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Sylow permutable subnormal subgroups of finite groups

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Dedicated to John Cossey on the occasion of his sixtieth birthday

Abstract

An extension of the well-known Frobenius’ criterion of $p$-nilpotence in groups with modular Sylow $p$-subgroups is proved in the paper. This result is useful to get information about the classes of groups in which every subnormal subgroup is permutable and Sylow permutable.

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1 Introduction and statements of results

Throughout the paper, the word group means finite group.

A celebrated theorem of Frobenius ([9, Satz IV.5.8]) asserts that if \( p \) is a prime and \( G \) is a group such that \( N_G(H) \) is \( p \)-nilpotent for every \( p \)-subgroup \( H \) of \( G \), then \( G \) is \( p \)-nilpotent.

Our first main result can be considered as an extension of Frobenius’ theorem in groups with modular Sylow \( p \)-subgroups.

**Theorem 1.** Let \( p \) be a prime and let \( G \) be a group with a modular Sylow \( p \)-subgroup \( P \). Then \( G \) is \( p \)-nilpotent if and only if \( N_G(P) \) is \( p \)-nilpotent.

This result turns out to be useful to study the classes of PST-groups and PT-groups.

Recall that a subgroup \( H \) of a group \( G \) is said to be S-permutable (or S-quasinormal, or \( \pi \)-quasinormal) in \( G \) if \( HP = PH \) for all Sylow subgroups \( P \) of \( G \). It is clear that S-permutability is weaker than permutability and normality. According to a theorem of Kegel [10, Satz 1], every S-permutable subgroup is subnormal. S-permutability, like normality and permutability, is not a transitive relation.

We say that a group \( G \) is a PST-group if S-permutability is transitive in \( G \), that is, if \( A \) is an S-permutable subgroup of \( B \) and \( B \) is an S-permutable subgroup of \( G \), then \( A \) is S-permutable in \( G \). Applying Kegel’s theorem, PST-groups are exactly the groups in which every subnormal subgroup is S-permutable. This class contains the class of all groups in which normality is transitive (\( T \)-groups) and the class of all groups in which permutability is transitive (PT-groups). The last two classes have been widely studied ([1, 4, 6, 7, 10, 11, 15]).

The structure of soluble PST-groups was obtained by Agrawal in [1]. It is proved there that a group \( G \) is a soluble PST-group if and only if \( G \) has an abelian normal Hall subgroup of odd order \( N \) such that \( G/N \) is nilpotent and the elements of \( G \) induce power automorphisms in \( N \). In that result, if we force \( G/N \) to be a Dedekind group, we find Gaschütz’s characterisation of soluble \( T \)-groups ([7]), and if we impose that \( G/N \) is a nilpotent modular group, then we obtain Zacher’s characterisation of soluble PT-groups ([15]).

The above results show that, in the soluble universe, the difference between these three classes is simply the Sylow structure. Our second result supports that claim and provides a unified viewpoint for the classes of PST, PT and \( T \)-groups in the general finite case.

**Theorem 2.** Let \( G \) be a group.
1. Suppose that $p$ is a prime number and that $H$ is an $S$-permutable $p$-subgroup of $G$. If the Sylow $p$-subgroups of $G$ are modular (respectively, Dedekind), then $H$ is permutable (respectively, normal) in $G$.

2. Assume that $H$ is an $S$-permutable subgroup of $G$. If the Sylow subgroups of $G$ are modular (respectively, Dedekind), then $H$ is permutable (respectively, normal) in $G$.

Taking this result into account, it seems natural to look for characterisations of the above classes in terms of the Sylow structure. This was done by Robinson ([11]) for the class of $T$-groups and by Beidleman, Brewster and Robinson ([4]) for the class of $PT$-groups.

One of the purposes of this paper is to provide necessary and sufficient conditions on the Sylow structure for a group to be a soluble $PST$-group.

As in the $PT$ and $T$-cases, the procedure of defining local versions in order to simplify the study of the global properties has revealed itself as considerably useful.

Since our approach depends heavily on a previous analysis of the classes of $PT$-groups and $T$-groups, the following definition needs to be stated.

**Definition 1.** Let $G$ be a group and $p$ a prime. We say that $G$:

1. Enjoys property $C_p$ (see [11]) if each subgroup of a Sylow $p$-subgroup $P$ of $G$ is normal in the normaliser $N_G(P)$.

