McShane-Whitney extensions for fuzzy Lipschitz maps

E. Jiménez Fernández

Departamento de Economía
Universitat Jaume I, Campus del Riu Sec, s/n
12071 Castelló de la Plana. Spain

J. Rodríguez López

Instituto Universitario de Matemática Pura y Aplicada
Universitat Politècnica de València, Camino de Vera s/n
46022 Valencia. Spain

E. A. Sánchez-Pérez

**Instituto Universitario de Matemática Pura y Aplicada*
Universitat Politècnica de València, Camino de Vera s/n
46022 Valencia. Spain.
Tel.0034-963877660, Fax. 0034-963877669

Abstract

We present a McShane-Whitney extension theorem for real-valued fuzzy Lipschitz maps defined between fuzzy metric spaces. Motivated by the potential applications of the obtained results, we generalize the mathematical theory of extensions of Lipschitz functions to the fuzzy context. We develop the problem in its full generality, explaining the similarities and differences with the classical case of extensions on metric spaces.

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*Corresponding author

Email addresses: jimeneze@uji.es (E. Jiménez Fernández), jrlopez@mat.upv.es (J. Rodríguez López), easancpe@mat.upv.es (E. A. Sánchez-Pérez)

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1. Introduction and basic definitions

A classic and simple model for many machine learning developments is based on controlled extensions of real valued functions that act in subspaces of metric spaces. For example, suppose we have a metric model for a given problem —i.e., the individuals in the model are elements of a subspace (S_0, d) of a metric space (X, d)—, and assume that the learning tool is a real function I. A basic reinforcement learning scheme is then provided by an increasing subset of metric subspaces $S_0 \subseteq S_1 \subseteq S_2 \cdots \subseteq X$ and the corresponding extensions of the index $I_0, I_1, I_2, \cdots$, which improve as the process progresses, since we incorporate new information about the system at each step. Similar arguments provide standard tools in distance learning (see for example [5]). This and other issues on the relation among machine learning and Lipschitz functions are of current interest; the reader can find concrete information about in [1, 5, 6, 16, 28, 21] and the references therein.

On the other hand, nowadays it seems to be an objective of the scientific community to enrich the set of mathematical tools for machine learning by introducing increasingly sophisticated fuzzy methods. The nature of the topic itself and the multiple applications of the different areas of the machine learning motivated an early introduction of techniques of fuzzy mathematics in these areas, both to justify theoretical developments and for concrete applications (see for example [15, 31]).

Motivated in part by this situation, the aim of this paper is to unify both theoretical and practical contexts to produce new useful tools in machine learning. In particular, our aim is to adapt a central theorem of the theory of Lipschitz functions —the McShane-Whitney extension theorem for real valued Lipschitz functions— to the framework of fuzzy metric spaces.

Let us recall some fundamental definitions. A function $f : (X, d) \to (Y, q)$ between metric spaces is said to be Lipschitz if there is a constant $K > 0$ such that

$$q(f(x), f(y)) \leq K d(x, y), \quad x, y \in X.$$

The Lipschitz constant of f is the infimum of all the constants K satisfying the inequality. It is well known that Lipschitz real functions acting in metric subspaces can always be extended to the whole metric space preserving the Lipschitz constant. Indeed, the so called McShane-Whitney Theorem establishes that if S is a subspace of a metric space (X, d) and $f : S \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant K, there exists an extension to X with the same Lipschitz constant. In fact, a lot of extensions are available.

For example, the following function

$$f^M(x) := \sup_{s\in S}\{f(s) - K\, d(s,x)\}, \quad x \in X,$$

that is called the McShane extension of f, gives one of them, and also the Whitney extension defined as

$$f^W(x) := \inf_{s\in S}\{f(s) + K\, d(s,x)\}, \quad x \in X.$$

The aim of the present paper is to extend this result to the setting of real-valued fuzzy Lipschitz functions $f : (X, M, *) \to (\mathbb{R}, N, \circledast)$ between fuzzy metric spaces as introduced in [32]: f is fuzzy Lipschitz if for every $t > 0$, the supremum

$$\sup_{x\neq y} \frac{1 - N(f(x), f(y), t)}{1 - M(x, y, t)}$$

is finite. However, in order to extend the result in a coherent way, we will first consider functions $f : X \times (0, \infty) \to \mathbb{R}$, that is, we will consider explicitly the dependence of the original map on the parameter t. Our main theoretical result —Corollary 43— gives an extension theorem for fuzzy Lipschitz maps. If $(X, M, *)$ is a fuzzy metric space and $(\mathbb{R}, N, \circledast)$ is what we call a Euclidean fuzzy metric space, suppose $S \subseteq X$ and $f : (S, M, *) \times (0, \infty) \to (\mathbb{R}, N, \circledast)$ is a fuzzy Lipschitz map. Then, under some requirements on M, f can be extended to a fuzzy Lipschitz map $f : X \times (0, \infty) \to \mathbb{R}$.

Although this result seems to be the most adequate generalization after understanding the nature of fuzzy Lipschitz maps, from the point of view of the applications seems to be relevant the case of extending fuzzy Lipschitz functions non-depending on t acting in S to fuzzy Lipschitz functions non-depending on t acting in X. We will show that this is also possible using our construction.

We intend in this way to provide new tools for opening the door to a new methodological approach to machine learning, including fuzzy notions in the design of algorithms of artificial intelligence. Following the principles underlying the fuzzy philosophy, our technique allows us to modulate, using the "t" parameter, the level of uncertainty under which the function fulfills the Lipschitz inequality. This idea is automatically translated into the context of machine learning, thus constructing in future research work a technique that allows probabilistic interpretations to be introduced into the forecasts methods based on reinforcement learning.

The structure of the paper is as follows. We begin recalling in Section 2 some necessary concepts about fuzzy metric spaces. Our technique for extending a fuzzy Lipschitz map is based on a method for constructing a family of metrics from a fuzzy metric. The problem of obtaining a compatible metric from a fuzzy metric has already treated in the literature [3, 9, 23, 24]. We summarize some results in Section 3 and provide a new method in Lemma 18. Section 4 is devoted to introduce the concept of fuzzy Lipschitz function as considered in [32]. For completeness, we provide in this section some results about the relationship of this concept with other notions of Lipschitzness in the fuzzy framework coming from fuzzy contractiveness notions. Finally, in Section 5 we obtain a McShane-Whitney extension theorem for fuzzy Lipschitz mappings taking values in what we call a Euclidean fuzzy metric space.

2. Fuzzy metric spaces

The origins of fuzzy metric spaces are due mainly to Menger [18] (see also [26]) who introduced the concept of probabilistic metric space which gives a probabilistic interpretation of the distance between two points by assigning a distribution function with every pair of elements. This concept has evolved in the last decades to various concepts of fuzzy metrics. One of the most widespread notion is that due to George and Veeramani [7] (see also [8]) and this will be the notion that we will work with. Let us recall its definition.

A function $* : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-norm if it satisfies the following conditions:

(1) $*$ is associative and commutative;

(2) $*$ is continuous;

(3) $a * 1 = a$ for all $a \in [0,1]$, and

(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for $a, b, c, d \in [0,1]$.

Example 1. *Canonical examples of t-norms are:*

- $a *_P b = a \cdot b$.
- $a *_{\min} b = \min\{a, b\}$.
- $a *_L b = \max\{a + b - 1, 0\}$.

It is also well-known and easy to see that $* \leq *_{\min}$ *for every t-norm* $*$.

