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A CLASS OF SUMMING OPERATORS ACTING IN SPACES OF OPERATORS

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Abstract. Let X, Y and Z be Banach spaces and let U be a subspace of $\mathcal{L}(X^*, Y)$, the Banach space of all operators from X^* to Y. An operator $S\colon U \to Z$ is said to be (ℓ_p^s, ℓ_p)-summing (where $1 \leq p < \infty$) if there is a constant $K \geq 0$ such that

$$\left(\sum_{i=1}^{n} \|S(T_i)\|_Z^p\right)^{1/p} \leq K \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^{n} \|T_i(x^*)\|_Y^p\right)^{1/p}$$

for every $n \in \mathbf{N}$ and all $T_1, \ldots, T_n \in U$. In this paper we study this class of operators, introduced by Blasco and Signes as a natural generalization of the (p, Y)-summing operators of Kislyakov. On the one hand, we discuss Pietsch-type domination results for (ℓ_p^s, ℓ_p)-summing operators. In this direction, we provide a negative answer to a question raised by Blasco and Signes, and we also give new insight on a result by Botelho and Santos. On the other hand, we extend to this setting the classical theorem of Kwapień characterizing those operators which factor as $S_1 \circ S_2$, where S_2 is absolutely p-summing and S_1^* is absolutely q-summing ($1 < p, q < \infty$ and $1/p + 1/q \leq 1$).

1. Introduction

Summability of series in Banach spaces is a classical central topic in the field of mathematical analysis. This study is faced from an abstract point of view as a part of the general analysis of the summability properties of operators, using some remarkable results of the theory of operator ideals. Pietsch's Factorization Theorem is nowadays the central tool in this topic, and different versions of this result adapted to other contexts are currently known. This theorem establishes that operators that transform weakly p-summable sequences into absolutely p-summable ones can always be dominated by an integral, and factored through a subspace of an L_p-space. Some related relevant results can also be formulated in terms of integral domination and factorization of operators. For example, recall that an operator between Banach spaces $S\colon X \to Y$ is said to be (p, q)-dominated (where $1 < p, q < \infty$ and $1/p + 1/q = 1/r \leq 1$) if for every couple of finite sequences $(x_i)_{i=1}^n$ in X and $(y_i^*)_{i=1}^n$ in Y^*, the strong ℓ_r-norm of the sequence $(\langle S(x_i), y_i^* \rangle)_{i=1}^n$ is bounded above by the product of the weak ℓ_p-norm of $(x_i)_{i=1}^n$ and the weak ℓ_q-norm of $(y_i^*)_{i=1}^n$ (up to a multiplying constant independent of both sequences and their length). Kwapień's Factorization Theorem [18] states that an operator is (p, q)-dominated if and only if it can be

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written as the composition $S_1 \circ S_2$ of operators such that S_2 is absolutely p-summing and the adjoint S_1^* is absolutely q-summing (cf. [9, §19]).

The aim of this paper is to continue with the specific study of the summability properties of operators defined on spaces of operators. Throughout this paper X, Y and Z are Banach spaces.

Definition 1.1. [3, Blasco–Signes] Let $1 \leq p < \infty$ and let U be a subspace of $\mathcal{L}(X^*, Y)$. An operator $S\colon U \to Z$ is said to be (ℓ_p^s, ℓ_p)-*summing* if there is a constant $K \geq 0$ such that

$$\left(\sum_{i=1}^{n} \|S(T_i)\|_Z^p\right)^{1/p} \leq K \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^{n} \|T_i(x^*)\|_Y^p\right)^{1/p} \tag{1.1}$$

for every $n \in \mathbf{N}$ and all $T_1, \dots, T_n \in U$.

Some fundamental properties of this type of operators are already known, as well as the main picture of their summability properties. The works of Blasco and Signes [3] and Botelho and Santos [5] fixed the framework and solved a great part of the natural problems appearing in this context. In the particular case when U is the injective tensor product $X \hat{\otimes}_\varepsilon Y$ (naturally identified as a subspace of $\mathcal{L}(X^*, Y)$), (ℓ_p^s, ℓ_p)-summing operators had been studied earlier by Kislyakov [17] as "(p, Y)-summing" operators. In particular, he gave a Pietsch-type domination theorem for (ℓ_p^s, ℓ_p)-summing operators defined on $X \hat{\otimes}_\varepsilon Y$ (see [17, Theorem 1.1.6]). This led to the natural question of whether a Pietsch-type domination theorem holds for arbitrary (ℓ_p^s, ℓ_p)-summing operators, see [3, Question 5.2]. Botelho and Santos extended Kislyakov's result by showing that this is the case when U is Schwartz's ε-product $X \varepsilon Y$, i.e. the subspace of all operators from X^* to Y which are (w^*-to-norm) continuous when restricted to B_{X^*} (see [5, Theorem 3.1]).

This paper is organized as follows. In Section 2 we give new insight on the Botelho-Santos theorem and we provide a negative answer to the aforementioned question, see Example 2.10. To this end, we characterize those (ℓ_p^s, ℓ_p)-summing operators admitting a Pietsch-type domination by means of the strong operator topology (Theorem 2.9). All of this is naturally connected with a discussion on measurability properties of operators which might be of independent interest.

In Section 3 we start a general analysis of the summability properties of operators defined on spaces of operators that imply similar properties for the adjoint maps. Our main result along this way is a Kwapień-type theorem involving the special summation that arises in this setting related to the strong operator topology, see Theorem 3.2.

Notation and terminology. All our Banach spaces are real and all our topological spaces are Hausdorff. By a *subspace* of a Banach space we mean a norm-closed linear subspace. By an *operator* we mean a continuous linear map between Banach spaces. The norm of a Banach space X is denoted by $\|\cdot\|_X$ or simply $\|\cdot\|$. We write $B_X = \{x \in X\colon \|x\| \leq 1\}$ (the closed unit ball of X). The topological dual of X is denoted by X^* and we write w^* for its weak*-topology. The evaluation of a functional $x^* \in X^*$ at $x \in X$ is denoted by either $\langle x, x^* \rangle$ or $\langle x^*, x \rangle$. We write $X \not\supseteq \ell_1$ to say that X does not contain subspaces isomorphic to ℓ_1. We denote by $\mathcal{L}(X^*, Y)$ the Banach space of all operators from X^* to Y, equipped with the operator norm. The *strong operator topology* (*SOT* for short) on $\mathcal{L}(X^*, Y)$ is the locally convex topology for which the sets

$$\{T \in \mathcal{L}(X^*, Y) \colon \|T(x^*)\|_Y < \varepsilon\}, \ x^* \in X^*, \ \varepsilon > 0,$$

are a subbasis of open neighborhoods of 0. That is, a net (T_α) in $\mathcal{L}(X^*, Y)$ is SOT-convergent to 0 if and only if $\|T_\alpha(x^*)\|_Y \to 0$ for every $x^* \in X^*$. Given a compact topological space L, we denote by $C(L)$ the Banach space of all real-valued continuous functions on L, equipped with the supremum norm. Thanks to Riesz's representation theorem, the elements of $C(L)^*$ are identified with regular Borel signed measures on L. We denote by $P(L) \subseteq C(L)^*$ the convex w^*-compact set of all regular Borel probability measures on L. For each $t \in L$, we write $\delta_t \in P(L)$ to denote the evaluation functional at t, i.e. $\delta_t(h) := h(t)$ for all $h \in C(L)$. A function defined on L with values in a Banach space is said to be *universally strongly measurable* if it is strongly μ-measurable for all $\mu \in P(L)$. We will mostly consider the case when L is the dual closed unit ball B_{X^*} equipped with the weak*-topology.

