UNBOUNDED BERGMAN PROJECTIONS ON WEIGHTED SPACES WITH RESPECT TO EXPONENTIAL WEIGHTS

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Abstract. There are recent results concerning the boundedness and also unboundedness of Bergman projections on weighted spaces of the unit disc in special cases of rapidly decreasing weights, i.e. "large" Bergman spaces. The aim of our paper is to show that the cases of boundedness are largely exceptional: in general the Bergman projections are unbounded. In addition we give a new, more functional analytic proof for the known central boundedness case which also enables us to transfer our results to harmonic Bergman spaces.

1. Introduction

In this note we study weighted Bergman spaces on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane. We deal with radial weights, i.e. continuous, decreasing functions $v : [0,1[\to \mathbb{R}_+$ with $\lim_{\to 1} v(r) = 0$. Let, for $1 \le p < \infty$, L^p_v be the L^p-space over the unit disc $\mathbb{D}$ for the weighted area measure $dm_v = (2\pi)^{-1} v(r) r dr d\varphi$ (with respect to the polar coordinates) and put $A^p_v = \{f \in L^p_v : f \text{ holomorphic on } \mathbb{D}\}$. We consider the norms

$$\|f\|_{p,v} = \Big(\int_0^1 M_p^p(f,r) r v(r) dr\Big)^{1/p} = \Big(\int_{\mathbb{D}} |f(\zeta)|^p dm_v(\zeta)\Big)^{1/p}$$

$$\text{and} \quad \|f\|_{\infty,v} = \sup_{|z|<1} |f(z)| v(|z|)$$

where, for $1 \le p < \infty$,

$$M_p(f,r) = \Big(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\varphi})|^p d\varphi\Big)^{1/p} \quad \text{and} \quad M_\infty(f,r) = \sup_{|z|=r} |f(z)|.$$

Finally put $L^\infty_v = \{f : \mathbb{D} \to \mathbb{C} : f \text{ Borel-measurable and } \|f\|_{\infty,v} < \infty\}$ and $H^\infty_v = \{h \in L^\infty_v : h \text{ holomorphic }\}$. The orthogonal projection $P_v : L^2_v \to A^2_v$ is called the *Bergman projection*. It follows from the boundedness of the point evaluation functionals in the space A^p_v that P_v can be presented as an integral operator with the Bergman reproducing kernel $K(z,\cdot) \in A^p_v$ as

$$P_v f(z) = \int_{\mathbb{D}} K(z,\zeta) f(\zeta) dm_v(\zeta). \tag{1.1}$$

The boundedness of the Bergman projection and the related pointwise kernel estimates have been the topic of intensive research, see e.g. the monographs [11], [19]

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and the survey [18]. Here, we investigate the boundedness question for weights of exponential type. Let $\alpha, \tilde\alpha, \beta, \tilde\beta > 0$ and

$$v(r) = \exp\left(-\frac{\alpha}{(1-r)^{\beta}}\right) \quad \text{and} \quad w(r) = \exp\left(-\frac{\tilde\alpha}{(1-r)^{\tilde\beta}}\right). \tag{1.2}$$

It was shown in [6], [8] that P_w is never bounded on L^p_v if $v = w$ and $p \neq 2$. But according to [5], for $\tilde\alpha = 2\alpha/p$ and $\beta = 1 = \tilde\beta$ the Bergman projection P_w is indeed bounded with respect to $\|\cdot\|_{p,v}$ for all $1 \leq p < \infty$ and if $\tilde\alpha = 2\alpha$ then P_w is bounded on L^∞_v. This result, replacing r by r^2, was extended to the cases

$$\begin{aligned} &1 < p < \infty, \quad \tilde\alpha = 2\alpha/p, \quad 0 < \beta = \tilde\beta < \infty \quad \text{and} \\ &p = \infty, \quad \tilde\alpha = 2\alpha, \quad 0 < \beta = \tilde\beta < \infty \end{aligned} \tag{1.3}$$

in [1], Theorems 4.4., 4.25 and the beginning of Section 4.7. We emphasize that in the paper [14] the second and third named authors constructed concrete bounded projection operators $L^p_v \to A^p_v$ for a large class of weights satisfying condition (B), which includes the weights (1.2) but also much more general rapidly decreasing weights. (Condition (B) is used to pick up increasing sequences of powers and radii such that the values of the weight and the power functions on these radii are related. This leads, see e.g. Theorem 1 of [14], to a representation of the weighted sup-norm as a vector valued, weighted ℓ^∞-norm, which is crucial for the results. More information and examples about condition (B) can be seen in [13]) The projection operators were presented by using r-dependent Fourier series of L^p_v functions and thus were not expressed by known kernel functions. It remains as a (slightly vaguely formulated) open problem to study, if bounded projections for these large weight classes can be described using some "natural" kernel functions. Recently, in [10], Theorem 4.1. it was shown for a large class of weights v, including those in (1.2)–(1.3), that the Bergman projection P_v is bounded in the spaces $L^p_{v^{p/2}}$, $1 \leq p < \infty$, and $L^\infty_{v/2}$. The weights were of the form $v(z) = e^{-\varphi(z)}$ with certain growth conditions for the (not necessarily radial) function φ.

One might expect that P_w remains bounded on L^p_v provided that $\tilde\alpha$ is large enough but our aim is to show that the preceding cases are the only cases of boundedness of P_w. Indeed, in Theorems 1.1 and 1.2 we show that, for $\beta \neq \tilde\beta$ or for $\beta = \tilde\beta$ and $\tilde\alpha \neq 2\alpha/p$, if $1 \leq p < \infty$, and $\tilde\alpha \neq 2\alpha$, if $p = \infty$ the Bergman projection P_w is always unbounded. In Theorem 1.3 we give a new, functional analytic and relatively short proof that, for $\beta = \tilde\beta$ and $\tilde\alpha = 2\alpha/p$ the Bergman projection P_w is always bounded on L^p_v. More generally we obtain the same results if we consider $w(r^\ell)$ instead of $w(r)$ and $v(r^\ell)$ instead of $v(r)$ for some $\ell > 0$.

To prove our theorems we need to make an excursion into the corresponding weighted spaces of harmonic functions (which will be defined in section 3). There we recollect and extend known results concerning the weighted L^p-norms which seem to be of independent interest. They also show that our main results of section 1 remain true for the corresponding weighted spaces of harmonic functions which we present in the final Theorem 4.5.

In the following we denote $v_\ell(r) = v(r^\ell)$ and $w_\ell = w(r^\ell)$ for any $\ell > 0$.

At first we consider the case $\beta = \tilde\beta$.

Theorem 1.1. *Let $0 < \alpha, \tilde{\alpha}, \beta < \infty$, and $1 \le p < \infty$. Put*

$$v(r) = \exp\left(-\frac{\alpha}{(1-r)^\beta}\right) \quad \text{and} \quad w(r) = \exp\left(-\frac{\tilde{\alpha}}{(1-r)^\beta}\right).$$

If $\tilde{\alpha} \neq 2\alpha/p$ then P_{w_ℓ} is unbounded on $L^p_{v_\ell}$ with respect to $\|\cdot\|_{p,v_\ell}$ for all $\ell > 0$. If $\tilde{\alpha} \neq 2\alpha$ then P_{w_ℓ} is unbounded on $L^\infty_{v_\ell}$ with respect to $\|\cdot\|_{\infty,v_\ell}$ for all $\ell > 0$.

In particular, P_w is unbounded on L^p_v if $\alpha = \tilde{\alpha}$ and $p \neq 2$. Hence Theorem 1.1. includes results of [6]. Now we turn to the case $\beta \neq \tilde{\beta}$.

Theorem 1.2. *Let $0 < \alpha, \tilde{\alpha}, \beta, \tilde{\beta} < \infty$ and $1 \le p \le \infty$. Put*

$$v(r) = \exp\left(-\frac{\alpha}{(1-r)^\beta}\right) \quad \text{and} \quad w(r) = \exp\left(-\frac{\tilde{\alpha}}{(1-r)^{\tilde{\beta}}}\right).$$

If $\tilde{\beta} \neq \beta$ then P_{w_ℓ} is unbounded on $L^p_{v_\ell}$ with respect to $\|\cdot\|_{p,v_\ell}$ for all $\ell > 0$.