Definition 2 ([7]). *A triple* $(X, M, *)$ *is called a fuzzy metric space if* X *is a nonempty set,* $*$ *is a continuous t-norm, and* M *is a fuzzy set on* $X^2 \times (0, \infty)$ *such that for each* $x, y, z \in X$ *and* $t, s > 0$,

(1) $M(x, y, t) > 0$,

(2) $M(x, y, t) = 1$ *if and only if* $x = y$,

(3) $M(x, y, t) = M(y, x, t)$,

(4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$, *and*

(5) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$, *is continuous.*

The pair $(M, *)$ *is said to be a fuzzy metric on* X.

Example 3 ([7]). *Given a metric space* (X, d), *let* M_d *be the fuzzy set on* $X^2 \times (0, \infty)$ *defined by*

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

For every continuous t-norm $*$, $(M_d, *)$ *is a fuzzy metric on* X *which is called the standard fuzzy metric induced by* d.

Observe that if $(M, *)$ is a fuzzy metric on X then, by property (4) of a fuzzy metric, the function $M(x, y, \cdot)$ is increasing for every $x, y \in X$.

Remark 4. *George and Veeramani [7] showed that every fuzzy metric space generates a Hausdorff topology* $\tau(M)$ *on* X *having as base the family of open balls* $\{B(x, r, t) : x \in X, r > 0, t > 0\}$ *where*

$$B(x, r, t) := \{y \in X : M(x, y, t) > 1 - r\}.$$

Furthermore, in [11] the authors proved that $(X, \tau(M))$ *is metrizable since it possesses a compatible uniformity* $\mathcal{U}_M$ *with a countable base formed by all sets*

$$U_n := \{(x, y) \in X \times X : M(x, y, 1/n) > 1 - 1/n\},$$

where $n \in \mathbb{N}$.

The following class of fuzzy metric spaces was introduced in [12] when the authors were studying completability of fuzzy metric spaces.

Definition 5 ([12]). *A fuzzy metric space $(X, M, *)$ is said to be stationary (or $(M, *)$ is a stationary fuzzy metric on X) if the function $M(x, y, \cdot)$ is constant for every $x, y \in X$.*

Example 6 ([10]). *The pair $(M, \cdot)$ where $M : \mathbb{N} \times \mathbb{N} \times (0, +\infty) \to [0, 1]$ is given by*

$$M(x, y, t) = \frac{\min\{x, y\}}{\max\{x, y\}}$$

for all $x, y \in \mathbb{N}$ and all $t > 0$, is a stationary fuzzy metric on $\mathbb{N}$.

We can extend the previous concept in the following way.

Definition 7. *A fuzzy metric space $(X, M, *)$ is said to be eventually stationary (or $(M, *)$ is a eventually stationary fuzzy metric on X) if we can find $t_0 > 0$ such that the function $M(x, y, \cdot) : [t_0, +\infty) \to [0, 1]$ is constant for every $x, y \in X$.*

Example 8 ([10, Example 16]). *The pair $(M, \cdot)$ where $M : \mathbb{N} \times \mathbb{N} \times (0, +\infty) \to [0, 1]$ is given by*

$$M(x, y, t) = \begin{cases} 1 & \text{if } x = y \\ \dfrac{\min\{x, y\}}{\max\{x, y\}} \cdot t & \text{if } x \neq y, t < 1 \\ \dfrac{\min\{x, y\}}{\max\{x, y\}} & \text{if } x \neq y, t \geq 1 \end{cases}$$

for all $x, y \in \mathbb{N}$ and all $t > 0$, is a eventually stationary fuzzy metric on $\mathbb{N}$.

Another class of fuzzy metric spaces which includes the stationary fuzzy metric spaces are the so-called strong fuzzy metric spaces introduced and studied in [9].

Definition 9 ([9]). *A fuzzy metric space $(X, M, *)$ is said to be strong if*

$$M(x, y, t) * M(y, z, t) \leq M(x, z, t)$$

for all $x, y, z \in X$ and all $t > 0$.

It is obvious that every stationary fuzzy metric space is strong. Furthermore, $(M,*)$ is strong if and only if $\{(M_t,*) : t>0\}$ is a family of stationary fuzzy metrics on X associated to M, where $M_t : X\times X\times(0,+\infty)\to[0,1]$ is given by $M_t(x,y,s)=M(x,y,t)$, for all $x,y\in X$ and all $s>0$.

Example 10 ([14])**.** *If (X,d) is a metric space and $*$ is a t-norm such that $*\leq\cdot$ then $(M_d,*)$ is a strong fuzzy metric.*

3. Metrics from fuzzy metrics

Remark 4 naturally leads to the problem of constructing in an easy way a metric d on a fuzzy metric space $(X,M,*)$ compatible with the topology $\tau(M)$ such that we can infer results for the fuzzy metric $(M,*)$ from classic results about metrics. In this way Radu [23] constructed such a metric d which has been successfully applied to prove fixed point theorems for complete fuzzy metric spaces from classic results in the context of metric spaces. His construction is as follows:

Theorem 11 ([23])**.** *Let $(X,M,*)$ be a fuzzy metric space such that $*\geq *_L$. The function $d_R : X\times X\to[0+\infty)$ given by*

$$d_R(x,y):=\sup\{t\geq 0 : M(x,y,t)\leq 1-t\}$$

is a metric on X which also satisfies

$$d_R(x,y)<\varepsilon \Leftrightarrow M(x,y,\varepsilon)>1-\varepsilon$$

for all $\varepsilon\in(0,1)$. Hence the uniformities and topologies generated by $(M,*)$ and d_R coincide on X. Consequently, $(X,M,*)$ is complete if and only if (X,d_R) is complete.

Example 12. *Let (X,d) be a metric space and let us consider the standard fuzzy metric $(M_d,\cdot)$ on X. Then*

$$\begin{aligned}
d_R(x,y) &= \sup\{t\geq 0 : M_d(x,y,t)\leq 1-t\} = \sup\left\{t\geq 0 : \frac{t}{t+d(x,y)}\leq 1-t\right\}\\
&= \sup\left\{t\geq 0 : 0\leq 1-t-\frac{t}{t+d(x,y)}\right\}\\
&= \sup\left\{t\geq 0 : 0\leq \frac{-t^2-td(x,y)+d(x,y)}{t+d(x,y)}\right\}\\
&= \sup\left\{t\geq 0 : 0\leq -t^2-td(x,y)+d(x,y)\right\} = \frac{1}{2}\left(d(x,y)+\sqrt{d(x,y)^2+4d(x,y)}\right)
\end{aligned}$$

for all $x, y \in X$.

The construction of Theorem 11 was improved and modified in [3, 24] which allowed the authors to prove several fixed point theorems for different types of contractions in the context of fuzzy metric spaces. Their main tool is the construction of a metric from a fuzzy metric such that it preserves a certain contraction notion.

In a different way, Gregori, Morillas and Sapena [9] developed a method for constructing a metric from a strong fuzzy metric as follows.

Proposition 13 ([9])**.** *Let $(X, M, *)$ be a strong fuzzy metric space such that $* \geq *_L$. Let $\{M_t : t > 0\}$ be the family of stationary fuzzy metrics associated to $(M, *)$. Then*

(i) $\{d_t^M : t > 0\}$ is a family of metrics on X where $d_t^M(x, y) = 1 - M(x, y, t)$ for all $x, y \in X$ and all $t > 0$.

(ii) $d = \sup_{t>0} d_t^M$ is a metric on X such that $\tau(M_t) \subseteq \tau(d)$.

Remark 14. *Observe that part (i) of the above proposition is a particular case of Theorem 11. In fact, if $(X, M, *)$ is a strong fuzzy metric space then $(M_t, *)$ is a stationary fuzzy metric for all $t > 0$. In this case, the metric d_R^t associated with M_t is given by*

$$\begin{aligned} d_R^t(x, y) &= \sup\{s \geq 0 : M_t(x, y, s) \leq 1 - s\} = \sup\{s \geq 0 : M(x, y, t) \leq 1 - s\} \\ &= \sup\{s \geq 0 : s \leq 1 - M(x, y, t)\} = 1 - M(x, y, t) = d_t^M(x, y) \end{aligned}$$

for all $x, y \in X$.