2. Pietsch-type domination of (ℓ_p^s, ℓ_p)-summing operators

Throughout this section we fix $1 \leq p < \infty$. The aforementioned Pietsch-type domination theorem for (ℓ_p^s, ℓ_p)-summing operators proved in [5, Theorem 3.1] reads as follows:

Theorem 2.1. (Botelho–Santos) *Let U be a subspace of $X\varepsilon Y$ and let $S \colon U \to Z$ be an (ℓ_p^s, ℓ_p)-summing operator. Then there exist a constant $K \geq 0$ and $\mu \in P(B_{X^*})$ such that*

$$\|S(T)\|_Z \leq K \left(\int_{B_{X^*}} \|T(\cdot)\|_Y^p \, d\mu \right)^{1/p} \tag{2.1}$$

for every $T \in U$.

A first comment is that the integral of inequality (2.1) is always well-defined for any $T \in X\varepsilon Y$ and $\mu \in P(B_{X^*})$. Indeed, the restriction $T|_{B_{X^*}}$ is (w^*-to-norm) continuous, so it is universally strongly measurable. Since in addition $T|_{B_{X^*}}$ is bounded, it belongs to the Lebesgue–Bochner space $L_p(\mu, Y)$.

Remark 2.2. Actually, Theorem 2.1 is proved in [5, Theorem 3.1] for operators S defined on a subspace U contained in

$$\mathcal{L}_{w^*,\|\cdot\|}(X^*, Y) = \{T \in \mathcal{L}(X^*, Y) \colon T \text{ is } (w^*\text{-to-norm) continuous}\}.$$

The proof given there is based on the abstract Pietsch-type domination theorem of Botelho, Pellegrino and Rueda [4], and the argument works for subspaces of $X\varepsilon Y$ as well. We stress that $\mathcal{L}_{w^*,\|\cdot\|}(X^*, Y)$ consists of finite rank operators, one has

$$\overline{\mathcal{L}_{w^*,\|\cdot\|}(X^*, Y)}^{\|\cdot\|} = X \hat{\otimes}_\varepsilon Y \subseteq X\varepsilon Y$$

and, in general, $\mathcal{L}_{w^*,\|\cdot\|}(X^*, Y) \neq X\varepsilon Y$.

We next provide a more direct proof of Theorem 2.1. While the underlying idea is similar, we include the details for the reader's convenience. Yet another approach will be presented at the end of this section.

Proof of Theorem 2.1. For any $n \in \mathbf{N}$ and $\bar{T} = (T_1, \ldots, T_n) \in U^n$, we define

$$\Delta_{\bar{T}} \colon P(B_{X^*}) \to \mathbf{R}, \quad \Delta_{\bar{T}}(\mu) := \sum_{i=1}^n \|S(T_i)\|_Z^p - K^p \int_{B_{X^*}} \sum_{i=1}^n \|T_i(\cdot)\|_Y^p \, d\mu,$$

where $K \geq 0$ is a constant as in Definition 1.1. Clearly, $\Delta_{\bar{T}}$ is convex and w^*-continuous, because the real-valued function

$$x^* \mapsto \sum_{i=1}^{n} \|T_i(x^*)\|_Y^p$$

is w^*-continuous on B_{X^*}. This function attains its supremum at some $x^*_{\bar{T}} \in B_{X^*}$. Bearing in mind that S is (ℓ_p^s, ℓ_p)-summing, we get $\Delta_{\bar{T}}(\delta_{x^*_{\bar{T}}}) \leq 0$.

Note also that the collection of all functions of the form $\Delta_{\bar{T}}$ is a convex cone in $\mathbf{R}^{P(B_{X^*})}$. Indeed, given $\bar{T} = (T_1, \dots, T_n) \in U^n$, $\bar{R} = (R_1, \dots, R_m) \in U^m$, $\alpha \geq 0$ and $\beta \geq 0$, we have $\alpha\Delta_{\bar{T}} + \beta\Delta_{\bar{R}} = \Delta_{\bar{H}}$, where

$$\bar{H} = (\alpha^{1/p}T_1, \dots, \alpha^{1/p}T_n, \beta^{1/p}R_1, \dots, \beta^{1/p}R_m).$$

Therefore, by Ky Fan's Lemma (see e.g. [11, Lemma 9.10]), there is $\mu \in P(B_{X^*})$ such that $\Delta_{\bar{T}}(\mu) \leq 0$ for all functions of the form $\Delta_{\bar{T}}$. In particular, inequality (2.1) holds for all $T \in U$. □

Clearly, in order to extend the statement of Theorem 2.1 to other subspaces U of $\mathcal{L}(X^*, Y)$, the real-valued map $\|T(\cdot)\|_Y$ needs to be μ-measurable for every $T \in U$. This holds automatically if U is a subspace of

$$\mathcal{UM}(X^*, Y) := \{T \in \mathcal{L}(X^*, Y) \colon T|_{B_{X^*}} \text{ is universally strongly measurable}\}.$$

Note that $\mathcal{UM}(X^*, Y)$ is a SOT-sequentially closed subspace of $\mathcal{L}(X^*, Y)$.

Example 2.3. (i) We have $X\varepsilon Y \subseteq \mathcal{UM}(X^*, Y)$ according to the comment preceding Remark 2.2.

(ii) More generally, *every (w^*-to-weak) continuous operator from X^* to Y belongs to $\mathcal{UM}(X^*, Y)$.* Indeed, just bear in mind that any weakly continuous function from a compact topological space to a Banach space is universally strongly measurable, see [1, Proposition 4]. We stress that, by the Banach–Dieudonné theorem, an operator $T \colon X^* \to Y$ is (w^*-to-weak) continuous if and only if the restriction $T|_{B_{X^*}}$ is (w^*-to-weak) continuous.

(iii) In particular, *if X is reflexive, then $\mathcal{L}(X^*, Y) = \mathcal{UM}(X^*, Y)$.*

Example 2.4. *If $X \not\supseteq \ell_1$, then every $T \in \mathcal{L}(X^*, Y)$ with separable range belongs to $\mathcal{UM}(X^*, Y)$.* Indeed, a result of Haydon [15] (cf. [23, Theorem 6.9]) states that $X^{**} = \mathcal{UM}(X^*, \mathbf{R})$ if and only if $X \not\supseteq \ell_1$. The conclusion now follows from Pettis' measurability theorem applied to $T|_{B_{X^*}}$ and each $\mu \in P(B_{X^*})$, see e.g. [12, p. 42, Theorem 2].