We prove Theorems 1.1. and 1.2. in Section 2; cf. also the partially related Proposition 19 of the interesting paper [17].

It is known that in the case $\tilde{\alpha} = 2\alpha/p$ the Bergman projection is bounded. This was shown for $\beta = \ell = 1$ in [5] and in [1] for $0 < \beta < \infty$, $\ell = 2$. For a more general result, see [10]. We give here a different and in a way simpler and shorter proof.

Theorem 1.3 ([5],[10])**.** *Let $0 < \alpha, \tilde{\alpha}, \beta < \infty$ and $1 \le p < \infty$. Put*

$$v(r) = \exp\left(-\frac{\alpha}{(1-r)^\beta}\right) \quad \text{and} \quad w(r) = \exp\left(-\frac{\tilde{\alpha}}{(1-r)^\beta}\right). \tag{1.4}$$

If $\tilde{\alpha} = 2\alpha/p$ then P_{w_ℓ} is a bounded operator $L^p_{v_\ell} \to A^p_{v_\ell}$ for all $\ell > 0$. If $\tilde{\alpha} = 2\alpha$ then P_{w_ℓ} is a bounded operator $L^\infty_{v_\ell} \to H^\infty_{v_\ell}$ for all $\ell > 0$.

In this paper we will use standard notation and terminology; see e.g. [19]. Let us only mention that if X and Y are for example positive quantities depending on a parameter $k \in \mathbb{N}$, the expression $X \cong Y$ means that there are constants $c_2 \ge c_1 > 0$, independent of k, such that $c_1 X \le Y \le c_2 X$ holds for all k.

2. Monomial estimates.

To prove Theorem 1.1. we need the following integral estimate. Similar estimates can be found e.g. in Lemma 2.2. in [6], Lemma 1 in [7], Lemma 7 in [5], Lemma 4.28 in [1].

Lemma 2.1. *Let $\alpha > 0$, $\beta > 0$ and $\ell > 0$ be fixed, and let $k > 0$ be arbitrary. Then we have*

$$d_\ell k^{-\frac{2+\beta}{2(\beta+1)}} \exp\big(-B(\alpha,\beta,\ell) k^{\frac{\beta}{\beta+1}}\big) \le \int_0^1 r^k \exp\Big(-\frac{\alpha}{(1-r^\ell)^\beta}\Big) dr$$

$$\le \tilde{d}_\ell k^{-\frac{2+\beta}{2(\beta+1)}} \exp\big(-B(\alpha,\beta,\ell) k^{\frac{\beta}{\beta+1}}\big)) \tag{2.1}$$

where $0 < d_\ell < \tilde{d}_\ell$ are constants not depending on k and

$$B(\alpha,\beta,\ell) = \ell^{\frac{-\beta}{\beta+1}} \alpha^{\frac{1}{\beta+1}} \big(\beta^{\frac{1}{\beta+1}} + \beta^{\frac{-1}{\beta+1}}\big). \tag{2.2}$$

Proof. Lemma 1 in [7] yields the estimates

$$c_1 k^{-\frac{2+\beta}{2(\beta+1)}} \exp\big(-Bk^{\frac{\beta}{\beta+1}}\big) \le \int_0^1 r^k \exp\Big(-\frac{\alpha}{(1-r)^\beta}\Big)dr$$

$$\le c_2 k^{-\frac{2+\beta}{2(\beta+1)}} \exp\big(-Bk^{\frac{\beta}{\beta+1}}\big)) \tag{2.3}$$

where B equals $B(\alpha,\beta,1)$ of (2.2). The substitution $r^\ell \to s$ turns the integral in (2.1) into

$$\frac{1}{\ell}\int_0^1 s^{\frac{k}{\ell}+\frac{1-\ell}{\ell}} \exp\Big(-\frac{\alpha}{(1-s)^\beta}\Big)ds, \tag{2.4}$$

hence, expanding in the exponent

$$\Big(\frac{k}{\ell}+\frac{1-\ell}{\ell}\Big)^{\frac{\beta}{\beta+1}} = \Big(\frac{k}{\ell}\Big)^{\frac{\beta}{\beta+1}} + O(1)$$

for large k, the lemma follows from (2.3). $\square$

We continue with two more lemmas.

Lemma 2.2. *Let $\tilde\alpha \ge \alpha > 0$ and $\beta > 0$. Then we have*

$$2^{\frac{\beta}{\beta+1}}\tilde\alpha^{\frac{1}{\beta+1}} - (\tilde\alpha-\alpha)^{\frac{1}{\beta+1}} - \alpha^{\frac{1}{\beta+1}} = 0 \quad \text{if } \tilde\alpha = 2\alpha,$$

and

$$2^{\frac{\beta}{\beta+1}}\tilde\alpha^{\frac{1}{\beta+1}} - (\tilde\alpha-\alpha)^{\frac{1}{\beta+1}} - \alpha^{\frac{1}{\beta+1}} > 0 \quad \text{if } \tilde\alpha \ne 2\alpha. \tag{2.5}$$

Proof. Put

$$f(x) = 2^{\frac{\beta}{\beta+1}}x^{\frac{1}{\beta+1}} - (x-\alpha)^{\frac{1}{\beta+1}} - \alpha^{\frac{1}{\beta+1}}.$$

Then

$$f'(x) = \frac{1}{\beta+1}\Big(2^{\frac{\beta}{\beta+1}}x^{-\frac{\beta}{\beta+1}} - (x-\alpha)^{-\frac{\beta}{\beta+1}}\Big).$$

We obtain that $x = 2\alpha$ is the only zero of f as well as of f' and 2α is the only minimum of f. In particular $f(x) > 0$ for $x \ne 2\alpha$ which proves Lemma 2.2. $\square$

Lemma 2.3. *Let $\tilde\alpha \ge \alpha > 0$, $\ell > 0$ and put, for $k > 0$,*

$$D_k = \int_0^1 r^{k+1}\exp\big(-\alpha(1-r^\ell)^{-\beta}\big)dr \left(\frac{\int_0^1 r^{k+1}\exp\big(-(\tilde\alpha-\alpha)(1-r^\ell)^{-\beta}\big)dr}{\int_0^1 r^{2k+1}\exp\big(-\tilde\alpha(1-r^\ell)^{-\beta}\big)dr}\right).$$

If $\overline{\alpha} \ne 2\alpha$, then there is a constant $c_1 > 0$, independent of k such that

$$D_k \ge \exp\Big(c_1 k^{\frac{\beta}{\beta+1}}\Big). \tag{2.6}$$

Proof. We estimate all integrals in the definition of D_k by (2.1) in Lemma 2.1 and obtain that D_k is proportional to

$$k^{-\frac{2+\beta}{2(\beta+1)}} \exp\Bigg(\Big(\frac{k}{\ell}\Big)^{\frac{\beta}{\beta+1}}\big(\beta^{\frac{1}{\beta+1}} + \beta^{\frac{-1}{\beta+1}}\big)\Big(2^{\frac{\beta}{\beta+1}}\tilde\alpha^{\frac{1}{\beta+1}} - \alpha^{\frac{1}{\beta+1}} - (\tilde\alpha-\alpha)^{\frac{1}{\beta+1}}\Big)\Bigg). \tag{2.7}$$

This together with the assumptions on α and $\tilde\alpha$ and Lemma 2.2 yield (2.6). $\square$

Proof of Theorem 1.1. Let $v(r) = \exp(-\alpha/(1-r)^\beta)$ and $w(r) = \exp(-\tilde\alpha/(1-r)^\beta)$ and fix $1 \le p < \infty$ and $\ell > 0$.