Furthermore, part (i) of the previous result has been extended in [22, Corollary 4.4], where the authors present another method for constructing a metric from a fuzzy metric based on additive generators of t-norms.

Example 15 ([9, Example 25])**.** *Let (X, d) be a metric space and let us consider the fuzzy metric $(M, \cdot)$ on X given by*

$$M(x, y, t) := e^{(-d(x,y)/t)}, \quad x, y \in X, \; t > 0.$$

It is easy to see that $(M, \cdot)$ is strong. By the above result $\{d_t^M : t > 0\}$ is a family of metrics on X where

$$d_t^M(x, y) = 1 - M(x, y, t) = 1 - e^{(-d(x,y)/t)}$$

for all $x, y \in X$.

Example 16. *Let (X, d) be a metric space. As we have commented previously, if $*$ is a t-norm such that $* \leq \cdot$ then $(M_d, *)$ is a strong fuzzy metric on X [14]. Then, given $t > 0$ the function $d_t^M : X \times X \to [0, +\infty)$ given by*

$$d_t^M(x, y) = 1 - M_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}$$

for all $x, y \in X$ is a metric on X equivalent to d.

We notice that part (i) of Proposition 13 provides the following characterization of strong fuzzy metrics with respect to the Łukasiewicz t-norm.

Proposition 17. *A fuzzy metric space $(X, M, *_L)$ is strong if and only if $\{d_t^M : t > 0\}$ is a family of metrics on X where $d_t^M(x, y) = 1 - M(x, y, t)$ for all $x, y \in X$ and all $t > 0$.*

Proof. Necessity follows from Proposition 13.

For the converse, fix $t > 0$. Then given $x, y, z \in X$ we have that

$$\begin{aligned} d_t^M(x, y) + d_t^M(y, z) &\geq d_t^M(x, z) \\ 1 - M(x, y, t) + 1 - M(y, z, t) &\geq 1 - M(x, z, t) \\ 1 - M(x, y, t) - M(y, z, t) &\geq -M(x, z, t) \\ M(x, y, t) + M(y, z, t) - 1 &\leq M(x, z, t) \\ \max\{M(x, y, t) + M(y, z, t) - 1, 0\} &\leq M(x, z, t) \\ M(x, y, t) *_{\text{L}} M(y, z, t) &\leq M(x, z, t). \end{aligned}$$

Since t is arbitrary then $(M, *_{\text{L}})$ is strong. □

Observe that the method provided by Proposition 13 of constructing a metric d from a fuzzy metric $(M, *)$ is only valid when $(M, *)$ is strong. Moreover d does not preserve important properties of M since, for example, $\tau(d) \neq \tau(M)$ in general.

Next we present a new method for constructing a family of metrics from a fuzzy metric which is better behaved for our purposes. This method is based on an standard "convexification" process inspired by classical metrization theorems [30, Theorem 23.4] and the obtention of a subadditive function by means of the inf-convolution [4] (it is a particular case of the metrics d_ε considered in [17] when $\varepsilon = \infty$).

Lemma 18. *Let $(X, M, *)$ be a fuzzy metric space, and let $\varphi : [0,1) \to [0,1)$ be an increasing function such that $\varphi^{-1}(0) = \{0\}$. Fix $t > 0$ and consider the function $p_t : X \times X \to \mathbb{R}^+$ defined by*

$$p_t(x, y) := \inf \left\{ \sum_{i=1}^{n} \varphi\big(1 - M(x_i, x_{i+1}, t)\big) \ : \ x_1 = x,\, x_{n+1} = y,\ x_i \in X \right\}$$

for every $x, y \in X$. Then

(i) p_t is a pseudo metric on X, and

(ii) if $(x, y) \mapsto \varphi\big(1 - M(x, y, t)\big)$ satisfies the triangular inequality, then

$$p_t(\cdot, \cdot) = \varphi\big(1 - M(\cdot, \cdot, t)\big),$$

and it defines a metric.

Proof. (i) The function is clearly symmetric, due to the symmetry of M. On the other hand, a direct calculation using the properties of the infimum gives the triangular inequality.

(ii) Note that, if $\varphi\big(1 - M(\cdot, \cdot, t)\big)$ satisfies the triangular inequality then it is a pseudo metric. Moreover given $x, y \in X$ we have that for every $x_1, \ldots, x_{n+1} \in X$ such that $x_1 = x$, $x_{n+1} = y$ then

$$\varphi\big(1 - M(x, y, t)\big) \leq \sum_{i=1}^{n} \varphi\big(1 - M(x_i, x_{i+1}, t)\big),$$

and so the infimum $p_t(x, y)$ coincides with $\varphi\big(1 - M(x, y, t)\big)$. On the other hand, we have that $\varphi\big(1 - M(x, y, t)\big) = 0$ if and only if $1 - M(x, y, t) = 0$, and by the definition of fuzzy metric this happens if and only if $x = y$. This gives (ii). □

In the literature we can find a lot of examples of fuzzy metrics but many of them are constructed starting from a classic metric [10, 25]. We next show that the above Lemma allows to recover the metric from the fuzzy metric in some cases.

Example 19 (cf. [9, Example 25]). *Let (X, d) be a metric space and let us consider, following [10, Example 5], the fuzzy metric $(M, \cdot)$ on X given by*

$$M(x, y, t) := e^{(-d(x,y)/g(t))}, \quad x, y \in X, \ t > 0,$$

where $g : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing continuous function.

Let us consider $\varphi : [0, 1) \to [0, 1)$ given by $\varphi(x) = -\log(1 - x)$. It is obvious that $\varphi^{-1}(0) = \{0\}$ and that φ is subadditive since it is concave. Following Lemma 18, given $t > 0$ we can construct the following metric p_t on X :

$$p_t(x, y) := \varphi(1 - M(x, y, t)) = -\log(1 - 1 + e^{-d(x,y)/g(t)}) = \frac{d(x, y)}{g(t)}.$$

Example 20. *If (X, d) is a metric space and $n \in \mathbb{N}$ the pair $(M, \wedge)$ where*

$$M(x, y, t) = \frac{t^n}{t^n + d(x, y)}, \quad x, y \in X, t > 0$$

is a fuzzy metric on X [10, Example 4].

Consider the function $\varphi : [0, 1) \to [0, 1)$ given by $\varphi(x) = \frac{x}{1-x}$. Then $\varphi^{-1}(0) = \{0\}$ and φ is subadditive. Following Lemma 18, given $t > 0$ we can construct the following metric p_t on X :

$$p_t(x, y) := \varphi(1 - M(x, y, t)) = \frac{1 - M(x, y, t)}{1 - 1 + M(x, y, t)} = \frac{1 - \frac{t^n}{t^n+d(x,y)}}{\frac{t^n}{t^n+d(x,y)}} = \frac{d(x, y)}{t^n}.$$

Notice that $(M, \wedge)$ is the standard fuzzy metric associated with the metric $\frac{d(x,y)}{t^{n-1}}$.

Observe that if $(X, M, *)$ is a strong fuzzy metric space, then by Proposition 13 $\{d_t^M : t > 0\}$ is a family of metrics on X. Thus, if φ is a metric preserving function [4] then $\varphi(1-M(\cdot, \cdot, t))$ is also a metric on X for all $t > 0$ so we obtain (ii) of the previous result. We next present an example borrowed from [14], where Lemma 18 can be applied but Proposition 13 cannot.