So, we will look for conditions ensuring that an (ℓ_p^s, ℓ_p)-summing operator defined on a subspace of $\mathcal{UM}(X^*, Y)$ is (ℓ_p^s, ℓ_p)-controlled, according to the following:

Definition 2.5. Let U be a subspace of $\mathcal{UM}(X^*, Y)$. An operator $S \colon U \to Z$ is said to be (ℓ_p^s, ℓ_p)-*controlled* if there exist a constant $K \geq 0$ and $\mu \in P(B_{X^*})$ such that

$$\|S(T)\|_Z \leq K \left(\int_{B_{X^*}} \|T(\cdot)\|_Y^p \, d\mu \right)^{1/p} \tag{2.2}$$

for every $T \in U$.

The next characterization is straightforward.

Proposition 2.6. *Let U be a subspace of $\mathcal{UM}(X^*, Y)$ and let $S \colon U \to Z$ be an operator. Then S is (ℓ_p^s, ℓ_p)-controlled if and only if there exist $\mu \in P(B_{X^*})$, a* *subspace $W \subseteq L_p(\mu, Y)$ and an operator $\tilde{S}\colon W \to Y$ such that S factors as*

$$\begin{array}{ccc} U & \xrightarrow{\;S\;} & Z \\ {\scriptstyle i_\mu|_U}\downarrow & \nearrow{\scriptstyle \tilde{S}} & \\ W & & \end{array}$$

where $i_\mu\colon \mathcal{UM}(X^*, Y) \to L_p(\mu, Y)$ is the operator that maps each $T \in \mathcal{UM}(X^*, Y)$ to the equivalence class of $T|_{B_{X^*}}$ in $L_p(\mu, Y)$.

Proof. It is clear that such factorization implies that S is (ℓ_p^s, ℓ_p)-controlled. Conversely, inequality (2.2) in Definition 2.5 allows us to define a linear continuous map $\tilde{S}_0\colon i_\mu(U) \to Z$ by declaring $\tilde{S}_0(i_\mu(T)) := S(T)$ for all $T \in U$. Now, we can extend $\tilde{S}_0$ to an operator $\tilde{S}$ from $W := \overline{i_\mu(U)}$ to Z. Clearly, we have $\tilde{S} \circ i_\mu|_U = S$. $\square$

We next give a couple of applications of Proposition 2.6 related to topological properties of (ℓ_p^s, ℓ_p)-controlled operators.

The class of Banach spaces X such that $L_1(\mu)$ is separable for every $\mu \in P(B_{X^*})$ is rather wide. It contains, for instance, all weakly compactly generated Banach spaces (cf. [13, Theorem 13.20 and Corollary 14.6]) as well as all Banach spaces not containing subspaces isomorphic to ℓ_1 (see [2, Proposition B.1]). For such spaces we have:

Corollary 2.7. *Suppose that $L_1(\mu)$ is separable for every $\mu \in P(B_{X^*})$ and that Y is separable. Let U be a subspace of $\mathcal{UM}(X^*, Y)$ and let $S\colon U \to Z$ be an (ℓ_p^s, ℓ_p)-controlled operator. Then S has separable range.*

Proof. Under such assumptions, $L_p(\mu, Y)$ is separable for any $\mu \in P(B_{X^*})$. The result now follows from Proposition 2.6. $\square$

A subset of a Banach space is said to be *weakly precompact* if every sequence in it admits a weakly Cauchy subsequence. Rosenthal's ℓ_1-theorem [22] (cf. [13, Theorem 5.37]) characterizes weakly precompact sets as those which are bounded and contain no sequence equivalent to the unit basis of ℓ_1. An operator between Banach spaces is said to be *weakly precompact* if it maps bounded sets to weakly precompact sets; this is equivalent to saying that it factors through a Banach space not containing subspaces isomorphic to ℓ_1. For more information on weakly precompact operators we refer the reader to [14].

Corollary 2.8. *Let U be a subspace of $\mathcal{UM}(X^*, Y)$ and let $S\colon U \to Z$ be an (ℓ_p^s, ℓ_p)-controlled operator. Then:*

(i) *S is weakly compact whenever Y is reflexive.*
(iii) *S is weakly precompact whenever $Y \not\supseteq \ell_1$.*

Proof. We consider a factorization of S as in Proposition 2.6 and we distinguish two cases:

Case $1 < p < \infty$. If Y is reflexive, then so is $L_p(\mu, Y)$ (see e.g. [12, p. 100, Corollary 2]) and the same holds for W, hence S is weakly compact. On the other hand, if $Y \not\supseteq \ell_1$, then $L_p(\mu, Y) \not\supseteq \ell_1$ (see e.g. [7, Theorem 2.2.2]) and so $W \not\supseteq \ell_1$, hence S is weakly precompact.

Case $p = 1$. Let $j\colon L_2(\mu, Y) \to L_1(\mu, Y)$ be the identity operator. Since

$$i_\mu(B_U) \subseteq j(B_{L_2(\mu,Y)}),$$

we deduce that $i_\mu(B_U)$ is relatively weakly compact (resp. weakly precompact) whenever Y is reflexive (resp. $Y \not\supseteq \ell_1$), and the same holds for $S(B_U) = \tilde{S}(i_\mu(B_U))$. □

The following result shows the link between (ℓ_p^s, ℓ_p)-controlled and (ℓ_p^s, ℓ_p)-summing operators.

Theorem 2.9. *Let U be a subspace of $\mathcal{UM}(X^*, Y)$ and let $S\colon U \to Z$ be an operator. Let us consider the following statements:*

(i) *S is (ℓ_p^s, ℓ_p)-controlled.*

(ii) *S is (ℓ_p^s, ℓ_p)-summing and (SOT-to-norm) sequentially continuous.*

Then (i) $\Rightarrow$ (ii). *Moreover, both statements are equivalent whenever $U \cap X\varepsilon Y$ is SOT-sequentially dense in U.*

Proof. Suppose first that S is (ℓ_p^s, ℓ_p)-controlled and consider a factorization of S as in Proposition 2.6. We will deduce that S is (ℓ_p^s, ℓ_p)-summing and (SOT-to-norm) sequentially continuous by checking that so is i_μ. On the one hand, i_μ is (ℓ_p^s, ℓ_p)-summing, because for every $n \in \mathbf{N}$ and $T_1, \dots, T_n \in \mathcal{UM}(X^*, Y)$ we have

$$\sum_{i=1}^n \|i_\mu(T_i)\|^p_{L_p(\mu,Y)} = \int_{B_{X^*}} \sum_{i=1}^n \|T_i(\cdot)\|_Y^p \, d\mu \le \sup_{x^* \in B_{X^*}} \sum_{i=1}^n \|T_i(x^*)\|_Y^p.$$

On the other hand, i_μ is (SOT-to-norm) sequentially continuous. Indeed, let (T_n) be a sequence in $\mathcal{UM}(X^*, Y)$ which SOT-converges to 0, i.e. $\|T_n(x^*)\|_Y \to 0$ for every $x^* \in X^*$. By the Banach–Steinhaus theorem, $\sup\{\|T_n\| \colon n \in \mathbf{N}\} < \infty$. From Lebesgue's dominated convergence theorem it follows that $(i_\mu(T_n))$ converges to 0 in the norm topology of $L_p(\mu, Y)$.