Consider $f \in L^p_{v_\ell}$ with $\|f\|_{p,v_\ell} \le 1$ and put

$$a_k(r) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\varphi})e^{-ik\varphi}d\varphi, \qquad i.e.\ \ f(re^{i\varphi}) \sim \sum_{k\in\mathbb{Z}} a_k(r)e^{ik\varphi}, \tag{2.8}$$

regarding $\sum_{k\in\mathbb{Z}} a_k(r)e^{ik\varphi}$ as the Fourier series of $f(re^{i\varphi})$ for fixed r.

We have

$$P_{w_\ell}f(z) = \frac{1}{2\pi}\int_0^1\int_0^{2\pi} K(z,re^{i\varphi})f(re^{i\varphi})rw(r^\ell)d\varphi dr,$$

where

$$K(z,\zeta) = \sum_{n=0}^\infty \frac{(z\bar\zeta)^n}{\int_0^1 r^{2n+1}w(r^\ell)dr}.$$

Using the orthogonality of $e^{ik\varphi}$ and $e^{ij\varphi}$ for $k \ne j$ and (2.8) we obtain

$$(P_{w_\ell}f)(z) = \sum_{k=0}^\infty z^k \frac{\int_0^1 r^{k+1}w(r^\ell)a_k(r)dr}{\int_0^1 r^{2k+1}w(r^\ell)dr}. \tag{2.9}$$

Consider $f_k(re^{i\varphi}) = v(r^\ell)^{-1/p}e^{ik\varphi}$, $k = 0,1,2,\ldots$. Then $f_k \in L^p_{v_\ell}$ for all k and $\|f_k\|^p_{p,v_\ell} = \int_0^1 rdr = 1/2$. Formula (2.8) implies $a_k(r) = v(r^\ell)^{-1/p}$ and $a_j(r) = 0$ for all $j \ne k$. With (2.9) and the Jensen inequality we obtain

$$\begin{aligned}
&\|P_{w_\ell}f_k\|_{p,v_\ell}\\
&= \left(\int_0^1 r^{kp+1}v(r^\ell)dr\right)^{\frac{1}{p}} \frac{\int_0^1 r^{k+1}w(r^\ell)a_k(r)dr}{\int_0^1 r^{2k+1}w(r^\ell)dr}\\
&\ge \left(\int_0^1 r^{k+\frac{1}{p}}v(r^\ell)^{\frac{1}{p}}dr\right) \frac{\int_0^1 r^{k+1}w(r^\ell)a_k(r)dr}{\int_0^1 r^{2k+1}w(r^\ell)dr}\\
&\ge \left(\int_0^1 r^{k+1}v(r^\ell)^{\frac{1}{p}}dr\right) \frac{\int_0^1 r^{k+1}w(r^\ell)a_k(r)dr}{\int_0^1 r^{2k+1}w(r^\ell)dr}
\end{aligned}$$

$$= \int_0^1 r^{k+1}\exp\Big(-\frac{\alpha}{p(1-r^\ell)^\beta}\Big)dr \frac{\int_0^1 r^{k+1}\exp\big(-(\tilde\alpha - \frac{\alpha}{p})(1-r^\ell)^{-\beta}\big)dr}{\int_0^1 r^{2k+1}\exp\big(-\tilde\alpha(1-r^\ell)^{-\beta}\big)dr}. \tag{2.10}$$

If $\alpha/p \le \tilde\alpha$ then we use Lemma 2.3. with α/p instead of α and find a constant $c_1 > 0$ with

$$\|P_w f_k\|_{p,v_\ell} \ge \exp\left(c_1 k^{\beta/(\beta+1)}\right).$$

This implies $\lim_{k\to\infty}\|P_{w_\ell}f_k\|_{p,v_\ell} = \infty$ which shows that P_{w_ℓ} is unbounded on $L^p_{v_\ell}$ provided that $\tilde\alpha \ne 2\alpha/p$.

Now assume $\tilde\alpha < \alpha/p$. We obtain from Lemma 2.1 that

$$\int_0^1 r^{k+1}\exp\Big(-\frac{\alpha}{p(1-r^\ell)^\beta}\Big) \ge \exp\left(-c_3 k^{\frac{\beta}{\beta+1}}\right)$$

as well as

$$\int_0^1 r^{2k+1}\exp\Big(-\frac{\tilde\alpha}{(1-r^\ell)^\beta}\Big)dr \le \exp\left(-c_4 k^{\frac{\beta}{\beta+1}}\right)$$

for some constants $c_3, c_4 > 0$. Hence the right-hand side of (2.10) is not smaller than

$$\begin{aligned}
&\exp\Big(-c_5 k^{\frac{\beta}{\beta+1}}\Big)\int_{(1-1/k)^{1/\ell}}^1 r^{k+1}\exp\Big(\Big(\frac{\alpha}{p}-\tilde\alpha\Big)(1-r^\ell)^{-\beta}\Big)dr\\
&\geq \exp\Big(-c_5 k^{\frac{\beta}{\beta+1}}\Big)\Big(1-\frac{1}{k}\Big)^{\frac{k+1}{\ell}}\Big(1-\Big(1-\frac{1}{k}\Big)^{\frac{1}{\ell}}\Big)\exp\Big(\Big(\frac{\alpha}{p}-\tilde\alpha\Big)k^\beta\Big)\\
&\geq c_6\exp\Big(-c_5 k^{\frac{\beta}{\beta+1}}+c_7 k^\beta-\log k\Big)
\end{aligned}$$

where c_5, c_6 are positive constants and $c_7 = \alpha/p - \tilde\alpha > 0$. We used that the exponential function in the last integral is increasing and $1-(1-1/k)^{1/\ell} \geq c_8/k$ for some constant c_8. But this shows again, together with (2.10), that $\lim_{k\to\infty}\|P_{w_\ell} f_k\|_{p,v_\ell} = \infty$ and hence P_{w_ℓ} is unbounded in this case, too.

Finally consider $p=\infty$. Here we take $f_k(re^{i\varphi}) = v(r^\ell)^{-1}e^{ik\varphi}$. Using (2.9) and

$$\sup_{0\leq r<1} r^k v(r^\ell) \quad\geq\quad \int_0^1 r^{k+1}v(r^\ell)dr$$

we again derive (2.10), i.e.

$$\begin{aligned}
&\|P_{w_\ell} f_k\|_{\infty,v_\ell}\\
&\geq \int_0^1 r^{k+1}\exp\Big(-\frac{\alpha}{(1-r^\ell)^\beta}\Big)dr\,\frac{\int_0^1 r^{k+1}\exp\big(-(\tilde\alpha-\alpha)(1-r^\ell)^{-\beta}\big)dr}{\int_0^1 r^{2k+1}\exp\big(-\tilde\alpha(1-r^\ell)^{-\beta}\big)dr}.
\end{aligned}$$

Then we proceed exactly as before (with 1 instead of p) to show $\lim_{k\to\infty}\|P_{w_\ell} f_k\|_{\infty,v_\ell} = \infty$ which proves that P_w is unbounded also in this case. $\Box$

Proof of Theorem 1.2. The proof is similar to the proof of Theorem 1.1. At first we consider $1 \leq p < \infty$. Let $f_k(re^{i\varphi}) = v(r^\ell)^{-1/p}e^{ik\varphi}$, $k = 0,1,2,\ldots$. Then $f_k \in L^p_{v_\ell}$ for all k and $\|f_k\|^p_{p,v_\ell} = 1/2$. With (2.9) we see that

$$(P_{w_\ell} f_k)(z) = \frac{\int_0^1 r^{k+1}\exp\Big(\frac{\alpha}{p}(1-r^\ell)^{-\beta}-\tilde\alpha(1-r^\ell)^{-\tilde\beta}\Big)\,dr}{\int_0^1 r^{2k+1}\exp\big(-\tilde\alpha(1-r^\ell)^{-\tilde\beta}\big)\,dr}z^k.$$

If $\beta > \tilde\beta$ then the integral in the numerator is equal to ∞. Hence P_{w^ℓ} cannot be bounded.