Example 21 ([14, Example 2]). *Let $X = \{a, b, c\}$ and $M : X \times X \times (0, +\infty) \to [0, 1]$ given by*

$$M(x, y, t) = M(y, x, t) = \begin{cases} 1 & \text{if } x = y \\ \dfrac{2t + 1}{2t + 2} & \text{if } x = a \text{ or } x = b \text{ and } y = c \\ \dfrac{t}{t + 2} & \text{if } x = a, y = b \end{cases}.$$

Then it was proved in [14] that $(M, *_L)$ *is a fuzzy metric on* X *which is not strong. Notice that by Proposition 17, there has to be* $t > 0$ *such that* d_t^M *is not a metric. In fact,* d_t^M *is not a metric for every* $t > 0$, *since*

$$d_t^M(a,b) = 1-M(a,b,t) = \frac{2}{t+2} \not\leq d_t^M(a,b)+d_t^M(b,c) = 1-M(a,b,t)+1-M(b,c,t) = \frac{2}{2t+2}.$$

It is easy to check that if we consider the function $\varphi : [0,1) \to [0,1)$ *given by* $\varphi(x) = x$ *then Lemma 18 provides*

$$p_t(x,y) = \begin{cases} \dfrac{2}{2t+2} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

for every $t > 0$.

It is straightforward to check that in this case

$$d_R(x,y) = d_R(y,x) = \begin{cases} 0 & \text{if } x = y \\ -1+\sqrt{3} & \text{if } x = a \text{ and } y = b \\ \dfrac{-1+\sqrt{3}}{2} & \text{if } x = a \text{ or } x = b \text{ and } y = c \end{cases}.$$

Consequently, there is no relationship between d_R *and the family of metrics* $\{p_t : t > 0\}$.

4. Fuzzy Lipschitz maps

In this section, we analyze some of the different notions of fuzzy Lipschitz maps that have been proposed in the mathematical literature. In [32] the authors introduced the following definition of fuzzy Lipschitz map between fuzzy metric spaces. We will consider this notion as a benchmark for the other concepts that will be introduced.

Definition 22 ([32])**.** *Let* $(X, M, *)$ *and* $(Y, N, \circledast)$ *be two fuzzy metric spaces. A map* $f : (X, M, *) \to (Y, N, \circledast)$ *is said to be* fuzzy Lipschitz *if*

$$dil(f,t) := \sup_{x \neq y} \frac{1 - N(f(x), f(y), t)}{1 - M(x, y, t)} < \infty$$

for each $t > 0$. *The number* $dil(f,t)$ *is called the* t-dilation *of* f.

If $dil(f,t) < 1$ for every $t > 0$ then we say that f is fuzzy contractive. *The corresponding notions fuzzy expansive and fuzzy nonexpansive maps are defined when $dil(f,t) > 1$ and $dil(f,t) = 1$ for every $t > 0$, respectively.*

Moreover f is said to be stationary fuzzy Lipschitz *if*

$$dil(f) = \sup_{t>0} dil(f,t) < \infty.$$

Example 23. *Let us consider $X = [0,1]$ and the fuzzy metrics $(M, *_L)$, $(N, \cdot)$ on X given by*

$$M(x,y,t) = e^{-\frac{|x-y|}{t}}$$
$$N(x,y,t) = 1 - |x-y|$$

for all $x, y \in X$ and all $t > 0$. Then the map $f : (X, M, *_L) \to (X, N, \cdot)$ given by $f(x) = x$ is fuzzy Lipschitz. In fact, we have that

$$dil(f,t) = \sup_{x \neq y} \frac{1 - N(f(x), f(y), t)}{1 - M(x,y,t)} = \sup_{x \neq y} \frac{|x-y|}{1 - e^{-\frac{|x-y|}{t}}} = \frac{1}{1 - e^{-\frac{1}{t}}} < \infty.$$

Yun, Hwang and Chang also proved in [32] that a fuzzy Lipschitz map is always continuous, and they studied the relationship of fuzzy Lipschitz maps with fuzzy contractive self-maps as defined by Gregori and Sapena [13]. For example, they proved [32, Corollary 18] that every GS-fuzzy contractive self-maps is fuzzy nonexpansive or fuzzy contractive. For completeness, we study more in deep the relationship between the concept of fuzzy Lipschitz map and other concepts of Lipschitz maps between fuzzy metric spaces derived from some notions of fuzzy contractive functions like that of Gregori and Sapena.

Definition 24 (cf. [13, Definition 3.5])**.** *Let $(X, M, *)$ and $(Y, N, \circledast)$ be two fuzzy metric spaces. A map $f : X \to Y$ is said to be* GS-fuzzy Lipschitz *with constant $k > 0$ if*

$$\frac{1}{N(f(x), f(y), t)} - 1 \leq k \left(\frac{1}{M(x,y,t)} - 1 \right)$$

for each $x, y \in X$ and $t > 0$.

In this case we say that f is

- GS-fuzzy contractive *if* $0 < k < 1$;

- GS-fuzzy expansive *if* $k > 1$;
- GS-fuzzy nonexpansive *if* $k = 1$.

Remark 25. *Notice that if $f : X \to Y$ is GS-fuzzy contractive then it is GS-fuzzy nonexpansive which also implies that it is GS-fuzzy expansive.*

The following proposition, whose proof is straightforward, appears in [13] in case of GS-fuzzy contractive self-maps.

Proposition 26. *A map $f : (X, d) \to (Y, q)$ between two metric spaces is Lipschitz with constant $k > 0$ if and only if $f : (X, M_d, *) \to (X, M_q, \circledast)$ is GS-fuzzy Lipschitz with the same constant k for any t-norms $*, \circledast$.*

Remark 27. *Miheţ [19] considered GS-fuzzy contractive functions in the context of Kramosil and Michalek fuzzy metric spaces by rewriting the above contractive condition as*

$$N(f(x), f(y), t) \geq \frac{M(x, y, t)}{M(x, y, t) + k(1 - M(x, y, t))}.$$

Furthermore, this class of fuzzy contractive functions has been extended by Wardowski in [29].

Another notion of contractivity for functions between fuzzy metric spaces was considered in [27]. Again, we can consider its natural associated notion of fuzzy Lipschitz function by modifying the range to which the constant k belongs.

Definition 28 (cf. [27])**.** *Let $(X, M, *)$ and $(Y, N, \circledast)$ be two fuzzy metric spaces. A map $f : X \to Y$ is said to be* SBR-fuzzy Lipschitz *with constant $k > 0$ if*

$$N(f(x), f(y), kt) \geq M(x, y, t)$$

for each $x, y \in X$ and $t > 0$.

The corresponding notions of SBR-fuzzy contractive, SBR-fuzzy expansive and SBR-fuzzy nonexpansive maps are defined in the obvious manner.

Remark 29. *Since $N(x, y, \cdot)$ is increasing, if $f : X \to Y$ is SBR-fuzzy contractive then it is SBR-fuzzy nonexpansive which also implies that it is SBR-fuzzy expansive.*

The following results establish some relationships between the concepts of GS-fuzzy Lipschitz and SBR-fuzzy Lipschitz map.

Proposition 30. *Let $f : (X, M, *) \to (Y, N, \circledast)$ be a map between two fuzzy metric spaces.*

(i) If f is GS-fuzzy contractive then it is SBR-fuzzy nonexpansive.

(ii) If f is SBR-fuzzy contractive then it is GS-fuzzy nonexpansive.

(iii) f is GS-fuzzy nonexpansive if and only if f is SBR-fuzzy nonexpansive.

(iv) f is SBR-fuzzy nonexpansive (equivalently, GS-fuzzy nonexpansive) if and only if it is fuzzy contractive or fuzzy nonexpansive.