Suppose now that (ii) holds and that $U \cap X\varepsilon Y$ is SOT-sequentially dense in U. The restriction $S|_{U\cap X\varepsilon Y}$ is (ℓ_p^s, ℓ_p)-summing and so Theorem 2.1 and Proposition 2.6 ensure the existence of $\mu \in P(B_{X^*})$, a subspace $W \subseteq L_p(\mu, Y)$ and an operator $\tilde{S}\colon W \to Z$ such that $i_\mu(U \cap X\varepsilon Y) \subseteq W$ and

$$\tilde{S} \circ i_\mu|_{U\cap X\varepsilon Y} = S|_{U\cap X\varepsilon Y}.$$

Then we have $i_\mu(U) \subseteq W$ and $\tilde{S} \circ i_\mu|_U = S$, because S and i_μ are (SOT-to-norm) sequentially continuous and $U \cap X\varepsilon Y$ is SOT-sequentially dense in U. Therefore, S is (ℓ_p^s, ℓ_p)-controlled. □

We are now ready to present a negative answer to [3, Question 5.2]:

Example 2.10. Suppose that X is not reflexive and X^* is separable (e.g. $X = c_0$). Then $X^{**} = \mathcal{UM}(X^*, \mathbf{R})$, every $S \in X^{***}$ is (ℓ_p^s, ℓ_p)-summing, but no $S \in X^{***} \setminus X^*$ is (ℓ_p^s, ℓ_p)-controlled (as operators from X^{**} to $\mathbf{R}$).

Proof. The equality $X^{**} = \mathcal{UM}(X^*, \mathbf{R})$ follows from the fact that $X \not\supseteq \ell_1$, according to Haydon's result which we already mentioned in Example 2.4. Every $S \in X^{***}$ is easily seen to be (ℓ_p^s, ℓ_p)-summing as an operator from X^{**} to $\mathbf{R}$ (use that B_{X^*} is w^*-dense in $B_{X^{***}}$, by Goldstine's theorem). On the other hand, if $S \in X^{***}$ is (ℓ_p^s, ℓ_p)-controlled, then it is w^*-sequentially continuous by Theorem 2.9 (bear in mind that SOT$= w^*$ on X^{**}). Since $(B_{X^{**}}, w^*)$ is metrizable (because X^* is separable), the restriction $S|_{B_{X^{**}}}$ is w^*-continuous and so, by the Banach–Dieudonné theorem, S is w^*-continuous, i.e. $S \in X^*$. □

In order to apply Theorem 2.9, there are many examples of subspaces U of $\mathcal{UM}(X^*, Y)$ for which $U \cap X\varepsilon Y$ is SOT-sequentially dense in U. An operator

$T\colon X^* \to Y$ is said to be *affine Baire-1* (we write $T \in \mathcal{AB}(X^*, Y)$ for short) if there is a sequence in $X\varepsilon Y$ which SOT-converges to T. Affine Baire-1 operators were studied by Mercourakis and Stamati [20] and Kalenda and Spurný [16]. We present below some examples. Recall first that a Banach space Y has the *approximation property* (*AP*) if for each norm-compact set $C \subseteq Y$ and each $\varepsilon > 0$ there is a finite rank operator $R\colon Y \to Y$ such that $\|R(y)-y\|_Y \leq \varepsilon$ for all $y \in C$. If in addition R can be chosen in such a way that $\|R\| \leq \lambda$ for some constant $\lambda \geq 1$ (independent of C and ε), then Y is said to have the *λ-bounded approximation property* (*λ-BAP*). A Banach space is said to have the *bounded approximation property* (*BAP*) if it has the λ-BAP for some $\lambda \geq 1$. For instance, every Banach space with a Schauder basis has the BAP. In general, the AP and the BAP are different. However, a separable dual Banach space has the AP if and only if it has the 1-BAP. For more information on these properties we refer the reader to [6].

Example 2.11. *Suppose that Y has the BAP. If $T \in \mathcal{L}(X^*, Y)$ is (w^*-to-weak) continuous and has separable range, then $T \in \mathcal{AB}(X^*, Y)$.*

Proof. Let $\lambda \geq 1$ be a constant such that Y has the λ-BAP. Given any countable set $D \subseteq Y$, there is a sequence (R_n) of finite rank operators on Y such that $\|R_n\| \leq \lambda$ for all $n \in \mathbf{N}$ and $\|R_n(y) - y\|_Y \to 0$ for every $y \in D$. Therefore, $\|R_n(y) - y\|_Y \to 0$ for every $y \in \overline{D}$ (the norm-closure of D). In particular, if this argument is applied to any countable set D such that $D \subseteq T(X^*) \subseteq \overline{D}$, we get that the sequence $(R_n \circ T)$ is SOT-convergent to T in $\mathcal{L}(X^*, Y)$. Note that each $R_n \circ T$ is (w^*-to-weak) continuous (because so is T) and has finite rank, hence it belongs to $\mathcal{L}_{w^*,\|\cdot\|}(X^*, Y) \subseteq X\varepsilon Y$. $\square$

Example 2.12. *Suppose that X^* is separable and that either X^* or Y has the BAP. Then*

$$\mathcal{L}(X^*, Y) = \mathcal{AB}(X^*, Y),$$

see [20, Theorems 2.18 and 2.19]. The proofs of these results contain a gap which was commented and corrected in [16, Remark 4.4]. Note that the separability assumption on Y that appears in the statement of [20, Theorem 2.19] can be removed by using the arguments of [16].

Clearly, $\mathcal{AB}(X^*, Y)$ is a linear subspace of $\mathcal{L}(X^*, Y)$. It is norm-closed whenever Y has the BAP, as we next show. To this end, we use an argument similar to the usual proof that the uniform limit of a sequence of real-valued Baire-1 functions is Baire-1 (see e.g. [19, Proposition A.126]). However, some technicalities arise since we need to approximate with operators instead of arbitrary continuous maps.

Lemma 2.13. *If Y has the BAP, then $\mathcal{AB}(X^*, Y)$ is norm-closed in $\mathcal{L}(X^*, Y)$.*

Proof. Fix $\lambda \geq 1$ such that Y has the λ-BAP. Let $T \in \overline{\mathcal{AB}(X^*, Y)}^{\|\cdot\|}$ with $\|T\| = 1$. Let (U_k) be a sequence in $\mathcal{AB}(X^*, Y)$ such that $\|U_k\| \leq 2^{-k+1}$ for all $k \in \mathbf{N}$ and $T = \sum_{k\in\mathbf{N}} U_k$ in the operator norm. Given $k \in \mathbf{N}$, we can apply to U_k the vector-valued version of Mokobodzki's theorem proved in [16, Theorem 2.2] to obtain a sequence $(S_{k,n})_{n\in\mathbf{N}}$ in $X\varepsilon Y$ such that

- $(S_{k,n})_{n\in\mathbf{N}}$ SOT-converges to U_k;
- $\|S_{k,n}\| \leq \lambda 2^{-k+1}$ for all $n \in \mathbf{N}$.