Now let $\beta < \tilde\beta$. With (2.10) we have

$$\begin{aligned}
&\|P_{w_\ell} f_k\|_{p,v_\ell}\\
&\geq \int_0^1 r^{k+1}\exp\Big(-\frac{\alpha}{p(1-r^\ell)^\beta}\Big)dr\,\frac{\int_0^1 r^{k+1}\exp\Big(\frac{\alpha}{p}(1-r^\ell)^{-\beta}-\tilde\alpha(1-r^\ell)^{-\tilde\beta}\Big)\,dr}{\int_0^1 r^{2k+1}\exp\big(-\tilde\alpha(1-r^\ell)^{-\tilde\beta}\big)\,dr}\\
&\geq \int_0^1 r^{k+1}\exp\Big(-\frac{\alpha}{p(1-r^\ell)^\beta}\Big)dr\,\frac{\int_0^1 r^{k+1}\exp\Big(-\tilde\alpha(1-r^\ell)^{-\tilde\beta}\Big)\,dr}{\int_0^1 r^{2k+1}\exp\big(-\tilde\alpha(1-r^\ell)^{-\tilde\beta}\big)\,dr}.
\end{aligned}$$

Now we apply Lemma 2.1 to the right-hand side. This yields constants $c_1, c_2 > 0$ with

$$\|P_{w_\ell} f_k\|_{p,v_\ell}$$

$$(2.11) \geq c_1 k^{c_2} \exp\left(-B(\alpha,\beta,\ell))k^{\frac{\beta}{\beta+1}} - B(\tilde\alpha,\tilde\beta,\ell))k^{\frac{\tilde\beta}{\tilde\beta+1}} + 2^{\frac{\tilde\beta}{\tilde\beta+1}} B(\tilde\alpha,\tilde\beta,\ell)k^{\frac{\tilde\beta}{\tilde\beta+1}}\right)$$

Since $\beta < \tilde\beta$ we obtain $\beta/(\beta+1) < \tilde\beta/(\tilde\beta+1)$, hence, the term with $k^{\beta/(\beta+1)}$ in the exponent is negligible in comparison with the two others, since we moreover have

$$-B(\tilde\alpha,\tilde\beta,\ell))k^{\frac{\tilde\beta}{\tilde\beta+1}} + 2^{\frac{\tilde\beta}{\tilde\beta+1}} B(\tilde\alpha,\tilde\beta,\ell)k^{\frac{\tilde\beta}{\tilde\beta+1}} \geq c_3 k^{\frac{\tilde\beta}{\tilde\beta+1}}$$

for some positive constant c_3 not depending on k. Thus, (2.11) yields $\lim_{k\to\infty} \|P_{w_\ell} f_k\|_{p,v_\ell} = \infty$. Again, P_{w_ℓ} is unbounded on $L^p_{v_\ell}$.

For $p = \infty$ the proof is similar. □

3. Banach spaces of harmonic functions with exponential weights.

At first we recall some well-known facts concerning our exponential weights. So, again, let $v_\ell(r) = \exp(-\alpha(1-r^\ell)^{-\beta}))$ for some constants $\alpha, \beta, \ell > 0$. For all $n \geq 1$, let $r_n \in [0,1[$ be the maximum point of $r^n v_\ell(r)$. It is easily seen that the number r_n is unique, increases with n and tends to 1 as $n \to \infty$.

Proposition 3.1. *Put*

$$m_n = \ell\beta^2 \left(\frac{\beta}{\alpha}\right)^{\frac{1}{\beta}} n^{2+\frac{2}{\beta}} - \ell\beta^2 n^2, \quad n = 0,1,2,\dots.$$

Then

$$r_{m_0} = 0 \quad \textit{and} \quad r_{m_n} = \left(1 - \left(\frac{\alpha}{\beta}\right)^{\frac{1}{\beta}} n^{-\frac{2}{\beta}}\right)^{\frac{1}{\ell}} \quad \textit{for } \; n = 1,2,\dots$$

and there are numbers $2 < b < K$ *which satisfy*

$$(3.1) \quad b \leq \left(\frac{r_{m_n}}{r_{m_{n+1}}}\right)^{m_n} \frac{v_\ell(r_{m_n})}{v_\ell(r_{m_{n+1}})}, \left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{m_{n+1}} \frac{v_\ell(r_{m_{n+1}})}{v_\ell(r_{m_n})} \leq K \quad \textit{for all } n.$$

Proof. The numbers m_n were computed in [2], (3.15), (3.16) and (3.30) for $\ell = 1$. Indeed, for $\ell = 1$ we have

$$m_n = \beta^2 \left(\frac{\beta}{\alpha}\right)^{\frac{1}{\beta}} n^{2+\frac{2}{\beta}} - \beta^2 n^2 \qquad \text{and} \qquad r_{m_n} = 1 - \left(\frac{\alpha}{\beta}\right)^{\frac{1}{\beta}} n^{-\frac{2}{\beta}}.$$

By substitution, using $\max_r r^k v_\ell(r) = \max_s s^{s/\ell} v(s)$, we obtain Proposition 3.1 □

In this section we want to study harmonic functions on $\mathbb{D}$. Put

$$h^p_{v_\ell} = \{f \in L^p_{v_\ell} : f \text{ harmonic on } \mathbb{D}\}.$$

For a harmonic function $f(z) = \sum_{k=0}^\infty a_k z^k + \sum_{k=1}^\infty a_k \bar z^k$ we define the operators

$$Q_0 f = 0, \quad Q_n f = \sum_{k=0}^{n-1} a_k z^k + \sum_{k=1}^{n-1} a_k \bar z^k, \quad n = 1,2,\dots \quad \text{and} \quad Rf = \sum_{k=0}^{\infty} a_k z^k.$$

The operator R is called the Riesz projection. We have

Proposition 3.2. *Let* $1 < p < \infty$. *Then there are constants* $c_1, c_2 > 0$ *such that*

$$c_1 \|f\|_{p,v_\ell} \leq \left(\sum_{n=0}^\infty M_p^p\left((Q_{[m_{(n+1)}/p]} - Q_{[m_n/p]})f, r_{m_n}\right) v_\ell(r_{m_n})(r_{m_{n+1}} - r_{m_n})\right)^{1/p} \leq c_2 \|f\|_{p,v_\ell}$$

for all $f \in h^p_{v_\ell}$.

Here $[m]$ is the largest integer not larger than m.

Proof. For holomorphic functions this was shown in [3], Theorem 3.1. with

$$I_n := \int_{r_{m_n}}^{r_{m_{n+1}}} r dr = \frac{1}{2}(r_{m_{n+1}}^2 - r_{m_n}^2) = \frac{1}{2}(r_{m_{n+1}} - r_{m_n})(r_{m_{n+1}} + r_{m_n})$$

instead of $(r_{m_{n+1}} - r_{m_n})$ in the statement of Proposition 3.2. Here we need that

$$\limsup_{n\to\infty} \frac{I_n}{\min(I_{n-1}, I_{n+1})} < b.$$

But it follows from Proposition 3.1 that this limsup is equal to 1. Now we have

$$\frac{1}{2}(r_{m_{n+1}} - r_{m_n}) \le I_n \le (r_{m_{n+1}} - r_{m_n})$$

for large enough n, which yields Proposition 3.2. for holomorphic f.

The case for harmonic functions follows from this by taking into account the following facts. There is a universal constant $d_p > 0$ with $M_p(Rf, r) \le d_p M_p(f, r)$ whenever f is harmonic, $0 \le r < 1$ and $1 < p < \infty$. Denoting by id the identity operator and $(Sf)(z) = f(\bar{z})$ for a harmonic function f, we have $M_p(Sf, r) = M_p(f, r)$ and $(\mathrm{id} - R)f = SRSf - f(0)$. $\square$

Proposition 3.2. remains true for $p = 1$ and $p = \infty$ with some modifications. To this end put, for $f(re^{i\varphi}) = \sum_{k=-\infty}^{\infty} a_k r^{|k|} e^{ik\varphi}$,

$$T_0 f = \sum_{|k|\le m_1} \frac{[m_1] - k}{[m_1]} a_k r^{|k|} e^{ik\varphi}$$

and, for $n = 1, 2, \ldots$,

$$T_n f = \sum_{m_{n-1}\le|k|\le m_n} \frac{k - [m_{n-1}]}{[m_n] - [m_{n-1}]} a_k r^{|k|} e^{ik\varphi} + \sum_{m_n\le|k|\le m_{n+1}} \frac{[m_{n+1}] - k}{[m_{n+1}] - [m_n]} a_k r^{|k|} e^{ik\varphi}.$$

Then we have

Proposition 3.3. *There are constants* $c_k > 0$ *with*

(i) $\quad c_1 \sup_n M_\infty(T_n f, r_{m_n}) v_\ell(r_{m_n}) \le \|f\|_{v_\ell,\infty} \le c_2 \sup_n M_\infty(T_n f, r_{m_n}) v_\ell(r_{m_n})$,

for all $f \in h_{v_\ell}^\infty$ *and*

(ii) $\quad c_3 \|f\|_{1,v_\ell} \le \sum_{n=0}^\infty M_1(T_n f, r_{m_n}) v_\ell(r_{m_n})(r_{m_{n+1}} - r_{m_n}) \le c_4 \|f\|_{1,v_\ell}$

for all $f \in h_{v_\ell}^1$.

