\theta$ such that $\mathbf{x}_0^\theta=\mathbf{x}_0+\mathbf{x}_{(\epsilon,\theta)}$, where $|\mathbf{x}_{(\epsilon,\theta)}|=\epsilon$ with ϵ as an infinitesimally small quantity and $\angle\mathbf{x}_{(\epsilon,\theta)}=\theta\in[0,2\pi]$. Taking into account the properties of a phase singularity, the phase of the function is increased or decreased in an integer multiple of 2π along the circle ξ_0 obtained after fixing ϵ and varying θ from 0 to 2π. Now, let us apply the rotational elements of the group $C_n^t=e^{it(2\pi/n)}$, with $t=1,\ldots,n-1$, to the function in all the points in a circle such as ξ_0. The transformed functions are the function ψ evaluated at points $\mathbf{x}_t^\theta=\mathbf{x}_t+\mathbf{x}_{(\epsilon,\theta_t)}$ where $\theta_t=\theta+t\frac{2\pi}{n}$ of circles ξ_t. Taking into account Eq. (8), the phase of the transformed function at each point of the circle ξ_t and of the function itself at each point of the circle ξ_0 differs only in a fixed number α_t, which is the same for all the points at ξ_t. Consequently, the phase of the function still increases or decreases in an integer multiple of 2π along the circles ξ_t for any ϵ.

Then, on one hand, the function vanishes at points $\mathbf{x}_t$ and, on the other hand, the phase of the function is increased or decreased an integer multiple of 2π along any circle surrounding these points. This proves that if there exist an off-axis phase singularity, there are other $n-1$ off-axis phase singularities distributed symmetrically with respect to the rotation axis.

Then, one can rewrite Eq. (7) as

$$\gamma = m + \sum_{k=1}^{K} n v_k, \tag{9}$$

where v_k is the topological charge of each of the n phase singularities related by the symmetry conditions and K is the total number of these *rings* of singularities.

III. NUMERICAL RESULTS

Let us illustrate the previous results with numerical examples of optical DV with more than one singularity in photonic crystal fibers showing $\mathcal{C}_{6v}$ discrete symmetry. In Refs. [11,33], DVs with just one singularity in this kind of systems have been introduced. Additionally, the cutoff theorem for discrete-symmetry media introduced in Ref. [34] states that the only allowed values for the angular pseudomomentum m for a system with discrete symmetry of order $n=6$ are $m=\pm 1$ or $m=\pm 2$. In Figs. 1 and 2 we present two stationary solutions of Eq. (1) with self-focusing nonlinearity $L_{NL}(|\psi|)=|\psi|^2$ and $\mu=-0.08$. These solutions have been cal-

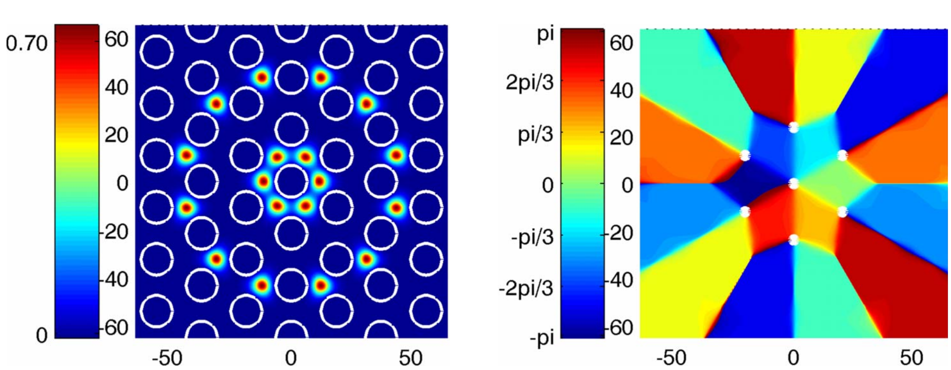

FIG. 2. (Color online) (a) Amplitude and (b) phase of a DV with angular pseudomomentum $m=-2$, $\mu=-0.08$, and more than one singularity. White circles in (a) represent the air holes in the photonic crystal fiber. White circles in (b) represent the positions of the singularities.

culated with the conventional finite-difference Newton-Raphson method using a phase-engineered seed function that presents the corresponding angular pseudomomentum and a distribution of off-axis singularities that fulfills the symmetry conditions. According with the transformations of the $\mathcal{C}_{6v}$ discrete-symmetry group, these solutions show angular pseudomomenta $m=-1$ and $m=-2$, respectively. The numerical calculations of the winding number along a closed curve near the boundary of the numerical solutions are $\gamma=5$ and $\gamma=4$, respectively. Then, in accordance with Eq. (7) there are $V=6$ off-axis singularities in each case and an undetermined number of vortex-antivortex pairs. According to Eq. (9), there is at least one ring of off-axis singularities. A thorough analysis of the phase of both stationary solutions shows, first, that there is a singularity in the symmetry axis. The calculation of the topological charge of this singularity is $v=-1$ and $v=-2$, for each case, and therefore equal to the corresponding angular pseudomomentum. On the other hand, there are a limited number of points where the off-axis singularities can be located, i.e., those where the phase seem to be undetermined. The calculation of the topological charge around these points shows that the singularities are located in the points marked with white circles in Figs. 1(b) and 2(b) and presents topological charge $v=+1$. Therefore, Eq. (7) is fulfilled, since $\gamma=5=-1+6$ for DV with $m=-1$ and $\gamma=4=-2+6$ for DV with $m=-2$. Finally, it is easy to check for both solutions that the positions of these off-axis singularities are related according to the transformations of the discrete group.

IV. CONCLUSIONS

In conclusion, we have obtained analytically the behavior of a DV near the symmetry axis. With this result, we have been able to establish a general relationship among angular pseudomomentum, winding number, and topological charge for DV. This rule permits one to study DV with more than one singularity in any discrete-symmetry media. Additionally, we have shown that the positions of the off-axis singularities are related according to symmetry arguments. Finally, the results have been illustrated with two numerical examples of high-charged discrete vortices in a system with $\mathcal{C}_{6v}$ discrete symmetry.

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