Define a sequence (T_n) in $X\varepsilon Y$ by

$$T_n := \sum_{k=1}^{n} S_{k,n} \quad \text{for all } n \in \mathbf{N}.$$

It is easy to check that (T_n) SOT-converges to T, hence $T \in \mathcal{AB}(X^*, Y)$. $\square$

As usual, we denote by $\mathcal{K}(X^*, Y)$ the subspace of $\mathcal{L}(X^*, Y)$ consisting of all compact operators from X^* to Y. Clearly, we have $X\varepsilon Y \subseteq \mathcal{K}(X^*, Y)$.

Example 2.14. *Suppose that X is separable and $X \not\supseteq \ell_1$.*

(i) *Every finite rank operator $T\colon X^* \to Y$ is affine Baire-1.*
(ii) *If Y has the BAP, then*

$$\mathcal{K}(X^*, Y) \subseteq \mathcal{AB}(X^*, Y).$$

Proof. (i) It suffices to check it for rank one operators. Fix $x^{**} \in X^{**}$ and $y \in Y$ in such a way that $T(x^*) = \langle x^{**}, x^* \rangle y$ for all $x^* \in X^*$. Since X is w^*-sequentially dense in X^{**} (by the Odell–Rosenthal theorem [21], cf. [23, Theorem 4.1]), there is a sequence (x_n) in X which w^*-converges to x^{**}. For each $n \in \mathbf{N}$ we define $T_n \in \mathcal{L}_{w^*,\|\cdot\|}(X^*, Y) \subseteq X\varepsilon Y$ by declaring $T_n(x^*) := \langle x_n, x^* \rangle y$ for all $x^* \in X^*$. Clearly, (T_n) is SOT-convergent to T.

(ii) Take any $T \in \mathcal{K}(X^*, Y)$. Since Y has the AP, there is a sequence (T_n) of finite rank operators from X^* to Y converging to T in the operator norm. Each T_n is affine Baire-1 by (i). An appeal to Lemma 2.13 ensures that $T \in \mathcal{AB}(X^*, Y)$. $\square$

The proof of Theorem 2.1 makes essential use of the w^*-continuity on B_{X^*} of the real-valued map $\|T(\cdot)\|_Y$ for $T \in X\varepsilon Y$. We next present an abstract Pietsch-type domination theorem for (ℓ_p^s, ℓ_p)-summing operators that does not require that continuity assumption, at the price of dominating with a *finitely additive* measure. As a consequence of this result, we will obtain another proof of Theorem 2.1.

Given a measurable space (Ω, Σ), we denote by $B(\Sigma)$ the Banach space of all bounded Σ-measurable real-valued functions on Ω, equipped with the supremum norm. The dual $B(\Sigma)^*$ can be identified with the Banach space $\mathrm{ba}(\Sigma)$ of all bounded finitely additive real-valued measures on Σ, equipped with the variation norm. The duality is given by integration, that is, $\langle h, \nu \rangle = \int_\Omega h \, d\nu$ for every $h \in B(\Sigma)$ and $\nu \in \mathrm{ba}(\Sigma)$, see e.g. [10, p. 77, Theorem 7].

Theorem 2.15. *Let Σ be a σ-algebra on B_{X^*} and let U be a subspace of $\mathcal{L}(X^*, Y)$ such that the restriction of $\|T(\cdot)\|_Y$ to B_{X^*} is Σ-measurable for every $T \in U$. Let $S\colon U \to Z$ be an (ℓ_p^s, ℓ_p)-summing operator. Then there exist a constant $K \geq 0$ and a finitely additive probability ν on Σ such that*

$$\|S(T)\|_Z \leq K\Big(\int_{B_{X^*}} \|T(\cdot)\|_Y^p \, d\nu \Big)^{1/p} \tag{2.3}$$

for every $T \in U$.

Proof. For each $T \in U$ we define $\psi_T \in B(\Sigma)$ by

$$\psi_T(x^*) := \|T(x^*)\|_Y^p \quad \text{for all } x^* \in B_{X^*}.$$

Let $L \subseteq \mathrm{ba}(\Sigma) = B(\Sigma)^*$ be the convex w^*-compact set of all finitely additive probabilities on Σ. For any $n \in \mathbf{N}$ and $\bar{T} = (T_1, \dots, T_n) \in U^n$, we define

$$\Delta_{\bar{T}} : L \to \mathbf{R}, \quad \Delta_{\bar{T}}(\nu) := \sum_{i=1}^n \|S(T_i)\|_Z^p - K^p \int_K \sum_{i=1}^n \psi_{T_i} \, d\nu,$$

where $K \geq 0$ is a constant as in Definition 1.1. Clearly, $\Delta_{\bar{T}}$ is convex and w^*-continuous. Moreover, by the Hahn–Banach theorem there is $\eta_{\bar{T}} \in \mathrm{ba}(\Sigma)$ with

$\|\eta_{\bar{T}}\|_{\mathrm{ba}(\Sigma)} = 1$ such that

$$\Big\langle \sum_{i=1}^{n} \psi_{T_i}, \eta_{\bar{T}} \Big\rangle = \Big\| \sum_{i=1}^{n} \psi_{T_i} \Big\|_{B(\Sigma)}.$$

Bearing in mind that $\sum_{i=1}^{n} \psi_{T_i} \geq 0$, it follows that the variation $|\eta_{\bar{T}}| \in L$ satisfies

$$\Big\langle \sum_{i=1}^{n} \psi_{T_i}, |\eta_{\bar{T}}| \Big\rangle = \sup_{x^* \in B_{X^*}} \sum_{i=1}^{n} \psi_{T_i}(x^*).$$

Therefore, inequality (1.1) in Definition 1.1 yields

$$\Delta_{\bar{T}}\big(|\eta_{\bar{T}}|\big) = \sum_{i=1}^{n} \|S(T_i)\|_Z^p - K^p \Big\langle \sum_{i=1}^{n} \psi_{T_i}, |\eta_{\bar{T}}| \Big\rangle \leq 0.$$

The collection of all functions of the form $\Delta_{\bar{T}}$ is easily seen to be a convex cone in $\mathbf{R}^L$. By Ky Fan's Lemma (see e.g. [11, Lemma 9.10]), there is $\nu \in L$ such that $\Delta_{\bar{T}}(\nu) \leq 0$ for all functions of the form $\Delta_{\bar{T}}$. In particular, (2.3) holds for every $T \in U$. □