2. Satisfies property $X_p$ (as in [4]) if each subgroup of a Sylow $p$-subgroup $P$ of $G$ is permutable in the normaliser $N_G(P)$.

Robinson ([11]) proved that a group $G$ is a soluble $T$-group if and only if $G$ satisfies property $C_p$ for all primes $p$ and, thirty-one years later, Beidleman, Brewster and Robinson proved that $G$ is a soluble $PT$-group if and only if $G$ satisfies property $X_p$ for all primes $p$.

These results would follow easily if one could prove that $C_p$ and $X_p$ are subgroup-closed. The subgroup-closed character of $C_p$ follows from the abnormality of the Sylow normalisers. Nevertheless, in the Beidleman, Brewster and Robinson approach, the subgroup-closed character of $X_p$ follows after an intensive study of the property $X_p$ and its consequences for the group structure (see [4, Corollary 3]). In the following, we show that the subgroup-closed character of the property $X_p$ follows as a natural consequence of Theorem 1 and a new property called $Y_p$, which can be considered as the “$PST$-version” of the properties $C_p$ and $X_p$.

**Definition 2.** Let $p$ be a prime number. A group $G$ is said to be a $Y_p$-$group$ when for all $p$-subgroups $H$ and $S$ of $G$ such that $H \leq S$, $H$ is $S$-permutable in $N_G(S)$.
The above property can be compared to property $S_p$ introduced by Beidleman and Heineken in [5].

**Theorem 3.** A group $G$ satisfies $X_p$ (respectively, $C_p$) if and only if $G$ satisfies $Y_p$ and the Sylow $p$-subgroups of $G$ are modular (respectively, Dedekind).

Since $Y_p$ is subgroup-closed, this result has the virtue of showing that the subgroup-closed character of $X_p$ depends exclusively on the modularity of the Sylow $p$-subgroups. It also shows that, in order to get a global characterisation of the soluble PST-groups, it is necessary to impose the subgroup-closed character in the definition $Y_p$, as in the PST-case there are no restrictions on the Sylow $p$-subgroups.

Assume that $G$ is a solubpe PST-group. If $H$ and $S$ are $p$-subgroups of $G$ such that $H \leq S$, then $H$ is subnormal in $N_G(S)$. Now, by Agrawal’s Theorem, $N_G(S)$ is a PST-group. Therefore $H$ is $S$-permutable in $N_G(S)$. Consequently every soluble PST-group has property $Y_p$. Our next result confirms that the converse is also true.

**Theorem 4.** A group $G$ is a soluble PST-group if and only if $G$ satisfies $Y_p$ for all primes $p$.

Note that Theorem A of [4] is a consequence of Theorems 3 and 4.

One of the main results of [4] is that a group $G$ satisfies $X_p$ if and only if $G$ has modular Sylow $p$-subgroups and either $G$ is $p$-nilpotent or a Sylow $p$-subgroup $P$ of $G$ is abelian and $G$ satisfies $C_p$.

This result is a consequence of Theorem 3 and the following:

**Theorem 5.** A group $G$ is a $Y_p$-group if and only if $G$ is either $p$-nilpotent, or $G$ has abelian Sylow $p$-subgroups and $G$ satisfies $C_p$.

Theorem C of [4] follows from Theorem 3 and

**Corollary 1.** If $p$ is the smallest prime divisor of the order of $G$, then $G$ is a $Y_p$-group if and only if $G$ is $p$-nilpotent.

Theorem 5 has revealed itself to be useful to prove some interesting results on PST-groups. For instance, it is proved in [5, Theorem H] that a soluble group $G$ is a PST-group if and only if every subnormal subgroup permutes with every Carter subgroup of $G$ and the subnormal subgroups are hypercentrally embedded in $G$. As an application of Theorem 5, we prove in [3, Corollary 2] that the permutability with the Carter subgroups can be removed.

**Theorem 6 ([3]).** A soluble group $G$ is a PST-group if and only if every subnormal subgroup of $G$ is hypercentrally embedded in $G$. 
Another application of Theorem 5 is the following structure theorem for $p$-soluble groups with property $\mathcal{Y}_p$.