Proof. (i) Let $0 < k < 1$ such that

$$N(f(x), f(y), t) \geq \frac{M(x, y, t)}{M(x, y, t) + k(1 - M(x, y, t))}$$

for each $x, y \in X$ and $t > 0$. Given any $k' \geq 1$ we have that

$$\begin{aligned} N(f(x), f(y), k't) \geq N(f(x), f(y), t) &\geq \frac{M(x, y, t)}{M(x, y, t) + k(1 - M(x, y, t))} \\ &> \frac{M(x, y, t)}{M(x, y, t) + (1 - M(x, y, t))} = M(x, y, t), \end{aligned}$$

so f is SBR-fuzzy nonexpansive.

(ii) Let $0 < k < 1$ such that $N(f(x), f(y), kt) \geq M(x, y, t)$ for all $x, y \in X$ and $t > 0$. Given any $k' \geq 1$ we have that

$$N(f(x), f(y), t) \geq N(f(x), f(y), kt) \geq M(x, y, t) \geq \frac{M(x, y, t)}{M(x, y, t) + k'(1 - M(x, y, t))}.$$

(iii) This is obvious since if $k = 1$ we have that

$$N(f(x), f(y), kt) = N(f(x), f(y), t)$$

and

$$M(x, y, t) = \frac{M(x, y, t)}{M(x, y, t) + k(1 - M(x, y, t))}.$$

(iv) The statement follows from the next equivalences.

$$\begin{aligned} N(f(x), f(y), t) \geq M(x, y, t) &\qquad \text{for all } x, y \in X \text{ and } t > 0, \Leftrightarrow \\ 1 - N(f(x), f(y), t) \leq 1 - M(x, y, t) &\qquad \text{for all } x, y \in X \text{ and } t > 0, \Leftrightarrow \\ \frac{1 - N(f(x), f(y), t)}{1 - M(x, y, t)} \leq 1 &\qquad \text{for all distinct } x, y \in X \text{ and } t > 0, \end{aligned}$$

which finally gives $dil(f, t) \leq 1$ for all $t > 0$.

□

Remark 31. *Notice that by the above proof, if f is GS-fuzzy contractive then f is an Edelstein mapping [20], i.e.*

$$N(f(x), f(y), t) > M(x, y, t)$$

for all $x, y \in X$ and all $t > 0$ (see [32, Lemma 15]). We also observe that GS-fuzzy contractiveness of a self-map implies fuzzy contractiveness; this follows from [32, Lemma 16, Theorem 17].

Corollary 32. *Let $f : (X, M, *) \to (Y, N, \circledast)$ be a map between two fuzzy metric spaces such that $(N, \circledast)$ is stationary. The following statements are equivalent.*

(i) *f is SBR-fuzzy Lipschitz;*
(ii) *f is SBR-fuzzy nonexpansive;*
(iii) *f is SBR-fuzzy contractive;*
(iv) *f is GS-fuzzy contractive;*
(v) *f is fuzzy contractive or fuzzy nonexpansive.*

Proof. Equivalence between (i), (ii) and (iii) is obvious from the definitions. (iii) implies (iv) is also clear since for any $0 < k < 1$ we have that

$$N(f(x), f(y), t) = N(f(x), f(y), kt) \geq M(x, y, t) \geq \frac{M(x, y, t)}{M(x, y, t) + k(1 - M(x, y, t))}.$$

The other implications are deduced from Proposition 30. □

Remark 33. *Notice that if f is fuzzy Lipschitz but not fuzzy contractive then f is not necessarily SBR-fuzzy Lipschitz. In fact, consider the function f of*

Example 23 which is fuzzy Lipschitz but not fuzzy contractive. Nevertheless, f is not SBR-fuzzy Lipschitz. Otherwise we would have

$$N(f(x), f(y), t) = 1 - |x-y| \geq M(x,y,t) = e^{-\frac{|x-y|}{t}}$$

for all $x, y \in X = [0,1]$ and all $t > 0$. Hence, the function $g(x,t) : [0,1] \times (0, +\infty) \to \mathbb{R}$ given by $g(x,t) = 1 - x - e^{-\frac{x}{t}}$ should be greater than 0 for every $t > 0$ and every $x \in [0,1]$. However, for example, $g'(x,1) = -1 + e^{-x} \leq 0$ for every $x \in X$. Since $g(0,1) = 0$ then $g(x,1) < 0$ for every $x \in]0,1]$, which is a contradiction.

We also notice that f is not GS-fuzzy Lipschitz.

Proposition 34. *A mapping $f : (X,d) \to (Y,q)$ between two metric spaces is Lipschitz with constant $k > 0$ if and only if $f : (X, M_d, *) \to (Y, M_q, \circledast)$ is SBR-fuzzy Lipschitz with the same constant k for any continuous t-norms $*, \circledast$.*

Proof. Fix $k > 0$. The statement of the theorem is a consequence of the following equivalences

$$\begin{aligned} M_q(f(x), f(y), kt) \geq M_d(x,y,t) &\Leftrightarrow \frac{kt}{kt + q(f(x), f(y))} \geq \frac{t}{t + d(x,y)} \\ &\Leftrightarrow \frac{kt}{t} \geq \frac{kt + q(f(x), f(y))}{t + d(x,y)} \\ &\Leftrightarrow kt + kd(x,y) \geq kt + q(f(x), f(y)) \\ &\Leftrightarrow kd(x,y) \geq q(f(x), f(y)). \end{aligned}$$

□

Next result is a direct consequence of the previous proposition and Proposition 26.

Corollary 35. *Let d, q be two metrics on a nonempty set X. A map $f : (X, M_d, *) \to (Y, M_q, \circledast)$ is GS-fuzzy Lipschitz with constant $k > 0$ if and only if f is SBR-fuzzy Lipschitz with the same constant k.*

Proposition 36 ([32, Lemma 8])**.** *If a mapping $f : (X,d) \to (Y,q)$ between two metric spaces is Lipschitz then $f : (X, M_d, *) \to (Y, M_q, \circledast)$ is fuzzy Lipschitz.*

Observe that, by Propositions 30 and 34, the converse of the previous proposition is true if $f : (X, M_d, *) \to (Y, M_q, \circledast)$ is fuzzy contractive or fuzzy nonexpansive. Unfortunately, it is not true in general as the next example shows.

Example 37. *Let us consider the real line $\mathbb{R}$, and the discrete metric d and the Euclidean metric e on $\mathbb{R}$. It is obvious that the map $f : (\mathbb{R}, d) \to (\mathbb{R}, e)$ given by $f(x) = x^2$ is not Lipschitz since $\{|x^2 - y^2| : x, y \in \mathbb{R}\}$ is not bounded. Nevertheless $f : (\mathbb{R}, M_d, \cdot) \to (\mathbb{R}, M_e, \cdot)$ is fuzzy Lipschitz since given $t > 0$ we have that*

$$dil(f,t) = \sup_{x \neq y} \frac{1 - M_e(f(x), f(y), t)}{1 - M_d(x, y, t)} = \sup_{x \neq y} \frac{\frac{e(f(x),f(y))}{t+e(f(x),f(y))}}{\frac{d(x,y)}{t+d(x,y)}}$$

$$= \sup_{x \neq y} \frac{e(f(x), f(y))}{d(x,y)} \frac{t + d(x,y)}{t + e(f(x), f(y))} = \sup_{x \neq y} \frac{|x^2 - y^2|}{t + |x^2 - y^2|}(t+1) \leq t + 1.$$

For the aim of this paper it is convenient to use a broader definition of fuzzy Lipschitz map including the dependence of the parameter t.

Definition 38. *given two fuzzy metric spaces $(X, M, *), (Y, N, \circledast)$, we will say that a map*

$$f : X \times (0, \infty) \to Y$$

is a fuzzy Lipschitz map if the "extended dilation"

$$dil(f,t) := \sup_{x \neq y} \frac{1 - N(f(x,t), f(y,t), t)}{1 - M(x, y, t)}$$

is finite for every $t > 0$.