Another proof of Theorem 2.1. Let $\Sigma := \mathrm{Borel}(B_{X^*}, w^*)$. Let K and ν be as in Theorem 2.15. Define $\varphi \in B(\Sigma)^*$ by $\langle h, \varphi \rangle := \int_{B_{X^*}} h\, d\nu$ for all $h \in B(\Sigma)$. Let $\mu \in C(B_{X^*})^*$ be the restriction of φ to $C(B_{X^*})$ (as a subspace of $B(\Sigma)$). Then $\mu \in P(B_{X^*})$ and (2.3) now reads as

$$\|S(T)\|_Z \leq K \left(\int_{B_{X^*}} \|T(\cdot)\|_Y^p \, d\mu \right)^{1/p}$$

for every $T \in U \subseteq X \varepsilon Y$. □

3. Kwapień-type theorem for (ℓ_p^s, ℓ_q^s)-dominated operators

Throughout this section we fix $1 < p, q < \infty$ such that $1/p + 1/q \leq 1$. Let $1 \leq r < \infty$ be defined by $1/p + 1/q = 1/r$. An operator $S\colon X \to Y$ is said to be (p,q)-*dominated* if there is a constant $K \geq 0$ such that

$$\left(\sum_{i=1}^{n} |\langle S(x_i), y_i^* \rangle|^r \right)^{1/r} \leq K \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^{n} |\langle x_i, x^* \rangle|^p \right)^{1/p} \cdot \sup_{y \in B_Y} \left(\sum_{i=1}^{n} |\langle y, y_i^* \rangle|^q \right)^{1/q}$$

for every $n \in \mathbf{N}$, all $x_1, \dots, x_n \in X$ and all $y_1^*, \dots, y_n^* \in Y^*$. The classical result of Kwapień [18] mentioned in the introduction says that an operator between Banach spaces is (p,q)-dominated if and only if it can be written as $S_1 \circ S_2$ for some operators S_1 and S_2 such that S_2 is absolutely p-summing and S_1^* is absolutely q-summing (cf. [9, §19]). Our aim in this section is to extend Kwapień's result to the framework of (ℓ_p^s, ℓ_p)-summing operators, see Theorem 3.2 below.

From now on we assume that Z is such that Z^* is a subspace of $\mathcal{UM}(E^*, F)$ for some fixed Banach spaces E and F. Accordingly, the adjoint of any operator taking values in Z is defined on a subspace of $\mathcal{UM}(E^*, F)$ and we can discuss whether it is (ℓ_q^s, ℓ_q)-summing or (ℓ_q^s, ℓ_q)-controlled.

Definition 3.1. Let U be a subspace of $\mathcal{L}(X^*,Y)$. An operator $S\colon U\to Z$ is said to be (ℓ_p^s,ℓ_q^s)-*dominated* if there is a constant $K\geq 0$ such that

$$\begin{aligned}(3.1)\qquad &\left(\sum_{i=1}^n |\langle S(T_i),z_i^*\rangle|^r\right)^{1/r}\\ &\leq K\sup_{x^*\in B_{X^*}}\left(\sum_{i=1}^n \|T_i(x^*)\|_Y^p\right)^{1/p}\cdot\sup_{e^*\in B_{E^*}}\left(\sum_{i=1}^n \|z_i^*(e^*)\|_F^q\right)^{1/q}\end{aligned}$$

for every $n\in\mathbf{N}$, all $T_1,\dots,T_n\in U$ and all $z_1^*,\dots,z_n^*\in Z^*$.

Theorem 3.2. *Let U be a subspace of $\mathcal{UM}(X^*,Y)$ and let $S\colon U\to Z$ be an operator. Consider the following statements:*

(i) *S is (ℓ_p^s,ℓ_q^s)-dominated.*

(ii) *There exist a constant $K\geq 0$ and measures $\mu\in P(B_{X^*})$ and $\eta\in P(B_{E^*})$ such that*

$$(3.2)\qquad |\langle S(T),z^*\rangle|\leq K\left(\int_{B_{X^*}}\|T(\cdot)\|_Y^p\,d\mu\right)^{1/p}\cdot\left(\int_{B_{E^*}}\|z^*(\cdot)\|_F^q\,d\eta\right)^{1/q}$$

for every $T\in U\cap X\varepsilon Y$ and every $z^*\in Z^*\cap E\varepsilon F$.

(iii) *There exist a constant $K\geq 0$ and measures $\mu\in P(B_{X^*})$ and $\eta\in P(B_{E^*})$ such that* (3.2) *holds for every $T\in U$ and every $z^*\in Z^*$.*

(iv) *There exist a Banach space W, an (ℓ_p^s,ℓ_p)-controlled operator $S_2\colon U\to W$ and an operator $S_1\colon W\to Z$ with (ℓ_q^s,ℓ_q)-controlled adjoint such that S factors as $S=S_1\circ S_2$.*

(v) *There exist a Banach space W, an (ℓ_p^s,ℓ_p)-summing operator $S_2\colon U\to W$ and an operator $S_1\colon W\to Z$ with (ℓ_q^s,ℓ_q)-summing adjoint such that S factors as $S=S_1\circ S_2$.*

Then (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v) $\Longrightarrow$ (i) $\Longrightarrow$ (ii). *All statements are equivalent if, in addition, we assume that:*

(a) *the identity map on Z^* is (SOT-to-w^*) sequentially continuous;*

(b) *$Z^*\cap E\varepsilon F$ is SOT-sequentially dense in Z^*;*

(c) *$U\cap X\varepsilon Y$ is SOT-sequentially dense in U;*

(d) *S is (SOT-to-norm) sequentially continuous.*

For the sake of brevity it is convenient to introduce the following:

Definition 3.3. We say that the triple (Z,E,F) is *admissible* if conditions (a) and (b) above hold.

Before embarking on the proof of Theorem 3.2 we present some examples of admissible triples. Recall that the *weak operator topology* (*WOT* for short) on $\mathcal{L}(E^*,F)$ is the locally convex topology for which the sets

$$\{R\in\mathcal{L}(E^*,F)\colon |\langle R(e^*),f^*\rangle|<\varepsilon\},\quad e^*\in E^*,\quad f^*\in F^*,\quad \varepsilon>0,$$

are a subbasis of open neighborhoods of 0. So, a net (R_α) in $\mathcal{L}(E^*,F)$ is WOT-convergent to 0 if and only if $(R_\alpha(e^*))$ is weakly null in F for every $e^*\in E^*$.

Example 3.4. *If $Z^*\subseteq E\varepsilon F$, then (Z,E,F) is admissible.* Indeed, (b) holds trivially, while (a) follows from the fact that a sequence in $E\varepsilon F$ is WOT-convergent to 0 if and only if it is weakly null in $E\varepsilon F\subseteq\mathcal{L}(E^*,F)$ (see e.g. [8, Theorem 1.3]).

Example 3.5. Suppose that $E \not\supseteq \ell_1$. Take $Z := E^*$ and $F := \mathbf{R}$. Then we have $Z^* = E^{**} = \mathcal{UM}(E^*, F)$ (see Example 2.4) and, of course, SOT $= w^*$ on Z^*, so that (a) holds. If in addition E is separable, then (b) also holds, i.e. $E\varepsilon F = E$ is w^*-sequentially dense in E^{**}, by the Odell–Rosenthal theorem [21] (cf. [23, Theorem 4.1]).

Example 3.6. Suppose that $F := X_0^*$ for a Banach space X_0. Take $Z := E^* \hat{\otimes}_\pi X_0$ (the projective tensor product of E^* and X_0). Then:

(i) $Z^* = \mathcal{L}(E^*, F)$ in the natural way (see e.g. [12, p. 230, Corollary 2]).