**Theorem 7** ([3]). A $p$-soluble group has property $\mathcal{Y}_p$ if and only if

1. either $G$ is $p$-nilpotent, or

2. $G(p)/O_{p'}(G(p))$ is an abelian normal Sylow $p$-subgroup of $G/O_{p'}(G(p))$ such that the elements of $G/O_{p'}(G(p))$ induce power automorphisms in $G(p)/O_{p'}(G(p))$.

Here, $G(p)$ denotes the $p$-nilpotent residual of $G$, that is, the smallest normal subgroup of $G$ such that $G/G(p)$ is $p$-nilpotent.

The paper is organised as follows. In Section 2 we study property $\mathcal{Y}_p$ and its relation with the properties $\mathcal{C}_p$ and $\mathcal{X}_p$. The local approach to the class of soluble $PST$-groups developed in [2] plays an important role. The proofs of the main results appear in Section 3. Finally, we give some non-soluble examples of groups with property $\mathcal{Y}_p$ and a remark to show that any hope of creating a similar landscape out of the soluble universe which leads to a characterisation of $PST$, $PT$ and $T$-groups is soon dispelled.

## 2 The property $\mathcal{Y}_p$

In the sequel $p$ will be a fixed prime.

Our first result confirms the subgroup-closed character of the property $\mathcal{C}_p$. This is a consequence of the abnormality of the normalisers of the Sylow subgroups.

**Lemma 1.** $\mathcal{C}_p$ is inherited by subgroups.

**Proof.** Assume that $G$ has the property $\mathcal{C}_p$ and let $B$ a subgroup of $G$. If $C$ is a Sylow $p$-subgroup of $B$ and $D$ is contained in $C$, then $D$ is normal in $N_G(P)$ for every Sylow $p$-subgroup $P$ of $G$ containing $C$. Therefore if $g \in N_G(C)$, then $D$ is normal in $\langle N_G(P), N_G(P^{g^{-1}}) \rangle$. Since $N_G(P)$ is abnormal, it follows that $g^{-1} \in \langle N_G(P), N_G(P^{g^{-1}}) \rangle$ and so $g \in N_G(D)$. Therefore $D$ is normal in $N_G(C)$ and $B$ has property $\mathcal{C}_p$. \(\blacksquare\)

Bryce and Cossey ([6]) established local versions of some results of soluble $T$-groups. In particular, they characterised the soluble groups with the property $\mathcal{C}_p$ as the groups $G$ in which every $p'$-perfect subnormal subgroup of $G$ is normal in $G$.  

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Following Bryce and Cossey’s approach, it is proved in [2] that the soluble groups with property $X_p$ are those whose $p'$-perfect subnormal subgroups are permutable with the Hall $p'$-subgroups and the Sylow $p$-subgroups are modular (see [2, Theorems 6 and 7]). Then the following definition arose:

**Definition 3 ([2]).** We say that a group $G$ is a $PST_p$-group if $G$ is $p$-soluble and every $p'$-perfect subnormal subgroup is permutable with the Hall $p'$-subgroups of $G$.

According to [2, Theorem 8], a soluble group $G$ is a $PST$-group if and only if $G$ is a $PST_p$-group for all primes $p$.

We say that a group $G \in U^*_p$ if it is $p$-soluble, and the $p$-chief factors of $G$ are cyclic groups and are $G$-isomorphic when regarded as $G$-groups by conjugation.

In [2, Theorem 6] it is proved that a soluble group $G$ belongs to $PST_p$ if and only if $G \in U^*_p$. The arguments used there still hold in the $p$-soluble universe. Therefore we have:

**Theorem 8.** $PST_p = U^*_p$.

In [2, Lemma 2] it is proved that the class of the $PST_p$-groups is quotient-closed. Theorem 8 shows that this class is also subgroup-closed.

The characterisation of soluble $PST$-groups in terms of the Sylow structure follows from the following:

**Theorem 9.** A $p$-soluble group is a $PST_p$-group if and only if it satisfies $Y_p$.

We need the following elementary lemma.