We will show that the extension results can be applied to this class of functions: if the original fuzzy Lipschitz function f depends on t, then the extension depends on t too. But we will also prove that, under reasonable requirements, if f does not depend on t —despite being a fuzzy Lipschitz map—, then we can choose an extension which also does not.

We can write the definition of our Lipschitz-type maps in the usual terms, that is, involving inclusions of sets and inequalities as in the classical presentation of Lipschitz maps. In this way, we can say that f is fuzzy Lipschitz if and only if for each $x \in X$ and $t > 0$ there is a constant $K(t)$ such that

$$f(B^M_\varepsilon(x,t), t) \subseteq K(t)\, B^N_\varepsilon(f(x,t), t),$$

which gives an inequality as the next one for each $t > 0$,

$$1 - N(f(x,t), f(y,t), t) \leq K(t)\Big(1 - M(x,y,t)\Big), \quad x, y \in X.$$

Notice that we can take $K(t) = dil(f,t)$.

5. The McShane-Whitney extension theorem for fuzzy Lipschitz maps

In what follows we show our main result, the McShane-Whitney extension theorem in the realm of fuzzy metric spaces. To achieve this, we will consider a special type of fuzzy metrics on $\mathbb{R}$.

Definition 39. *A fuzzy metric $(M,*)$ on $\mathbb{R}$ is said to be a* Euclidean fuzzy metric *if there are functions $\phi, g : \mathbb{R}^+ \to \mathbb{R}^+$ such that ϕ is increasing and*

$$M(x,y,t) = 1 - \phi(|x-y|)g(t).$$

In this case we will denote M by $M_{\phi,g}$ and we will say that $(\mathbb{R}, M_{\phi,g}, *)$ is a Euclidean fuzzy metric space.

Example 40. *Let us consider $X = [0,1]$ and $M : X^2 \times (0,+\infty) \to [0,1]$ given by $M(x,y,t) = 1-|x-y|$ for all $x, y \in X$ and all $t > 0$. It is easy to see that $(M, *_L)$ is a Euclidean stationary fuzzy metric induced by the functions $\phi, g : \mathbb{R}^+ \to \mathbb{R}^+$ given by*

$$\phi(x) = x$$
$$g(x) = 1$$

for all $x \in X$.

Example 41 (cf. [10, Example 6])**.** *Let us consider two functions $\phi, g : \mathbb{R}^+ \to \mathbb{R}^+$ such that ϕ is strictly increasing, subadditive, bounded by $k > 0$ and $\phi^{-1}(0) = \{0\}$, meanwhile g is decreasing, bounded by $\frac{1}{k}$ and greater than 0. Then it is easy to see that $(M, *_L)$ is a Euclidean fuzzy metric on $\mathbb{R}$ where*

$$M(x,y,t) = 1 - \phi(|x-y|)g(t).$$

We only check the triangle inequality. Let $x, y, z \in \mathbb{R}$ and $t, s > 0$. Since ϕ is subadditive and strictly increasing then $\phi(|x-y|) \leq \phi(|x-y| + |y-z|) \leq \phi(|x-z|) + \phi(|z-y|)$. Using that g is decreasing we have that

$$\begin{aligned}\phi(|x-y|)g(t+s) &\leq \phi(|x-z|)g(t+s) + \phi(|z-y|)g(t+s) \\ &\leq \phi(|x-z|)g(t) + \phi(|z-y|)g(s).\end{aligned}$$

That is,

$$1-\phi(|x-y|)g(t+s)\geq 1-\phi(|x-z|)g(t)-\phi(|z-y|)g(s),$$

what gives

$$M(x,y,t+s)\geq M(x,z,t)*_L M(z,y,s).$$

Let $(X,M,*)$ be a fuzzy metric space and consider a Euclidean fuzzy metric space $(\mathbb{R},N_{\phi,g},\circledast)$. If $f:(X,M,*)\times(0,\infty)\to(\mathbb{R},N_{\phi,g},\circledast)$ is a fuzzy Lipschitz map then for each $t>0$ there is a positive real number $K(t)$ such that

$$1-N_{\phi,g}(f(x,t),f(y,t),t)\leq K(t)\Big(1-M(x,y,t)\Big)$$
$$\phi(|f(x,t)-f(y,t)|)g(t)\leq K(t)\Big(1-M(x,y,t)\Big)$$

for all $x,y\in X$. By composing with the increasing function ϕ^{-1} we obtain

$$|f(x,t)-f(y,t)|\leq\phi^{-1}\Big(\frac{K(t)}{g(t)}\big(1-M(x,y,t)\big)\Big),\quad x,y\in X.$$

Due to the fact that $|\cdot|$ is a norm on $\mathbb{R}$, we automatically get from the inequality above that for each $t>0$, the inequality

$$|f(x,t)-f(y,t)|$$
$$\leq\inf_n\Big\{\sum_{i=1}^{n}\phi^{-1}\Big(\frac{K(t)}{g(t)}\big(1-M(x_i,x_{i+1},t)\big)\Big),x_1=x,x_2,...,x_n,x_{n+1}=y,\,x_i\in X\Big\},$$

where $x,y\in S$. We will write $d_{t,f}(x,y)$ for the second term of this inequality, that is clearly a pseudometric on X for each $t>0$. Therefore, we obtain in this way a typical Lipschitz inequality for the function $f(\cdot,t):X\to\mathbb{R}$, where the spaces X and $\mathbb{R}$ are assumed to be pseudometric spaces with the explained pseudometrics.

This construction gives the proof of the following extension theorem. Note that we have to take the elements x_i in all the set X in the definition of $d_{t,f}$ and not only the ones of S.

Theorem 42. *Let $(X,M,*)$ be a fuzzy metric space and $(\mathbb{R},N_{\phi,g},\circledast)$ be a Euclidean fuzzy metric space. Let $S\subseteq X$ and suppose that $f:(S,M,*)\times(0,\infty)\to(\mathbb{R},N_{\phi,g},\circledast)$ is a fuzzy Lipschitz map. Then the following statements are equivalent.*

(i) $f(\cdot,t):(S,d_{t,f})\to\mathbb{R}$ *is nonexpansive for all* $t>0$.

(ii) The function f *can be extended as a fuzzy Lipschitz map to* $X\times(0,\infty)$.

Proof. (i) $\Rightarrow$ (ii) Fix $t>0$. By assumption, the function $f(\cdot,t):(X,d_{t,f})\to\mathbb{R}$ is Lipschitz so, for example, the McShane formula provides an extension $\hat{f}(\cdot,t)$ of $f(\cdot,t)$ to all the set X. Therefore, we have in particular that for each $x,y\in X$,

$$|\hat{f}(x,t)-\hat{f}(y,t)|\le d_{t,f}(x,y)\le \phi^{-1}\left(\frac{K(t)}{g(t)}\left(1-M(x,y,t)\right)\right),$$

and so

$$1-N_{\phi,g}(\hat{f}(x,t),\hat{f}(y,t),t)=\phi(|f(x,t)-f(y,t)|)g(t)\le K(t)\left(1-M(x,y,t)\right).$$

Since $t>0$ is arbitrary, $\hat{f}$ is a fuzzy Lipschitz map.