Proof. (i): The operators T_n are uniformly bounded on $h_{v_\ell}^\infty$ by Proposition 3.4. of [13]. Hence the left-hand inequality of (i) is trivial. According to Proposition 5.2. of [13] we obtain

$$\|f\|_{\infty,v_\ell} \le c_1 \sup_n \sup_{r_{m_{n-1}}\le r\le r_{m_{n+1}}} M_\infty(T_n f, r) v_\ell(r).$$

For $r_{m_n} \le r \le r_{m_{n+1}}$ we have, by the maximum property of the number $r_{m_{n+1}}$ and [13], Lemma 3.1.,

$$M_\infty(T_n f, r) v_\ell(r) \le 2 \left(\frac{r}{r_{m_n}}\right)^{m_{n+1}} \frac{v_\ell(r)}{v_\ell(r_{m_n})} M_\infty(T_n f, r_{m_n}) v_\ell(r_{m_n})$$

$$\leq 2\left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{m_{n+1}} \frac{v_\ell(r_{m_{n+1}})}{v_\ell(r_{m_n})} M_\infty(T_n f, r_{m_n}) v_\ell(r_{m_n}).$$

Similarly, if $r_{m_{n-1}} \leq r \leq r_{m_n}$ we have

$$\begin{aligned} M_\infty(T_n f, r) v_\ell(r) &\leq \left(\frac{r}{r_{m_n}}\right)^{m_{n-1}} \frac{v_\ell(r)}{v_\ell(r_{m_n})} M_\infty(T_n f, r_{m_n}) v_\ell(r_{m_n}) \\ &\leq \left(\frac{r_{m_{n-1}}}{r_{m_n}}\right)^{m_{n-1}} \frac{v_\ell(r_{m_{n-1}})}{v_\ell(r_{m_n})} M_\infty(T_n f, r_{m_n}) v_\ell(r_{m_n}). \end{aligned}$$

In both cases (3.1) shows the right-hand inequality of (i).

(ii): Let $f \in h^1_{v_\ell}$. Then we have $f = \sum_n T_n f$ where the series converges pointwise on $\mathbb{D}$ and $T_n f \in \operatorname{span}\{r^{|k|} e^{ik\varphi} : m_{n-1} < |k| < m_{n+1}\}$. We use the following result (see [12], Lemma 3.1.): if $0 < r < s$ we have

$$M_1(T_n f, s) \leq \left(\frac{s}{r}\right)^{m_{n+1}} M_1(T_n f, r),$$

$$M_1(T_n f, r) \leq c_1 \left(\frac{r}{s}\right)^{m_{n-1}} M_1(T_n f, s) \tag{3.2}$$

for all n, where c_1 is a universal constant.

In view of Proposition 3.1. there are constants c_2, c_3 such that

$$c_2(m_n - m_{n-1}) \leq m_{n+1} - m_n \leq c_3(m_n - m_{n-1}) \quad \text{for all } n. \tag{3.3}$$

There is a constant c_4 such that for all r with $r_{m_n} \leq r \leq r_{m_{n+1}}$ we have

$$M_1(T_n f, r_{m_n}) v_\ell(r_{m_n}) \leq c_4 M_1(T_n f, r) v_\ell(r). \tag{3.4}$$

Indeed, we estimate

$$M_1(T_n f, r_{m_n}) v_\ell(r_{m_n}) \leq c_1 \left(\frac{r_{m_n}}{r}\right)^{m_{n-1}} \frac{v_\ell(r_{m_n})}{v_\ell(r)} M_1(T_n f, r) v_\ell(r)$$

$$\begin{aligned} &\leq c_1 \left(\frac{r_{m_n}}{r_{m_{n+1}}}\right)^{m_{n-1}} \frac{v_\ell(r_{m_n})}{v_\ell(r_{m_{n+1}})} M_1(T_n f, r) v_\ell(r) \\ &= c_1 \left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{m_n - m_{n-1}} \left(\frac{r_{m_n}}{r_{m_{n+1}}}\right)^{m_n} \frac{v_\ell(r_{m_n})}{v_\ell(r_{m_{n+1}})} M_1(T_n f, r) v_\ell(r) \\ &\leq c_1 \left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{(m_{n+1}-m_n)/c_2} K M_1(T_n f, r) v_\ell(r) \\ &\leq c_1 K^{2/c_2} K M_1(T_n f, r) v_\ell(r). \end{aligned}$$

where we first used (3.2), then the fact that $r^{m_{n-1}} v_\ell(r) \geq r_{m_{n+1}}^{m_{n-1}} v_\ell(r_{m_{n+1}})$ (since $r \leq r_{m_{n+1}}$ and the function $t \mapsto t^{m_{n-1}} v_\ell(t)$ is decreasing for t larger than its unique maximum $r_{m_{n-1}}$), and then, consequently, (3.3) and (3.1); finally, at the last step, we used the inequalities (3.1) another time, multiplying the two expressions in the middle of it with each other.

According to [9], Lemma 4.1., in view of (3.3), there is a constant c_5 such that

$$M_1(T_n f, r) \leq c_5 M_1(f, r) \quad \text{for all } f, r \text{ and } n. \tag{3.5}$$

Put $J_n = r_{m_{n+1}} - r_{m_n}$. Then we have

$$J_n \leq \frac{2}{r_{m_1}} \int_{r_{m_n}}^{r_{m_{n+1}}} r\,dr \leq \frac{2}{r_{m_1}} J_n \quad \text{for all } \ n.$$

The inequalities (3.4) and (3.5) imply with $c_6 = 2/r_{m_1}$

$$\sum_{n=0}^{\infty} M_1(T_n f, r_{m_n}) v_\ell(r_{m_n}) J_n \leq \sum_{n=0}^{\infty} c_6 \int_{r_{m_n}}^{r_{m_{n+1}}} M_1(T_n f, r_{m_n}) v_\ell(r_{m_n}) r dr$$
$$\leq \sum_{n=0}^{\infty} c_4 c_5 c_6 \int_{r_{m_n}}^{r_{m_{n+1}}} M_1(f, r) v_\ell(r) r dr = c_4 c_5 c_6 \int_0^1 M_1(f, r) v_\ell(r) r dr$$

which yields the right-hand inequality of (ii).