**Lemma 2.** Let $G$ be a group.

1. If $G$ has property $Y_p$ and $A$ is a normal $p$-subgroup of $G$, then $G/A$ has property $Y_p$.

2. If $G$ has property $Y_p$ and $N$ is a normal $p'$-subgroup of $G$, then $G/N$ has property $Y_p$.

**Proof.** 1. This follows immediately from the definition.

2. Assume that $G$ has property $Y_p$ and let $H/N \leq S/N$ be $p$-subgroups of $G/N$. Then there exist Sylow $p$-subgroups $H_1$ and $S_1$ of $H$ and $S$, respectively, such that $H_1$ is contained in $S_1$ and $H = H_1 N$ and $S = S_1 N$. Since $G$ has $Y_p$, it follows that $H_1$ is $S$-permutable in $N_G(S_1)$. Therefore $H/N = H_1 N/N$ is $S$-permutable in $N_G(S_1) N/N = N_{G/N}(S/N)$. This implies that $G/N$ has $Y_p$. $\square$
Proof of Theorem 9. Assume that $G$ satisfies $\mathcal{Y}_p$. We prove that $G$ is a $\text{PST}_{p^r}$-group by induction on $|G|$. Denote $O_{p^r}(G)$ by $A$ and suppose that $A \neq 1$. Let $H$ be a $p'$-perfect subnormal subgroup of $G$ and let $B$ be a Hall $p'$-subgroup of $G$. Then $A \leq B$ and $B/A$ is a Hall $p'$-subgroup of $G/A$. Since $G/A$ is a $\text{PST}_{p^r}$-group, it follows that $HA/A$ permutes with $B/A$. Consequently $H$ permutes with $B$ and hence $G$ is a $\text{PST}_{p^r}$-group. Therefore we may assume that $A = O_{p^r}(G) = 1$.

Let $N$ be a minimal normal subgroup of $G$. Then $N$ is a $p$-group because $G$ is $p$-soluble. If $N_0$ is a subgroup of $N$, then $N_0$ is $S$-permutable in $N_G(N) = G$. This means that if $Q$ is a Sylow $q$-subgroup of $G$ for $q \neq p$, then $N_0$ is a Sylow $p$-subgroup of $N_0Q$ and so $Q$ normalises $N_0$.

Therefore $O^p(G)$ normalises every subgroup of $N$. Let $P$ be a Sylow $p$-subgroup of $G$ and let $N_1$ be a minimal normal subgroup of $P$ contained in $N$. Then $PO^p(G) = G$ normalises $N_1$ and so $N_1 = N$. This means that $N$ is cyclic of order $p$. By Lemma 2, we know that $G/N$ has $\mathcal{Y}_p$. Therefore $G/N$ is a $\text{PST}_{p^r}$-group by induction.

Applying Theorem 8, we have that $G/N$ is a $\mathcal{U}_p^*$-group. In particular, $G/N$ is $p$-supersoluble. Since $N$ is cyclic, it follows that $G$ is $p$-supersoluble. Then $G$ has a normal Sylow $p$-subgroup $P$ containing the derived subgroup $G'$ by [2, Lemma 1]. Let $H$ be a $p'$-perfect subnormal subgroup of $G$. Then $P \cap H$ is a normal Sylow $p$-subgroup of $H$ and so $P \cap H = H$ since $H$ is $p'$-perfect. Hence $H$ is a $p$-group and $H \leq P$. Therefore $H$ is $S$-permutable in $N_G(P) = G$. In particular, $H$ permutes with the Hall $p'$-subgroups of $G$. Therefore $G$ is a $\text{PST}_{p^r}$-group.

Conversely, suppose that $G$ is a $\text{PST}_{p^r}$-group. Suppose that $H$ and $S$ are $p$-subgroups of $G$ such that $H \leq S$. Then $H$ is a subnormal subgroup of $N_G(S)$, $H$ is $p'$-perfect and $N_G(S)$ is a $\text{PST}_{p^r}$-group because the class of $\text{PST}_{p^r}$-groups is subgroup-closed. Thus $H$ permutes with every Hall $p'$-subgroup $Q$ of $N_G(S)$ and $X = HQ$ is a subgroup of $G$. Then $H \leq O_p(X)$ and $O_p(X) = H(O_p(X) \cap Q) = H$. Therefore $H$ is normalised by $Q$. Consequently, $O^p(N_G(S))$ normalises $H$ and $G$ has $\mathcal{Y}_p$. \hfill \Box

Note that every $p$-nilpotent group is $\mathcal{U}_p^*$-group. Therefore by Theorem 8 and 9 we have:

**Corollary 2.** If $G$ is $p$-nilpotent, then $G$ has $\mathcal{Y}_p$.

Another relevant property of groups with $\mathcal{Y}_p$ is:

**Lemma 3.** If $G$ has $\mathcal{Y}_p$ and if $P$ is a non-abelian Sylow $p$-subgroup of $G$, then $N_G(P)$ is $p$-nilpotent.
Proof. Let $H$ be a subgroup of $P$. If $Q$ is a Sylow $q$-subgroup of $N_G(P)$ for a prime $p \neq q$, then $HQ$ is a subgroup of $G$. This implies that $H$ is a subnormal Sylow $p$-subgroup of $HQ$ and then $Q$ normalises $H$. Therefore every $p'$-element of $N_G(P)$ normalises every subgroup of $P$. Since $P$ is non-abelian, we can apply [8, Hilfssatz 5] to conclude that every $p'$-element of $N_G(P)$ actually centralises $P$. Consequently, $N_G(P)$ is $p$-nilpotent.

Our proof of Theorem 5 depends on the relation between $\mathcal{Y}_p$ and $p$-normality.

Recall that if $p$ is a prime, a group $G$ is said to be $p$-normal if it satisfies the following property:

If $P$ is a Sylow $p$-subgroup of $G$ and $Z(P)$ is contained in $P^g$ for some $g \in G$, then $Z(P) = Z(P^g)$.

This property is closely related to property $\mathcal{Y}_p$. In fact, we have:

Lemma 4. If $G$ satisfies $\mathcal{Y}_p$, then $G$ is $p$-normal.

Proof. Suppose that $G$ satisfies $\mathcal{Y}_p$. Let $P$ be a Sylow $p$-subgroup of $G$ and let $g$ be an element of $G$ such that $Z = Z(P) \leq P^g$. Suppose that $Z$ is not a normal subgroup of $P^g$. Then (see Burnside's Theorem, [9, Satz IV.5.1]) there exists an element $g \in G$ of order $q^b$ for a prime $q \neq p$ such that $g \notin N_G(Z)$, $J = ZZ^g \cdots Z^{q^b}g^{-1}$ is a $p$-group and $g \in N_G(J) \setminus C_G(J)$. But $g$ is a $p'$-element of $N_G(J)$ and $G$ is a $\mathcal{Y}_p$-group. Consequently $g$ induces a power automorphism on $J$. In particular, we get the contradiction $g \in N_G(Z)$.

Therefore $Z(P)$ is a normal subgroup of $P^g$. Then $Z(P^{g^{-1}}) = (Z(P))^g$ is a normal subgroup of $P$. By [9, Hilfssatz IV.5.2], since $Z(P)$ is a characteristic subgroup of $P$, we have that $Z(P) = Z(P^{g^{-1}})$ and $Z(P) = Z(P^g)$. That proves that $G$ is $p$-normal.

3 Proofs of the main results

The next result is the $p$-soluble version of Theorem 1.

Lemma 5. Let $p$ be a prime. Assume that $G$ is a $p$-soluble group with modular Sylow $p$-subgroups. If $P$ is a Sylow $p$-subgroup of $G$ such that $N_G(P)$ is $p$-nilpotent, then $G$ is $p$-nilpotent.

Proof. Assume the result is false and let $G$ be a counterexample of least order. Then for each non-trivial normal subgroup $N$ of $G$, it follows that $G/N$ is $p$-nilpotent. Therefore, since the class of $p$-nilpotent groups is a saturated
formation, it follows that $G$ has a unique minimal normal subgroup $N$ such that $N$ is an elementary abelian $p$-group, $C_G(N) = N$ and $N$ is complemented in $G$ by a core-free maximal subgroup $M$. It is clear that $N$ is contained in $P$. Suppose that $N$ is a proper subgroup of $P$. Then, since $G = NM$, we have that $P = N(P \cap M)$ and $P \cap M \neq 1$. Let $x \in P \cap M$ be an element of order $p$. If $n \in N$, then $\langle n, x \rangle = \langle n \rangle \langle x \rangle$ is an elementary abelian $p$-group because $P$ is modular. Therefore $x \in C_G(n)$. This implies that $1 \neq P \cap M \cap C_G(N) = P \cap M \cap N$, a contradiction. Hence $P = N$ and so $G = N_G(P)$ is $p$-nilpotent, final contradiction.