(ii) $\Rightarrow$ (i) Suppose that f allows an extension $\hat{f}$ to the whole X which is fuzzy Lipschitz. Then given $t>0$ we have that

$$\begin{aligned}1-N_{\phi,g}(\hat{f}(x,t),\hat{f}(y,t),t)&\le K(t)(1-M(x,y,t))\\ \phi(|\hat{f}(x,t)-\hat{f}(y,t)|)g(t)&\le K(t)\left(1-M(x,y,t)\right)\\ |\hat{f}(x,t)-\hat{f}(y,t)|&\le\phi^{-1}\left(\frac{K(t)}{g(t)}\left(1-M(x,y,t)\right)\right)\end{aligned}\tag{1}$$

for all $x,y\in X$. Fix now $x,y\in S$. Given $n\in\mathbb{N}$ and $\{x_1,\dots,x_{n+1}\}\subseteq X$ such that $x_1=x$ and $x_{n+1}=y$ we can use inequality (1) to obtain

$$|f(x,t)-f(y,t)|=|\hat{f}(x,t)-\hat{f}(y,t)|$$

$$\le\sum_{i=1}^{n}|\hat{f}(x_i,t)-\hat{f}(x_{i+1},t)|\le\sum_{i=1}^{n}\phi^{-1}\Big(\frac{K(t)}{g(t)}\big(1-M(x_i,x_{i+1},t)\big)\Big).$$

Since this holds for all $n\in\mathbb{R}$ and for all $x_i{}'s$, we obtain that for $x,y\in S$,

$$|f(x,t)-f(y,t)|\le d_{t,f}(x,y).$$

Therefore, (i) holds.

□

The following result will be the main tool for working in concrete applications.

Corollary 43. *Let $(X, M, *)$ be a fuzzy metric space and $(\mathbb{R}, N_{\phi,g}, \circledast)$ be a Euclidean fuzzy metric space. Let $S \subseteq X$ and suppose that $f : (S, M, *) \times (0, \infty) \to (\mathbb{R}, N_{\phi,g}, \circledast)$ is a fuzzy Lipschitz map with extended dilation $K(t)$. Assume also that for every $t > 0$ the map $\rho_t : X \times X \to \mathbb{R}^+$ given by*

$$\rho_{t,f}(x,y) := \phi^{-1}\left(\frac{K(t)}{g(t)}\left(1 - M(x,y,t)\right)\right), \quad x, y \in X,$$

is a metric on X. Then the function f can be extended as a fuzzy Lipschitz map to $X \times (0, \infty)$.

Proof. If the above defined map is a metric on X, then we clearly have that it coincides with $d_{t,f}$. Then we directly get (i) in Theorem 42, and the result holds. □

Remark 44. *Observe that under the hypotheses of the previous corollary, we can provide directly an extension of f by using:*

- *the following modified McShane formula*

$$f^M(x,t) = \sup_{s \in S}\left\{f(s) - \phi^{-1}\left(\frac{K(t)}{g(t)}(1 - M(x,s,t))\right)\right\}$$

  for all $x \in X$ and all $t > 0$.

- *the following modified Whitney formula*

$$f^W(x,t) = \inf_{s \in S}\left\{f(s) + \phi^{-1}\left(\frac{K(t)}{g(t)}(1 - M(x,s,t))\right)\right\}$$

  for all $x \in X$ and all $t > 0$.

Remark 45. *The main problem to define the extension of a fuzzy Lipschitz function is that the Lipschitz extension obtained by our method depends on the parameter t. However, under some assumptions, it can be proved that we can obtain an extension not depending on t. For example, let $(X, M, *), (\mathbb{R}, N_{\phi,g}, \circledast)$ be two stationary fuzzy metric spaces such that $(N_{\phi,g}, \circledast)$ is a Euclidean fuzzy metric. Given $S \subseteq X$ and a fuzzy Lipschitz function $f : (S, M, *) \to (Y, N_{\phi,g}, \circledast)$, then f is a stationary fuzzy Lipschitz* *function. Furthermore, if ϕ^{-1} is subadditive then it is easy to see [4] that the function ρ_t considered in the previous corollary is a metric on X which does not depend on t, so the function f can be extended to the whole X without depending on t (see also [2]).*

Nevertheless, we can improve a little bit this result as follows.

Proposition 46. *Let $(X, M, *)$ be a eventually stationary fuzzy metric space with $* \geq *_L$ and $(\mathbb{R}, N_{\phi,g}, \circledast)$ be a Euclidean stationary fuzzy metric space such that ϕ is strictly increasing and ϕ^{-1} is subadditive. Let $S \subseteq X$ and suppose that $f : (S, M, *) \to (\mathbb{R}, N_{\phi,g}, \circledast)$ is fuzzy Lipschitz. Then f can be extended as a fuzzy Lipschitz map to X.*

Proof. Although we could use Corollary 43 to prove this proposition, for completeness, we present a direct proof by using a suitable McShane formula.

Since $(N_{\phi,g}, \circledast)$ is stationary then g must be constant. Otherwise, if $g(t_1) \neq g(t_2)$ for some $t_1, t_2 > 0$ then $N_{\phi,g}(0, 1, t_1) = 1 - \phi(|1-0|)g(t_1) \neq 1 - \phi(|1-0|)g(t_2) = N_{\phi,g}(0, 1, t_2)$ which contradicts stationarity of $(N_{\phi,g}, \circledast)$.

Furthermore, since $(X, M, *)$ is eventually stationary we can find $t_0 > 0$ such that $M(x, y, t) = M(x, y, s)$ for all $x, y \in X$ and all $t, s \geq t_0$. By assumption we can find $K(t_0) > 0$ such that

$$1 - N_{\phi,g}(f(s), f(s'), t_0) \leq K(t_0)(1 - M(s, s', t_0))$$

for all $s, s' \in S$. Observe that $d_{t_0} : X \times X \to [0, +\infty)$ given by $d_{t_0}(x, y) = \frac{K(t_0)}{g(t_0)}(1 - M(x, y, t_0))$ is a metric on X.

Consider $\hat{f} : X \to \mathbb{R}$ defined by the following modified McShane formula:

$$\hat{f}(x) = \sup_{s \in S} \left\{ f(s) - \phi^{-1}\left(\frac{K(t_0)}{g(t_0)}(1 - M(x, s, t_0)) \right) \right\}$$

for all $x \in X$. Let us check that $\hat{f}$ is fuzzy Lipschitz. Given $x, y \in X$ we have that

$$
\begin{aligned}
1-N_{\phi,g}(\hat{f}(x),\hat{f}(y),t) &= \phi(|\hat{f}(x)-\hat{f}(y)|)g(t_0)\\
&= \phi\Big(\Big|\sup_{s\in S}\left\{f(s)-\phi^{-1}\left(\frac{K(t_0)}{g(t_0)}(1-M(x,s,t_0))\right)\right\}\\
&- \sup_{s\in S}\left\{f(s)-\phi^{-1}\left(\frac{K(t_0)}{g(t_0)}(1-M(y,s,t_0))\right)\right\}\Big|\Big)g(t_0)\\
&\leq \phi\Big(\Big|\sup_{s\in S}\Big\{f(s)-\phi^{-1}\left(\frac{K(t_0)}{g(t_0)}(1-M(x,s,t_0))\right)\\
&- f(s)+\phi^{-1}\left(\frac{K(t_0)}{g(t_0)}(1-M(y,s,t_0))\right)\Big\}\Big|\Big)g(t_0)
\end{aligned}
$$

$$
\begin{aligned}
&= \phi\left(\left|\sup_{s\in S}\left\{\phi^{-1}\left(\frac{K(t_0)}{g(t_0)}(1-M(y,s,t_0))\right)-\phi^{-1}\left(\frac{K(t_0)}{g(t_0)}(1-M(x,s,t_0))\right)\right\}\right|\right)\\
&= \phi(\left(\left|\sup_{s\in S}\left\{\phi^{-1}(d_{t_0}(y,s))-\phi^{-1}(d_{t_0}(x,s))\right\}\right|\right)g(t_0)\\
&\leq \phi\left(\phi^{-1}(d_{t_0}(x,y))\right)g(t_0) = K(t_0)(1-M(x,y,t_0))\\
&\leq K(t_0)(1-M(x,y,t))
\end{aligned}
$$

where in the last inequality we have used that $M(x,y,t)=M(x,y,t_0)$ whenever $t\geq t_0$ and $M(x,y,t)\leq M(x,y,t_0)$ whenever $t<t_0$. Consequently, $\hat{f}$ is fuzzy Lipschitz.