To show the left-hand inequality of (ii) fix $1 < \rho < b$ (b of (3.1)) and $k_0 > 0$ such that $J_k/(\min(J_{k-1}, J_{k+1}) < \rho$ for all $k > k_0$ which is possible since the preceding left-hand side goes to 1 as $k \to \infty$ according to Proposition 3.1. Hence we find a constant $c_7 > 0$ with

$$J_k \leq c_7 \rho^{|k-n|} J_n \quad \text{for all } n \text{ and } k. \tag{3.6}$$

We have

$$\int_0^1 M_1(f, r) v_\ell(r) r dr \leq \sum_{n=0}^{\infty} \int_0^1 M_1(T_n f, r) v_\ell(r) r dr. \tag{3.7}$$

Moreover

$$\int_0^1 M_1(T_n f, r) v_\ell(r) r dr \leq \sum_{k=0}^{\infty} \int_{r_{m_k}}^{r_{m_{k+1}}} M_1(T_n f, r) v_\ell(r) dr$$
$$\leq \sum_{k<n} c_1 \left(\int_{r_{m_k}}^{r_{m_{k+1}}} \left(\frac{r}{r_{m_n}}\right)^{m_{n-1}} \frac{v_\ell(r)}{v_\ell(r_{m_n})} dr \right) M_1(T_n f, r_{m_n}) v_\ell(r_{m_n})$$
$$+ \sum_{k \geq n} \left(\int_{r_{m_k}}^{r_{m_{k+1}}} \left(\frac{r}{r_{m_n}}\right)^{m_{n+1}} \frac{v_\ell(r)}{v_\ell(r_{m_n})} dr \right) M_1(T_n f, r_{m_n}) v_\ell(r_{m_n}). \tag{3.8}$$

Here we used (3.2).

For further estimates we use the following well-known facts ([3], Lemma 3.7.): Fix k, n and $r_{m_k} \leq r \leq r_{m_{k+1}}$. Then

$$\left(\frac{r}{r_{m_n}}\right)^{m_n} \frac{v(r)}{v(r_{m_n})} \leq \left(\frac{1}{b}\right)^{n-k-1} \qquad \text{if } k < n \tag{3.9}$$

and

$$\left(\frac{r}{r_{m_n}}\right)^{m_{n+1}} \frac{v(r)}{v(r_{m_n})} \leq K \left(\frac{1}{b}\right)^{k-n-1} \qquad \text{if } k \geq n. \tag{3.10}$$

This implies

$$\left(\frac{r}{r_{m_n}}\right)^{m_{n-1}} \frac{v(r)}{v(r_{m_n})}$$
$$= \left(\frac{r_{m_{n-1}}}{r_{m_n}}\right)^{m_{n-1}} \frac{v(r_{m_{n-1}})}{v(r_{m_n})} \left(\frac{r}{r_{m_{n-1}}}\right)^{m_{n-1}} \frac{v(r)}{v(r_{m_{n-1}})} \leq K \left(\frac{1}{b}\right)^{n-k-2}$$

for all $k < n$. For $k < n-1$ this follows from (3.9) with $n-1$ instead of n. For $k = n-1$ this follows from (3.1) and

$$\left(\frac{r}{r_{m_{n-1}}}\right)^{m_{n-1}} \frac{v(r)}{v(r_{m_{n-1}})} \leq 1.$$

Moreover, (3.6) yields $J_k \le c_7\rho^{n-k}J_n$. If $k \ge n$ then we can apply (3.10) directly and we have $J_k \le c_7\rho^{k-n}J_n$. Hence

$$\sum_{k<n}\left(\int_{r_{m_k}}^{r_{m_{k+1}}}\left(\frac{r}{r_{m_n}}\right)^{m_{n-1}}\frac{v_\ell(r)}{v_\ell(r_{m_n})}dr\right)+\sum_{k\ge n}\left(\int_{r_{m_k}}^{r_{m_{k+1}}}\left(\frac{r}{r_{m_n}}\right)^{m_{n+1}}\frac{v_\ell(r)}{v_\ell(r_{m_n})}dr\right)$$
$$\le \sum_{k<n}c_7Kb\rho\left(\frac{\rho}{b}\right)^{n-k-1}J_n+\sum_{k<n}c_7K\rho\left(\frac{\rho}{b}\right)^{k-n-1}J_n \le c_8J_n$$

for some constant c_8 since $0<\rho/b<1$. Hence, by (3.8),

$$\int_0^1 M_1(T_nf,r)v_\ell(r)dr \le c_8J_nM_1(T_nf,r_{m_n})v_\ell(r_{m_n}).$$

Inserting this into (3.7) yields the left-hand inequality of (ii). $\square$

Corollary 3.4. *Let f be a harmonic function on $\mathbb{D}$. Assume that*

$$\sup_n M_\infty(T_nf,r_{m_n})v_\ell(r_{m_n})<\infty.$$

Then $f\in h^\infty_{v_\ell}$.

Proof. Let $f_s(z)=f(sz)$, $z\in\mathbb{D}$, for fixed $s\in]0,1[$. Then $f_s\in h^\infty_{v_\ell}$ for all s. Apply Proposition 3.3. (i) to f_s and use $\sup_s M_\infty(f_s,r)=M_\infty(f,r)$. $\square$

The last result of this section is known (see Theorem 1 in [16] for the case $1\le p<\infty$ and Theorem 2.1 of [15] for $p=\infty$) but for the sake of completeness we show how it also follows from the previous considerations.

Corollary 3.5. *The Riesz projection R is a bounded operator $h^p_{v_\ell}\to A^p_{v_\ell}$ for all p with $1\le p<\infty$ and also a bounded operator $h^\infty_{v_\ell}\to H^\infty_{v_\ell}$.*

Proof. For $1<p<\infty$ the result follows from the fact that there are universal constants $c_p>0$ such that

$$M_p(Rf,r)\le c_pM_p(f,r) \quad \text{for all } f\in h^p_{v_\ell} \quad \text{and } 0\le r<1.$$

and from Proposition 3.2.

To prove the corollary for $p=1$ and $p=\infty$ we infer from Proposition 3.1. that

$$0<\inf_n\frac{m_{n+1}-m_n}{m_n-m_{n-1}}\le\sup_n\frac{m_{n+1}-m_n}{m_n-m_{n-1}}<\infty.$$

We apply Lemma 3.3. of [13] and Lemma 4.1. of [9]. This yields constants c_p, $p=1,\infty$, such that

$$M_p(RT_nf,r)\le c_pM_p(T_nf,r) \quad \text{for all } f\in h^p_{v_\ell}, n=0,1,2,\ldots, \text{ and } 0\le r<1$$

where T_n are the operators of Proposition 3.3.. Hence Proposition 3.3. shows that the Riesz projection is also bounded for the exponential weights in the remaining cases $p=1,\infty$. $\square$

4. Estimates of the reproducing kernel.

We continue with a proposition, which can be conveniently proven by using the results of Section 3. In this way we will later avoid cumbersome proofs which would involve a direct estimation of the maximum point of the function $r^k v_\ell(r)$. Again, let $v(r) = \exp(-\alpha/(1-r)^\beta)$ and put $v_\ell(r) = v(r^\ell)$ for some $\ell > 0$.

Proposition 4.1. *There are constants $c_1, c_2 > 0$ such that*

$$c_1 k^{-\frac{2+\beta}{2+2\beta}} \sup_r r^k v_\ell(r) \le \int_0^1 r^{k+1} v_\ell(r) dr \le c_2 k^{-\frac{2+\beta}{2+2\beta}} \sup_r r^k v_\ell(r) \tag{4.1}$$

for every $k \ge 2\ell - 1$.

Proof. Fix $k \ge 1$ and at first consider $\ell = 1$. According to Lemma 2.1 we have $\int_0^1 r^k v(r) dr \cong \int_0^1 r^{k+1} v(r) dr \cong \int_0^1 r^{[k]} v(r) dr$, and, of course, $\sup_r r^k v(r) \cong \sup_r r^{k+1} v(r) \cong \sup_r r^{[k]} v(r)$. So it suffices to assume that k is an even integer. Put $f(z) = z^{k/2}$ and assume that $m_n \le k \le m_{n+1}$. Then $\|f\|_{2,v}^2 = \int_0^1 r^{k+1} v(r) dr$ and

$$M_2^2\big((Q_{[m_{n+1}/2]} - Q_{[m_n/2]})f, r_{m_n}\big) = r_{m_n}^k.$$

In view of Proposition 3.1. there are constants $d_1, d_2 > 0$ such that

$$d_1 n^{2+2/\beta} \le k \le d_2 n^{2+2/\beta}. \tag{4.2}$$

Moreover an application of Taylor's theorem reveals that

$$d_3 n^{-1-2/\beta} \le r_{m_{n+1}} - r_{m_n} \le d_4 n^{-1-2/\beta}$$

for some constants $d_3, d_4 > 0$ which implies with (4.2)

$$d_5 k^{-\frac{2+\beta}{2+2\beta}} \le r_{m_{n+1}} - r_{m_n} \le d_6 k^{-\frac{2+\beta}{2+2\beta}}. \tag{4.3}$$