(ii) The identity map on Z^* is (WOT-to-w^*) sequentially continuous.

(iii) If E^* is separable and either E^* or F has the BAP, then $Z^* = \mathcal{UM}(E^*, F)$ and (Z, E, F) is admissible.

Proof. (ii) Let (φ_n) be a sequence in $Z^* = \mathcal{L}(E^*, F)$ which WOT-converges to 0. Then it is bounded (by the Banach–Steinhaus theorem) and

$$\langle e^* \otimes x_0, \varphi_n \rangle = \langle x_0, \varphi_n(e^*) \rangle \to 0 \quad \text{for all } e^* \in E^* \text{ and } x_0 \in X_0,$$

hence (φ_n) is w^*-null.

(iii) Under such assumptions $E\varepsilon F$ is SOT-sequentially dense in $\mathcal{L}(E^*, F)$ (see Example 2.12). In particular, we have $\mathcal{L}(E^*, F) = \mathcal{UM}(E^*, F)$. Bearing in mind (ii) it follows that (Z, E, F) is admissible. $\square$

Proof of Theorem 3.2. (iii) $\Rightarrow$ (iv) By assumption we have

$$|\langle S(T), z^* \rangle| \le K \|i_\mu(T)\|_{L_p(\mu,Y)} \|z^*\|_{Z^*} \quad \text{for every } T \in U \text{ and } z^* \in Z^*,$$

hence

$$\|S(T)\|_Z \le K \|i_\mu(T)\|_{L_p(\mu,Y)} \quad \text{for every } T \in U.$$

Write $W := \overline{i_\mu(U)}$. By the previous inequality, there is an operator $S_1 \colon W \to Z$ such that $S_1 \circ i_\mu|_U = S$ (cf. the proof of Proposition 2.6). Of course, $S_2 := i_\mu|_U$ is (ℓ_p^s, ℓ_p)-controlled. We claim that $S_1^* \colon Z^* \to W^*$ is (ℓ_q^s, ℓ_q)-controlled. Indeed, inequality (3.2) reads as

$$|\langle i_\mu(T), S_1^*(z^*) \rangle| \le K \, \|i_\mu(T)\|_{L_p(\mu,Y)} \, \|i_\eta(z^*)\|_{L_q(\eta,F)}$$

for every $T \in U$ and $z^* \in Z^*$. Thus, $\|S_1^*(z^*)\|_{W^*} \le K \|i_\eta(z^*)\|_{L_q(\eta,F)}$ for every $z^* \in Z^*$, so that S_1^* is (ℓ_q^s, ℓ_q)-controlled.

(iv) $\Rightarrow$ (v) This follows from Theorem 2.9.

(v) $\Rightarrow$ (i) Fix $n \in \mathbf{N}$ and take $T_1, \dots, T_n \in U$ and $z_1^*, \dots, z_n^* \in Z^*$. Then Hölder's inequality and the fact that S_2 (resp. S_1^*) is (ℓ_p^s, ℓ_p)-summing (resp. (ℓ_q^s, ℓ_q)-summing) yield

$$\begin{aligned}
\left(\sum_{i=1}^n |\langle S(T_i), z_i^* \rangle|^r \right)^{1/r} &= \left(\sum_{i=1}^n |\langle S_2(T_i), S_1^*(z_i^*) \rangle|^r \right)^{1/r} \\
&\le \left(\sum_{i=1}^n \|S_2(T_i)\|_W^r \cdot \|S_1^*(z_i^*)\|_{W^*}^r \right)^{1/r} \\
&\le \left(\sum_{i=1}^n \|S_2(T_i)\|_W^p \right)^{1/p} \cdot \left(\sum_{i=1}^n \|S_1^*(z_i^*)\|_{W^*}^q \right)^{1/q} \\
&\le K \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^n \|T_i(x^*)\|_Y^p \right)^{1/p} \cdot \sup_{e^* \in B_{E^*}} \left(\sum_{i=1}^n \|z_i^*(e^*)\|_F^q \right)^{1/q}
\end{aligned}$$

for some constant $K \geq 0$ independent of the T_i's and z_i^*'s. This shows that S is (ℓ_p^s, ℓ_q^s)-dominated.

(i) $\Rightarrow$ (ii) Observe that $L := P(B_{X^*}) \times P(B_{E^*})$ is a compact convex set of the locally convex space $C(B_{X^*})^* \times C(B_{E^*})^*$, equipped with the product of the corresponding w^*-topologies. Fix $n \in \mathbf{N}$,

$$\bar{T} = (T_1, \dots, T_n) \in (U \cap X\varepsilon Y)^n \quad \text{and} \quad \bar{z^*} = (z_1^*, \dots, z_n^*) \in (Z^* \cap E\varepsilon F)^n.$$

Consider the function $\Delta_{\bar{T},\bar{z^*}} \colon L \to \mathbf{R}$ given by

$$\Delta_{\bar{T},\bar{z^*}}(\mu,\eta) := \sum_{i=1}^n |\langle S(T_i), z_i^*\rangle|^r - K^r \frac{r}{p} \int_{B_{X^*}} \sum_{i=1}^n \|T_i(\cdot)\|_Y^p \, d\mu - K^r \frac{r}{q} \int_{B_{E^*}} \sum_{i=1}^n \|z_i^*(\cdot)\|_F^q \, d\eta,$$

where $K \geq 0$ is a constant as in Definition 3.1. Clearly, $\Delta_{\bar{T},\bar{z^*}}$ is convex and continuous, because $T_i \in X\varepsilon Y$ and $z_i^* \in E\varepsilon F$ for every $i = 1, \dots, n$. We claim that $\Delta_{\bar{T},\bar{z^*}}(\mu,\eta) \leq 0$ for some $(\mu,\eta) \in L$. Indeed, since the functions

$$x^* \mapsto \sum_{i=1}^n \|T_i(x^*)\|_Y^p \quad \text{and} \quad e^* \mapsto \sum_{i=1}^n \|z_i^*(e^*)\|_F^q$$

are w^*-continuous on B_{X^*} and B_{E^*}, they attain their suprema at some $x_{\bar{T}}^* \in B_{X^*}$ and $e_{\bar{z^*}}^* \in B_{E^*}$, respectively. By taking into account Young's inequality, we have

$$\sum_{i=1}^n |\langle S(T_i), z_i^*\rangle|^r \overset{(3.1)}{\leq} K^r \left(\sum_{i=1}^n \|T_i(x_{\bar{T}}^*)\|_Y^p \right)^{r/p} \cdot \left(\sum_{i=1}^n \|z_i^*(e_{\bar{z^*}}^*)\|_F^q \right)^{r/q} \leq K^r \frac{r}{p} \sum_{i=1}^n \|T_i(x_{\bar{T}}^*)\|_Y^p + K^r \frac{r}{q} \sum_{i=1}^n \|z_i^*(e_{\bar{z^*}}^*)\|_F^q. \tag{3.3}$$

If we write $\mu := \delta_{x_{\bar{T}}^*} \in P(B_{X^*})$ and $\eta := \delta_{e_{\bar{z^*}}^*} \in P(B_{E^*})$, then (3.3) yields $\Delta_{\bar{T},\bar{z^*}}(\mu,\eta) \leq 0$, as required.