Moreover, $\hat{f}$ extends f to X. In fact, if $s_0\in S$ then

$$f(s_0)=f(s_0)-\phi^{-1}\left(\frac{K(t_0)}{g(t_0)}(1-M(s_0,s_0,t_0))\right)\leq \hat{f}(s_0).$$

On the other hand, since f is fuzzy Lipschitz for every $s\in S$ we have that

$$
\begin{aligned}
1-N_{\phi,g}(f(s),f(s_0),t_0) &\leq K(t_0)(1-M(s,s_0,t_0)),\\
\phi(|f(s)-f(s_0)|)g(t_0) &\leq K(t_0)(1-M(s,s_0,t_0)),\\
|f(s)-f(s_0)| &\leq \phi^{-1}\left(\frac{K(t_0)}{g(t_0)}(1-M(s,s_0,t_0))\right),
\end{aligned}
$$

and then

$$f(s)-\phi^{-1}\left(\frac{K(t_0)}{g(t_0)}(1-M(s,s_0,t_0))\right)\leq f(s_0).$$

Hence $\hat{f}(s_0)\leq f(s_0)$ so $\hat{f}(s_0)=f(s_0)$. □

Remark 47. *Notice that under the conditions of the previous proposition, we can provide directly an extension of f not depending on t by using*

- *the following modified McShane formula (as used in the proof)*

$$f^M(x) = \sup_{s \in S} \left\{ f(s) - \phi^{-1}\left(\frac{K(t_0)}{g(t_0)}(1 - M(x,s,t_0)) \right) \right\}$$

  for all $x \in X$, or

- *the following modified Whitney formula*

$$f^W(x) = \inf_{s \in S} \left\{ f(s) + \phi^{-1}\left(\frac{K(t_0)}{g(t_0)}(1 - M(x,s,t_0)) \right) \right\}$$

  for all $x \in X$.

6. Applications: the extension formulas for two fuzzy metric spaces

To finish the paper, let us explain how to obtain explicit formulas providing parameterized families of extensions. These families can be used for constructing machine learning algorithms as the ones cited in the Introduction, but integrating new fuzzy elements in them. Although the same construction that we present here can be done for a broader class of fuzzy metric spaces, we will center our attention in two standard cases for the aim of clarity.

Let us consider a strong fuzzy metric space $(X, M, *)$ with $* \geq *_{\mathrm{L}}$ and the Euclidean fuzzy metric space $(\mathbb{R}, N_E, *_{\mathrm{L}})$ (see Example 41), given by

$$N_E(x,y,t) = 1 - \min\{|x-y|, 1\} g(t), \quad x, y \in \mathbb{R},$$

where $g : [0, +\infty) \to (0, 1]$ is a decreasing function.

Let $S \subseteq X$ and $I : (S, M, *) \to (\mathbb{R}, N_E, *_{\mathrm{L}})$ be a fuzzy Lipschitz map. Since $(M, *)$ is a strong fuzzy metric, by using Proposition 13 we can obtain that the hypotheses of Corollary 43 are satisfied. Thus, we know (see Remark 44) that there are two canonical extensions of I, —provided by the McShane and Whitney formulas—, $I^M, I^W : X \times (0, +\infty) \to \mathbb{R}$. Parameter dependent interpolations of these functions can be considered as optimal extensions of I, and would be given by

$$I_\alpha(x,t) := \alpha(t)\, I^M(x,t) + \big(1 - \alpha(t)\big)\, I^W(x,t), \quad x \in X, \ t > 0.$$

Here, $\alpha : (0, +\infty) \to [0, 1]$.

- *The metric model depending on a parameter.* Take a metric space (X, d) and construct the associated strong fuzzy metric space $(X, M_k, *_{\mathrm{L}})$ (cf. [10, Example 6]) defined by

$$M_k(x, y, t) = 1 - \frac{\min\{d(x, y), k\}}{h(t)}, \quad x, y \in X,$$

  where $k > 0$ and $h : (0, +\infty) \to (k, +\infty)$ is an increasing continuous function. Let $S \subseteq X$. Suppose that the function $I : (S, M_k, *_{\mathrm{L}}) \to (\mathbb{R}, N_E, *_{\mathrm{L}})$ is a fuzzy Lipschitz map. Its corresponding Lipschitz inequality is

$$1 - N_E(I(x), I(y), t) \leq K(t)\big(1 - M_k(x, y, t)\big), \quad x, y \in S,$$

  for all $x, y \in X$ and all $t > 0$, which can be rewritten as

$$\min\{|I(x) - I(y)|, 1\} \leq \frac{K(t)}{g(t)h(t)} \min\{d(x, y), k\}.$$

  That is, there is a function $Q : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\min\{|I(x) - I(y)|, 1\} \leq Q(t) \min\{d(x, y), k\}$$

  for all $x, y \in S$ and all $t > 0$. Then the McShane and Whitney extensions of I to $X \times (0, \infty)$ are given by

$$I^M(x, t) := \sup_{s \in S}\{I(s) - Q(t)\, \min\{d(s, x), k\}\}, \quad x \in X,$$

  and

$$I^W(x, t) := \inf_{s \in S}\{I(s) + Q(t)\, \min\{d(s, x), k\}\}, \quad x \in X.$$

  Thus, a possible family of extensions would be given by functions as

$$I_{\alpha(t)}(x, t) = \alpha(t)\, I^M(x, t) + (1 - \alpha(t))\, I^W(x, t)$$

$$= \alpha(t) \sup_{s \in S}\{I(s) - Q(t)\, \min\{d(s, x), k\}\} + (1 - \alpha(t)) \inf_{s \in S}\{I(s) + Q(t)\, \min\{d(s, x), k\}\},$$

  $x \in X$, $t > 0$. An adequate function $\alpha(t)$ could be given for example by an optimization procedure, in order to define a machine learning method incorporating fuzzy elements.

- *The exponential fuzzy model.* In this case, we consider the stationary fuzzy metric $(M_1, \cdot)$ given by [10, Example 5],

  $$M_1(x, y, t) = e^{-d(x,y)}, \quad x, y \in X,$$

  where (X, d) is a metric space. As above, let $S \subseteq X$ and $I : (S, M_1, \cdot) \to (\mathbb{R}, N_E, *_{\mathrm{L}})$ be a fuzzy Lipschitz function. Then, for each $t > 0$ we can find $K(t) > 0$ such that

  $$1 - N_E(I(x), I(y), t) \leq K(t)\big(1 - M_1(x, y, t)\big), \quad x, y \in S,$$

  for all $x, y \in X$ and all $t > 0$, which can be rewritten as

  $$\min\{|I(x) - I(y)|, 1\} \leq \frac{K(t)}{g(t)}(1 - e^{-d(x,y)}).$$

  Notice that since $\cdot \geq *_{\mathrm{L}}$ we have by Proposition 13 that $1 - e^{-d(x,y)}$ is a metric on X.

  Using the same arguments than in the previous example, we obtain that the family of extensions would be given by functions as

  $$I_{\alpha(t)}(x, t) = \alpha(t)\, I^M(x, t) + (1 - \alpha(t))\, I^W(x, t), \quad x \in X,\ t > 0,$$

  where

  $$I^M(x, t) = \sup_{s \in S} \left\{ I(s) - \frac{K(t)}{g(t)} \left(1 - e^{-d(s,x)}\right) \right\}, \quad x \in X,\ t > 0,$$

  and

  $$I^W(x, t) = \inf_{s \in S} \left\{ I(s) + \frac{K(t)}{g(t)} \left(1 - e^{-d(s,x)}\right) \right\}, \quad x \in X,\ t > 0.$$

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