Proof of Theorem 1. Let $G$ be a group with modular Sylow $p$-subgroups and $p$-nilpotent Sylow normalisers with least order subject to not being $p$-nilpotent. Let $P$ be a Sylow $p$-subgroup of $G$. From Burnside’s $p$-nilpotence criterion ([9, Hauptsatz IV.2.6]) we have that $P$ is non-abelian. Assume that $P$ is Dedekind. Then $p = 2$ and $P$ is a direct product of a quaternion group and an elementary abelian $2$-group. Hence, if $\Omega_1(P)$ is the subgroup generated by the involutions of $P$, it follows that $\Omega_1(P) \leq Z(P)$. Suppose that $C_G(Z(P))$ is a proper subgroup of $G$. Then $C_G(Z(P))$ inherits the hypotheses of the theorem. By minimality of $G$, it follows that $C_G(Z(P))$ is $p$-nilpotent. The $p$-nilpotence of $G$ follows now from [16, Theorem 1]. Suppose that $C_G(Z(P)) = G$. Then $1 \neq Z(P)$ is central in $G$. From the minimality of $G$, it follows that $G/Z(P)$ is $p$-nilpotent and so $G$ is $p$-nilpotent, a contradiction.

Suppose now that $P$ is not Dedekind. Then, applying [14, Exercise 4.4.1], we have that $N = O^p(G) \neq G$. It is clear that $N_G(P) \leq N_G(P \cap N)$ and $P \cap N$ is a modular Sylow $p$-subgroup of $N$. Suppose that $P \cap N = 1$. Then $N$ is a normal Hall $p'$-subgroup of $G$ because $G = NP$. This implies that $G$ is $p$-nilpotent, a contradiction. Therefore $P \cap N \neq 1$. Suppose that $N_G(P \cap N) = G$. Then there exists a minimal normal subgroup $A$ of $G$ such that $A \leq P \cap N$. By minimality of $G$, it follows that $G/A$ is $p$-nilpotent. In particular, $G$ is $p$-soluble. Applying Lemma 5, we have that $G$ is $p$-nilpotent, a contradiction. Consequently $N_G(P \cap N)$ is a proper subgroup of $G$ and it inherits the properties of $G$. The minimal choice of $G$ implies that $N_G(P \cap N)$ is $p$-nilpotent. Then $N_N(P \cap N)$ is also $p$-nilpotent and so $N$ satisfies the hypotheses of the theorem. Since $N \neq G$, it follows that $N$ is $p$-nilpotent. Hence $G$ is $p$-nilpotent, a contradiction.

Proof of Theorem 2. 1. Let $A$ be a subgroup of $G$ and denote $T = \langle A, H \rangle$. Since $H$ is S-permutable in $T$, then $H$ is a subnormal subgroup of $T$ and $H$ is contained in $O_p(T)$, which is contained in every Sylow $p$-subgroup $P$ of $T$. Therefore $T = \langle H, A \rangle \leq (O_p(T), A) = O_p(T)A \leq T$. Let $A_q$ be a Sylow
\textit{q-subgroup of A} for a prime \( q \neq p \), and let \( G_q \) be a Sylow \( q \)-subgroup of \( G \) containing \( A_q \). We have that \( A_q \) is a Sylow \( q \)-subgroup of \( T \), and \( A_q = G_q \cap T \) because \( A_q \leq G_q \cap T \). Hence \( HA_q = H(G_q \cap T) = HG_q \cap T \) is a subgroup of \( T \). Moreover \( O_p(T) \cap HA_q = H \). Therefore \( H \) is normalised by \( A_q \). On the other hand, since \( P \) is modular (respectively, Dedekind), we have that \( H \) permutes with (respectively, is normalised by) a Sylow \( p \)-subgroup \( A_p \) of \( A \). Therefore \( H \) permutes with (respectively, is normalised by) all Sylow subgroups of \( A \). In particular, \( H \) permutes with \( A \) (respectively, \( H \) is normalised by \( A \)). This implies that \( H \) is a permutable (respectively, normal) subgroup of \( G \).