The collection $\mathcal{C}$ of all functions $\Delta_{\bar{T},\bar{z^*}}$ as above is a convex cone in $\mathbf{R}^L$. Indeed, $\mathcal{C}$ is obviously closed under sums and we have

$$\alpha \Delta_{\bar{T},\bar{z^*}} = \Delta_{(\alpha^{1/p}T_1,\dots,\alpha^{1/p}T_n),(\alpha^{1/q}z_1^*,\dots,\alpha^{1/q}z_n^*)}$$

for all $\alpha \geq 0$.

By Ky Fan's Lemma (see e.g. [11, Lemma 9.10]), there is $(\mu,\eta) \in L$ such that $\Delta_{\bar{T},\bar{z^*}}(\mu,\eta) \leq 0$ for every $\Delta_{\bar{T},\bar{z^*}} \in \mathcal{C}$. In particular,

$$|\langle S(T), z^*\rangle|^r \leq K^r \frac{r}{p} \int_{B_{X^*}} \|T(\cdot)\|_Y^p \, d\mu + K^r \frac{r}{q} \int_{B_{E^*}} \|z^*(\cdot)\|_F^q \, d\eta \tag{3.4}$$

for all $T \in U \cap X\varepsilon Y$ and $z^* \in Z^* \cap E\varepsilon F$.

Fix $T \in U \cap X\varepsilon Y$ and $z^* \in Z^* \cap E\varepsilon F$. We will check that (3.2) holds. Write

$$a := \left(\int_{B_{X^*}} \|T(\cdot)\|_Y^p \, d\mu \right)^{1/p} \quad \text{and} \quad b := \left(\int_{B_{E^*}} \|z^*(\cdot)\|_F^q \, d\eta \right)^{1/q}.$$

If either $a = 0$ or $b = 0$, then $\langle S(T), z^*\rangle = 0$. Indeed, if $a = 0$, then for each $n \in \mathbf{N}$ inequality (3.4) applied to the pair (nT, z^*) yields

$$|\langle S(T), z^*\rangle|^r = \frac{1}{n^r} \cdot |\langle S(nT), z^*\rangle|^r \leq \frac{1}{n^r} \cdot \frac{K^r r b^q}{q},$$

hence $\langle S(T), z^*\rangle = 0$. A similar argument works for the case $b = 0$. On the other hand, if $a \neq 0$ and $b \neq 0$, then inequality (3.4) applied to the pair $(\frac{1}{a}T, \frac{1}{b}z^*)$ yields

$$\begin{aligned} |\langle S(T), z^*\rangle|^r &= a^r\, b^r \left|\left\langle S\left(\frac{1}{a}T\right), \frac{1}{b}z^*\right\rangle\right|^r \\ &\leq K^r\, a^r\, b^r \left(\frac{r}{p\, a^p}\int_{B_{X^*}} \|T(\cdot)\|_Y^p\, d\mu + \frac{r}{q\, b^q}\int_{B_{E^*}} \|z^*(\cdot)\|_F^q\, d\eta\right) = K^r\, a^r\, b.^r \end{aligned}$$

This proves (3.2) when $T \in U \cap X\varepsilon Y$ and $z^* \in Z^* \cap E\varepsilon F$.

Finally, we prove the implication (ii) $\Rightarrow$ (iii) under the additional assumptions. Fix $T \in U$ and $z^* \in Z^*$. By (c) (resp. (b)), we can take a sequence (T_n) (resp. (z_n^*)) in $U \cap X\varepsilon Y$ (resp. $Z^* \cap E\varepsilon F$) which SOT-converges to T (resp. z^*). For each $n \in \mathbf{N}$ we have

$$|\langle S(T_n), z_n^*\rangle| \leq K\left(\int_{B_{X^*}} \|T_n(\cdot)\|_Y^p\, d\mu\right)^{1/p} \cdot \left(\int_{B_{E^*}} \|z_n^*(\cdot)\|_F^q\, d\eta\right)^{1/q}. \tag{3.5}$$

Since the operators i_μ and i_η are (SOT-to-norm) sequentially continuous (see the proof of Theorem 2.9), we have

$$\lim_{n\to\infty}\left(\int_{B_{X^*}} \|T_n(\cdot)\|_Y^p\, d\mu\right)^{1/p} = \left(\int_{B_{X^*}} \|T(\cdot)\|_Y^p\, d\mu\right)^{1/p}$$

and

$$\lim_{n\to\infty}\left(\int_{B_{E^*}} \|z_n^*(\cdot)\|_F^q\, d\eta\right)^{1/q} = \left(\int_{B_{E^*}} \|z^*(\cdot)\|_F^q\, d\eta\right)^{1/q}.$$

Moreover, S is (SOT-to-norm) sequentially continuous by assumption (d), so the sequence $(S(T_n))$ converges to $S(T)$ in the norm topology. Since (z_n^*) is w^*-convergent to z^* (by (a)), we conclude that

$$\begin{aligned} |\langle S(T), z^*\rangle| &= \lim_{n\to\infty} |\langle S(T_n), z_n^*\rangle| \\ &\overset{(3.5)}{\leq} K\left(\int_{B_{X^*}} \|T(\cdot)\|_Y^p\, d\mu\right)^{1/p} \cdot \left(\int_{B_{E^*}} \|z^*(\cdot)\|_F^q\, d\eta\right)^{1/q}, \end{aligned}$$

as we wanted. The proof is finished. □

Remark 3.7. Statement (iv) in Theorem 3.2 implies that S_2 is (SOT-to-norm) sequentially continuous (by Theorem 2.9) and so is S.

Corollary 3.8. *Suppose that $Z^* \subseteq E\varepsilon F$. Let U be a subspace of $X\varepsilon Y$ and let $S\colon U \to Z$ be an operator. Then the following statements are equivalent:*

(i) *S is (ℓ_p^s, ℓ_q^s)-dominated.*

(ii) *There exist a constant $K \geq 0$ and measures $\mu \in P(B_{X^*})$ and $\eta \in P(B_{E^*})$ such that*

$$|\langle S(T), z^*\rangle| \leq K\left(\int_{B_{X^*}} \|T(\cdot)\|_Y^p\, d\mu\right)^{1/p} \cdot \left(\int_{B_{E^*}} \|z^*(\cdot)\|_F^q\, d\eta\right)^{1/q}$$

for every $T \in U$ and every $z^* \in Z^*$.

(iii) *There exist a Banach space W, an (ℓ_p^s, ℓ_p)-summing operator $S_2\colon U \to W$ and an operator $S_1\colon W \to Z$ with (ℓ_q^s, ℓ_q)-summing adjoint such that S factors as $S = S_1 \circ S_2$.*

Since $E\hat{\otimes}_\varepsilon F \subseteq E\varepsilon F$, one can apply Corollary 3.8 whenever $Z^* \subseteq E\hat{\otimes}_\varepsilon F$ (in particular, it works when $E := Z^*$ and $F := \mathbf{R}$ or vice versa).

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