By Proposition 3.2. for $p = 2$ we have

$$c_3^2 \int_0^1 r^{k+1} v(r) dr \le r_{m_n}^k v(r_{m_n})(r_{m_{n+1}} - r_{m_n}) \le c_4^2 \int_0^1 r^{k+1} v(r) dr$$

so that (4.3) implies

$$\int_0^1 r^{k+1} v(r) dr \cong r_{m_n}^k v(r_{m_n}) k^{-\frac{2+\beta}{2+2\beta}}.$$

We clearly have, by the choice (maximum property) of the number r_k,

$$r_{m_n}^k v(r_{m_n}) \le r_k^k v(r_k). \tag{4.4}$$

Conversely,

$$\frac{r_k^k v(r_k)}{r_{m_n}^k v(r_{m_n})} = \left(\frac{r_{m_n}}{r_k}\right)^{m_{n+1}-k} \left(\frac{r_k}{r_{m_n}}\right)^{m_{n+1}} \frac{v(r_k)}{v(r_{m_n})} \le \frac{r_k^{m_{n+1}} v(r_k)}{r_{m_n}^{m_{n+1}} v(r_{m_n})}$$
$$\le \left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})} < K.$$

Here we used, consequently, the fact that $r_k \ge r_{m_n}$, the choice of the number $r_{m_{n+1}}$ and (3.1). Thus

$$r_k^k v(r_k) \le K r_{m_n}^k v(r_{m_n}). \tag{4.5}$$

The inequalities (4.4) and (4.5) together with the preceding prove the proposition for $\ell = 1$. The general case follows by the same substitution as in (2.4). $\square$

Lemma 4.2. *We have*

$$\sum_{k=0}^{\infty} \frac{t^k}{\int_0^1 r^{2k+1} v_\ell(r)dr} < \infty \quad \text{for every} \quad t \in]0,1[.$$

Proof. By Lemma 2.1, there are constants $c, c_1 > 0$ such that $\int_0^1 r^{2k+1} v_\ell(r)dr \geq c_1 \exp(-ck^{\beta/(\beta+1)})$. This implies for all k

$$\frac{t^k}{\int_0^1 r^{2k+1} v_\ell(r)dr} \leq \exp\left(-k\left(\log(1/t) - ck^{-\frac{1}{\beta+1}}\right)\right).$$

For suitable k_0 we obtain

$$\log(1/t) - ck^{-\frac{1}{\beta+1}} \geq \frac{\log(1/t)}{2} \quad \text{if} \ \ k \geq k_0.$$

Hence the series in the statement of the lemma converges. □

Now we turn to the proof of Theorem 1.3. Let $w(r) = \exp\left(-\tilde{\alpha}/(1-r)^\beta\right)$ for given constants $\tilde{\alpha} > 0$ and $\beta > 0$ and put $w_\ell(r) = w(r^\ell)$ for a fixed $\ell > 0$. We study the reproducing kernel

$$K_\ell(z,\zeta) = \sum_{n=0}^{\infty} \frac{(z\bar{\zeta})^n}{\int_0^1 r^{2n+1} w_\ell(r)dr}. \tag{4.6}$$

For the Bergman projection P_{w_ℓ} we obtain

$$P_{w_\ell} f(z) = \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} K_\ell(z, re^{i\varphi}) f(re^{i\varphi}) r w_\ell(r) d\varphi dr.$$

We use the following special case of a more general result in [5], Lemma 4.

Lemma 4.3. *Assume that*

$$\sup_{z\in\mathbb{D}} \int_0^1 \int_0^{2\pi} |K_\ell(z, re^{i\varphi})| \exp\Big(-\frac{\tilde{\alpha}}{2}\Big(\frac{1}{(1-r^\ell)^\beta} + \frac{1}{(1-|z|^\ell)^\beta}\Big)\Big) d\varphi r dr < \infty. \tag{4.7}$$

Then $P_{w_\ell} : L^p_{w_\ell^{p/2}} \to A^p_{w_\ell^{p/2}}$ *is bounded for* $1 \leq p < \infty$. *Moreover,* $P_{w_\ell} : L^\infty_{w_\ell^{1/2}} \to H^\infty_{w_\ell^{1/2}}$ *is bounded.*

Notice that, if $\tilde{\alpha} = 2\alpha/p$ for some $\alpha > 0$ then $w_\ell^{p/2}(r) = \exp(-\alpha/(1-r^\ell)^\beta)$.

Lemma 4.4. *Let*

$$H(te^{i\varphi}) = \frac{1}{\int_0^1 r w_\ell(r)dr} + \sum_{k=1}^{\infty} \left(\frac{t^k}{\int_0^1 r^{2k+1} w_\ell(r)dr}\right) \left(e^{ik\varphi} + e^{-ik\varphi}\right).$$

Then there is a constant $c > 0$ *such that*

$$\int_0^{2\pi} |H(te^{i\varphi})| d\varphi \leq c \sum_{k=0}^{\infty} \left(\frac{1}{k+1}\right) \left(\frac{t^k}{\int_0^1 r^{2k+1} w_\ell(r)dr}\right)$$

whenever $0 \leq t < 1$,

Proof. Put

$$a_k = \frac{1}{\int_0^1 r^{2k+1} w_\ell(r)dr}, \quad k = 0, 1, 2, \ldots,$$

and $a_{-1} = 0$. Then $a_k \le a_{k+1}$ for all k. We have

$$H(re^{i\varphi}) = \sum_{k=0}^{\infty}(a_k - a_{k-1})\sum_{j=k}^{\infty} t^j(e^{ij\varphi} + e^{-ij\varphi}).$$

Indeed, in view of Lemma 4.2., the preceding series converges absolutely whenever $0 \le t < 1$, and we have

$$\sum_{k=0}^{\infty}(a_k - a_{k-1})\sum_{j=k}^{\infty} t^j(e^{ij\varphi} + e^{-ij\varphi})$$
$$= \sum_{j=0}^{\infty}\sum_{k=0}^{j}(a_k - a_{k-1})t^j(e^{ij\varphi} + e^{-ij\varphi}) = \sum_{j=0}^{\infty} a_j t^j(e^{ij\varphi} + e^{-ij\varphi}).$$

Put $f_k(te^{i\varphi}) = \sum_{j=k}^{\infty} t^j(e^{ij\varphi} + e^{-ij\varphi})$. Then, for any $s > t$ we have

$$M_1(f_k, t) \le c_1 \left(\frac{t}{s}\right)^k M_1(f_k, s)$$

where c_1 is a universal constant ([12], Lemma 3.1.). Put

$$d_k(e^{i\varphi}) = \sum_{j=0}^{k-1} e^{ij\varphi} + \sum_{j=1}^{k-1} e^{-ij\varphi}.$$

Then d_k is the Dirichlet kernel and we obtain

$$\int_0^{2\pi} |d_k(e^{i\varphi})| d\varphi \le c_2 \log(k+2)$$

for some constant c_2 independent of k. Let

$$p_s(e^{i\varphi}) = \sum_{k=0}^{\infty} s^k e^{ik\varphi} + \sum_{k=1}^{\infty} s^k e^{-ik\varphi}$$

so that p_s is the Poisson kernel which implies

$$\int_0^{2\pi} |p_s(e^{i\varphi})| d\varphi = 2\pi \quad \text{for every } s \in [0, 1[.$$