2. Suppose that \( G \) is a counterexample of minimal order to the theorem. Then there exists an S-permutable subgroup \( H \) of \( G \) such that \( H \) is not permutable (respectively, normal) in \( G \). We take \( H \) of minimal order. Let \( N \) be a minimal normal subgroup of \( G \). Since \( HN/N \) is S-permutable in \( G/N \), we have that \( HN/N \) is permutable (respectively, normal) in \( G/N \). Assume that \( \text{Core}_G(H) = H \neq 1 \). Then we may suppose that \( N \leq H \) and then \( H \) is permutable (respectively, normal) in \( G \), a contradiction. Therefore we have that \( H \) is a \( p \)-group for some prime \( p \). By 1, we conclude that \( H \) is a permutable (respectively, normal) subgroup of \( G \).

\textbf{Proof of Theorem 3.} Suppose that \( G \) satisfies \( \mathcal{Y}_p \) and a Sylow \( p \)-subgroup \( P \) of \( G \) is modular. By Theorem 2, we have that every subgroup of \( P \) is permutable in \( N_G(P) \).

Conversely, suppose that \( G \) satisfies \( \mathcal{X}_p \). Then it is clear that every Sylow \( p \)-subgroup \( P \) of \( G \) is modular. Moreover, by [4, Lemma 2], every subgroup of \( P \) is normalised by the \( p' \)-elements of \( N_G(P) \). Therefore, if \( P \) is abelian, every subgroup of \( P \) is normal in \( N_G(P) \) and then \( G \) satisfies property \( \mathcal{C}_p \).

Since \( \mathcal{C}_p \) is subgroup closed by Lemma 1, \( G \) satisfies \( \mathcal{Y}_p \).

If \( P \) is non-abelian, then \( N_G(P) \) is \( p \)-nilpotent by [4, Corollary 2]. By Theorem 1, we have that \( G \) itself is \( p \)-nilpotent. Let \( H \leq S \) be \( p \)-subgroups of \( G \). Then \( N_G(S) \) is \( p \)-nilpotent. Therefore \( H \) is centralised by each \( p' \)-element of \( N_G(S) \). This implies that \( H \) is S-permutable in \( N_G(S) \). Consequently \( G \) has \( \mathcal{Y}_p \).

\textbf{Proof of Theorem 4.} Assume that \( G \) is a soluble \( PST \)-group. Then \( G \) is a \( p \)-soluble \( PST_p \)-group for all primes \( p \) by [2, Theorem 8]. By Theorem 9, it follows that \( G \) is a \( \mathcal{Y}_p \)-group for all primes \( p \).
Conversely, suppose that $G$ satisfies $\mathcal{Y}_p$ for all primes $p$. Then every subgroup of $G$ has the same property. Therefore if $G$ is a group with least order subject to not being a soluble $PST$-group, then every proper subgroup of $G$ is a soluble $PST$-group. According to Agrawal’s Theorem, every soluble $PST$-group is supersoluble. Therefore either $G$ is supersoluble, or $G$ is a minimal non-supersoluble group. In both cases, we have that $G$ is soluble (the solubility of $G$ follows from [9, Satz VI.9.6] in the second case). Since $\mathcal{Y}_p$ coincides with $PST_p$ in the $p$-soluble universe by Theorem 9, it follows that $G$ is a $PST_p$-group for all $p$. Then $G$ is a $PST$-group by [2, Theorem 8].

**Proof of Theorem 5.** Suppose that $G$ is $p$-nilpotent. Then $G$ satisfies $\mathcal{Y}_p$ by Corollary 2. Assume now that $G$ has abelian Sylow $p$-subgroups and that $G$ satisfies $C_p$. It follows from Theorem 3 that $G$ satisfies $\mathcal{Y}_p$.

Assume that the converse is not true and let $G$ be a counterexample of minimal order. If $G$ had an abelian Sylow $p$-subgroup, then $G$ would satisfy $C_p$ by Theorem 3. Therefore $G$ has a non-abelian Sylow $p$-subgroup $P$ and $G$ is not $p$-nilpotent. Suppose that $P_G = \text{Core}_G(P) = 1$. Therefore $N_G(Z(P))$ is a proper subgroup of $G$. Hence $N_G(Z(P))$ is $p$-nilpotent by the minimal choice of $G$. Applying Lemma 4 and [12, Exercise 594], we have that $G$ is $p$-nilpotent, a contradiction.

Consequently $P_G \neq 1$. Let $N$ be a minimal normal subgroup of $G$ contained in $P$. Since $G$ has minimal order and $G/N$ is a $\mathcal{Y}_p$-group, it follows that either $G/N$ is $p$-nilpotent or $P/N$ is abelian.

Suppose that $P/N$ is abelian. Since $P$ is non-abelian, then $N_G(P)$ is $p$-nilpotent by Lemma 3, and so $N_G(P)/N = P/N \times O_p(N_G(P))/N$ and $P/N$ lies in the center of $N_{G/N}(P/N)$. From Burnside’s $p$-nilpotence criterion (see [9, Hauptsatz IV.2.6]), we have that $G/N$ is $p$-nilpotent. But if $G/N$ is $p$-nilpotent, bearing in mind that $G$ is a $\mathcal{Y}_p$-group and hence a $U^*_p$-group by Theorems 8 and 9, we have that $|N| = p$ and $p$ divides $|G/N|$ (otherwise, $G$ would have an abelian Sylow $p$-subgroup $N = P$). It follows that $G$ is $p$-nilpotent, because $G$ acts centrally on the chief $p$-factors of $G/N$ and hence $G$ must act centrally on $N$. This contradiction proves the theorem.

**Proof of Corollary 1.** Suppose that $G$ is a non-$p$-nilpotent $\mathcal{Y}_p$-group of minimal order. Since all proper subgroups of $G$ satisfy $\mathcal{Y}_p$, from the minimality it follows that all the proper subgroups of $G$ are $p$-nilpotent. From Itô’s Theorem (see [9, Satz IV.5.4]), we have that $G$ has a normal Sylow $p$-subgroup $P$. But from Lemma 3, we have that $G = N_G(P)$ is $p$-nilpotent, a contradiction.
4 Examples and remarks

1. Property $\mathcal{Y}_p$ does not imply property $PST_p$ in general. The alternating group $G = A_5$ of degree 5 has Sylow 3-subgroups of order 3 and Sylow 5-subgroups of order 5. Hence it satisfies $\mathcal{C}_3$ and $\mathcal{C}_5$. By Theorem 5, $G$ satisfies $\mathcal{Y}_3$ and $\mathcal{Y}_5$. But $G$ is not 3-soluble nor 5-soluble. Hence it is clear that $G$ is not a $PST_3$-group nor a $PST_5$-group.

2. Theorem 5 indicates the way for constructing non-soluble examples of groups with property $\mathcal{Y}_p$ which are not $p$-nilpotent.

Let $A$ be an abelian $p$-group and let $B$ be a $p'$-group of power automorphisms of $A$. Denote by $H = [A]B$, the corresponding semidirect product. Suppose that $H$ is not $p$-nilpotent. If $S$ is any non-abelian simple group such that $p$ does not divide $S$, then the regular wreath product $G$ of $S$ by $H$ is a non-soluble group with property $\mathcal{Y}_p$ which is not $p$-nilpotent.

3. One might think that the most natural candidate for the “$PST$-version” of properties $\mathcal{C}_p$ and $\mathcal{X}_p$ could be:

A group $G$ satisfies $\mathcal{Y}_p^*$ if every subgroup of a Sylow $p$-subgroup $P$ is $S$-permutable in $N_G(P)$.

In $G = \Sigma_4$, the symmetric group of degree 4, the Sylow 2-subgroups are self-normalising, hence every subgroup of a Sylow 2-subgroup $P$ of $G$ is $S$-permutable in $N_G(P)$, and the subgroups of a Sylow 3-subgroup $Q$ of $G$ are $S$-permutable in $N_G(Q)$. Consequently $G$ satisfies $\mathcal{Y}_p^*$ for every $p$, but $G$ is not a $PST$-group, because the cyclic subgroups of the Klein 4-group are not permutable with the Sylow 3-subgroups of $G$. Note that the above example shows that $\mathcal{Y}_p^*$ is not subgroup-closed.

4. Any hope of creating a similar landscape outside of the soluble universe is soon dispelled. As soon as we have a local property $\mathcal{J}_p$ which is subgroup-closed and such that a finite group $G$ is a $PST$-group if and only if $G$ satisfies $\mathcal{J}_p$ for every prime $p$, then $\bigcap_{p \in P} \mathcal{J}_p$ is contained in the class of soluble groups. Therefore a group satisfying $\mathcal{J}_p$ for all $p$ should be soluble.

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References