Hence $f_k(se^{i\varphi}) = p_s(e^{i\varphi}) - (d_k * p_s)(e^{i\varphi})$ which implies, for $0 \le t < s < 1$,

$$M_1(f_k, t) \le c_2 \left(\frac{t}{s}\right)^k M_1(p_s - (d_k * p_s), s) \le c_3 \left(\frac{t}{s}\right)^k (1 + \log(k+2))$$
$$\le c_4 \left(\frac{t}{s}\right)^k \log(k+2)$$

for some universal constants c_3, c_4. Since s was arbitrary with $t < s < 1$ we obtain

$$M_1(f_k, t) \le c_4 t^k \log(k+2) \quad \text{for all } k.$$

Hence

$$\int_0^{2\pi} |H(te^{i\varphi})| d\varphi \le c_4 \sum_{k=0}^{\infty}(a_k - a_{k-1}) t^k \log(k+2)$$
$$= c_4 \sum_{k=0}^{\infty} a_k t^k (\log(k+2) - \log(k+1)) \le c \sum_{k=0}^{\infty} \left(\frac{1}{k+1}\right) a_k t^k$$

for some universal constant c. $\square$

Proof of Theorem 1.3. We show (4.7) of Lemma 4.3. To this end put

$$\tilde{K}_\ell(se^{i\psi}, re^{i\varphi}) = \frac{1}{\int_0^1 rw_\ell(r)dr} + \sum_{k=1}^{\infty} \left(\frac{s^k t^k}{\int_0^1 r^{2k+1} w_\ell(r)dr)} \right) \left(e^{ik(\psi-\varphi)} + e^{-ik(\psi-\varphi)} \right). \tag{4.8}$$

Hence we obtain with the Riesz projection R,

$$K_\ell(se^{i\psi}, re^{i\varphi}) = R\tilde{K}_\ell(se^{i\psi}, re^{i\varphi}) = RH(rse^{i(\psi-\varphi)})$$

where H is the function of Lemma 4.4. The function $g_s(re^{i\theta}) = H(rse^{i\theta})$ is an element of $h^1_{w_\ell^{1/2}}$ since $0 \leq s < 1$. Hence we obtain, in view of Corollary 3.5. and Lemma 4.4.,

$$\int_0^1 \int_0^{2\pi} |K_\ell(se^{i\psi}, re^{i\varphi})| d\varphi w_\ell^{1/2}(r) r dr = \int_0^1 \int_0^{2\pi} |RH(rse^{i(\psi-\varphi)})| d\varphi w_\ell^{1/2}(r) r dr$$

$$\leq c_1 \int_0^1 \int_0^{2\pi} |H(rse^{i\theta})| d\theta w_\ell^{1/2}(r) r dr \leq c_2 \sum_{k=0}^{\infty} \left(\frac{\int_0^1 r^{k+1} w_\ell^{1/2}(r) dr}{\int_0^1 r^{2k+1} w_\ell(r) dr} \right) \frac{s^k}{k+1}$$

where c_1, c_2 are universal constants. Proposition 4.1 yields

$$\frac{\int_0^1 r^{k+1} w_\ell^{1/2}(r)dr}{\int_0^1 r^{2k+1} w_\ell(r)dr} \leq c_3 \frac{\sup_r r^{k+1} w_\ell^{1/2}(r)}{\sup_r r^{2k+1} w_\ell(r)} \leq c_4 \frac{\sup_r r^k w_\ell^{1/2}(r)}{\sup_r r^{2k} w_\ell(r)}$$

for some constants c_3, c_4. Hence we arrive at

$$\int_0^1 \int_0^{2\pi} |K_\ell(se^{i\psi}, re^{i\varphi})| w_\ell^{1/2}(r) r d\varphi dr \leq c_5 \sum_{k=0}^{\infty} \left(\frac{1}{k+1} \right) \frac{\sup_r r^k w_\ell^{1/2}(r)}{\sup_r r^{2k} w_\ell(r)} s^k \tag{4.9}$$

for some constant c_5.

Now put

$$g(se^{i\psi}) = \sum_{k=0}^{\infty} \left(\frac{1}{k+1} \right) \frac{\sup_r r^k w_\ell^{1/2}(r)}{\sup_r r^{2k} w_\ell(r)} s^k e^{ik\psi}.$$

Then we easily see that

$$M_\infty(g, s) = \sum_{k=0}^{\infty} \left(\frac{1}{k+1} \right) \frac{\sup_r r^k w_\ell^{1/2}(r)}{\sup_r r^{2k} w_\ell(r)} s^k.$$

So, (4.9) implies that

$$\sup_s \int_0^1 \int_0^{2\pi} |K_\ell(se^{i\psi}, re^{i\varphi})| v_\ell(r) r d\varphi dr \, w_\ell^{1/2}(s) \leq \|g\|_{\infty, w_\ell^{1/2}} \tag{4.10}$$

In order to show that $\|g\|_{\infty, w_\ell^{1/2}}$ is bounded we apply Corollary 3.4. We have

$$M_\infty(T_n g, r_{m_n}) \leq \sum_{k=[m_{n-1}]}^{[m_{n+1}]} \gamma_k \left(\frac{1}{k+1} \right) \frac{\sup_r r^k w_\ell^{1/2}(r)}{\sup_r r^{2k} w_\ell(r)} r_{m_n}^k \tag{4.11}$$

for some $\gamma_k \in [0, 1]$. Here

$$m_n = \ell\beta^2 \left(\frac{2\beta}{\tilde{\alpha}} \right)^{\frac{1}{\beta}} n^{2+\frac{2}{\beta}} - \ell\beta^2 n^2, \quad \text{hence} \quad m_{n+1} - m_{n-1} \leq c_6 n^{1+\frac{2}{\beta}} \leq c_7 m_{n-1},$$

where c_6, c_7 are universal constants. Thus, (4.11) together with $v_\ell(r) \le w_\ell(r)^{1/2}$, see (1.4), and $(r^k w_\ell^{1/2}(r))^2 = r^{2k} w_\ell(r)$ yield

$$M_\infty(T_n g, r_{m_n}) v_\ell(r_{m_n}) \le c_6 \sup_{m_{n-1} \le k \le m_{n+1}} \begin{pmatrix} m_{n-1} \\ k+1 \end{pmatrix} \frac{\sup_r r^k w_\ell^{1/2}(r)}{\sup_r r^{2k} w_\ell(r)} r_{m_n}^k w_\ell^{1/2}(r_{m_n})$$

$$\le c_6 \frac{\sup_r r^k w_\ell^{1/2}(r) \sup_s s^k w_\ell^{1/2}(s)}{\sup_r r^{2k} w_\ell(r)} \le c_6.$$

So Corollary 3.4. shows that $\|g\|_{\infty, w_\ell^{1/2}} < \infty$. Finally, (4.10) shows that (4.7) is satisfied. □

We conclude the article with an extension of the results of Section 1 to weighted spaces of harmonic functions.

Theorem 4.5. *Let*

$$v_\ell(r) = \exp\Big(-\frac{\alpha}{(1-r^\ell)^\beta}\Big) \quad \textit{and} \quad w_\ell(r) = \exp\Big((-\frac{\tilde\alpha}{(1-r\ell)^{\tilde\beta}}\Big)$$

for some $\alpha, \tilde\alpha, \beta, \tilde\beta, \ell > 0$. Let $\tilde P_{w_\ell} : L^2_{w_\ell} \to h^2_{w_\ell}$ be the orthogonal projection. Then $\tilde P_{w_\ell}$ is a bounded operator $L^p_{v_\ell} \to h^p_{v_\ell}$ if and only if $\beta = \tilde\beta$ and $\tilde\alpha = 2\alpha/p$ in the case $1 \le p < \infty$, and $\beta = \tilde\beta$ and $\tilde\alpha = 2\alpha$ in the case $p = \infty$.

Proof. We have with (4.8)

$$\tilde P_{w_\ell}(z) = \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} \tilde K_\ell(z, re^{i\varphi}) f(re^{i\varphi}) r w_\ell(r) d\varphi dr.$$

Hence, in view of (4.6), $R\tilde P_{w_\ell} = P_{w_\ell}$ and $SP_{w_\ell}Sf = (\mathrm{id} - R)\tilde P_{w_\ell} f + (P_{w_\ell} f)(0)$ where $(Sf)(z) = f(\bar z)$. Since the Riesz projection and S are bounded we can easily transfer the results of Theorems 1.1., 1.3. and 1.2. to the space of harmonic functions. □

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