Composition operators on classes of holomorphic functions on Banach spaces

Author:
Daniel Santacreu Ferrà

Supervisors:
David Jornet Casanova
Pablo Sevilla Peris

Valencia, June 2022

UNIVERSITAT POLITÈCNICA DE VALÈNCIA

En primer lloc vull donar les gràcies a David i a Pablo. Sempre m'heu ajudat en tot assegurant-se de què no em faltara de res, no només durant el doctorat, sinó també en el màster. David, m'has donat l'espenteta necessària i m'has ensenyat a tindre cura pel que fem. I tu, Pablo, m'has ajudat a posar ordre en el meu cap i m'has mostrat que darrere les matemàtiques hi ha persones. Sense vosaltres no tindríem aquesta tesi en les nostres mans.

A tu, Pepe, vull agrair-te el suport que m'has donat de forma desinteressada des del principi. L'empremta que has deixat en la tesi és enorme i indispensable. Gràcies a tu no han faltat preguntes que ens guiaren ni resultats per a arribar on volíem.

Despite the fact that the stay did not take place under the circumstances we wished for, you, Thomas, were the best host I could have hoped for. I would even say that things had been going very smooth. Working with you will always be a pleasure.

M'agrada pensar que aquest viatge va començar l'estiu del 2007, dalt la Serena de camí al Baladrar. Amb una pregunta que desencadenaria tot... El "sí" va ser unànime, també vostre, Mamà i Papà, gràcies per no haver dubtat mai que aquest era el meu camí. També teu, Lídia, sempre seràs la meua primera companya de viatge. Poc després et vaig conèixer, Joan, i vull donar-te les gràcies perquè sempre has caminat al meu costat. Moltes gràcies Óscar, per caminar amb mi fins a Blagoevgrad... i fins on faça falta. Per al PDF, perquè siga refugi per a viatgers com nosaltres per molts anys més. Per als que un dia m'acompanyaren, Paquita, Álvaro i Paco. I per a totes aquelles persones amb qui he compartit el viatge.

Vull acabar amb tres agraïments especials: A Erik, perquè a pesar de la precarietat encara no ens hem rendit. A Alberto, perquè sempre ens quedarà CUBA! I a Esther, li done les gràcies amb el record d'un balcó i una pregunta sense formular.

Resum

L'objectiu principal d'aquesta tesi és l'estudi de diferents propietats (principalment ergòdiques) d'operadors de composició i de composició ponderats actuant en espais de funcions holomorfes en un espai de Banach de dimensió infinita.

Siga X un espai de Banach i U un subconjunt obert. Donada una aplicació $\varphi : U \to U$, l'acció $f \mapsto C_\varphi(f) = f \circ \varphi$ defineix un operador, anomenat *operador de composició* (i φ s'anomena *símbol* de l'operador). Considerem aquest operador actuant en diferents espais de funcions. La filosofia general és intentar caracteritzar en cada cas les propietats del nostre interés en funció de condicions en φ. També, donada $\psi : U \to \mathbb{C}$, l'*operador de multiplicació* es defineix com a $M_\psi(f) = \psi \cdot f$ i (amb φ com abans), l'*operador de composició ponderat* com a $C_{\psi,\varphi}(f) = \psi \cdot (f \circ \varphi)$ (en aquest cas ψ es coneix com el *pes* o *multiplicador* de l'operador). Novament, la idea és descriure propietats d'aquests operadors en termes de condicions sobre φ i/o ψ. Clarament $C_{\psi,\varphi} = M_\psi \circ C_\varphi$, i prenent $\varphi = \mathrm{id}_U$ (la identitat en U) o $\psi \equiv 1$ (la funció constant 1) recuperem M_ψ i C_φ.

Denotem per B la bola unitat oberta d'X. L'espai de funcions holomorfes $f : B \to \mathbb{C}$ es denota $H(B)$. Escrivim $H_b(B)$ per a l'espai de funcions holomorfes en B de tipus fitat i $H^\infty(B)$ per a l'espai de funcions holomorfes i fitades en B. Anem a considerar operadors de composició i de composició ponderats definits en cadascun d'aquests espais (prenent llavors $U = B$ en la definició). També considerem operadors de composició definits en l'espai vectorial de polinomis continus i m-homogenis (denotat $\mathscr{P}(^m X)$). En aquest cas prenem $U = X$.

La tesi consta de cinc capítols. En el Capítol 1 donem les definicions i resultats bàsics necessaris perquè el text siga autocontingut. En el Capítol 2 tractem amb operadors de composició ergòdics en mitjana i fitats en potències definits en $\mathscr{P}(^m X)$. En el Capítol 3 estudiem operadors de composició ergòdics en mitjana i fitats en potències definits en $H(B)$, $H_b(B)$ i $H^\infty(B)$; tractant també el cas particular en que B és la bola d'un espai de Hilbert. En el Capítol 4 estudiem la compacitat d'operadors de composició ponderats definits en $H^\infty(B)$, així com també la fitació, reflexivitat, quan és Montel i la compacitat (feble) en $H_b(B)$. Finalment, en el Capítol 5 obtenim resultats sobre la fitació en potències i ergodicitat en mitjana d'operadors de composició ponderats actuant en $H(B)$, $H_b(B)$ i $H^\infty(B)$; així com també sobre compacitat i ergodicitat en mitjana de l'operador de multiplicació.

Resumen

El objetivo principal de esta tesis es el estudio de diferentes propiedades (principalmente ergódicas) de operadores de composición y de composición ponderados actuando en espacios de funciones holomorfas definidas en un espacio de Banach de dimensión infinita.

Sea X un espacio de Banach y U un subconjunto abierto. Dada una aplicación $\varphi\colon U \to U$, la acción $f \mapsto C_\varphi(f) = f \circ \varphi$ define un operador, llamado *operador de composición* (y a φ se le llama *símbolo* del operador). Consideramos este operador actuando en diferentes espacios de funciones. La filosofía general es intentar caracterizar en cada caso las propiedades de nuestro interés en función de condiciones en φ. También, dada $\psi\colon U \to \mathbb{C}$, el *operador de multiplicación* se define como $M_\psi(f) = \psi \cdot f$ y (con φ como antes), el *operador de composición ponderado* como $C_{\psi,\varphi}(f) = \psi \cdot (f \circ \varphi)$ (en este caso ψ se conoce como el *peso* o *multiplicador* del operador). Nuevamente, la idea es describir propiedades de estos operadores en términos de condiciones sobre φ y/o ψ. Claramente $C_{\psi,\varphi} = M_\psi \circ C_\varphi$, y tomando $\varphi = \mathrm{id}_U$ (la identidad en U) o $\psi \equiv 1$ (la función constante 1) recuperamos M_ψ y C_φ.

Denotamos con B a la bola unidad abierta de X. El espacio de funciones holomorfas $f : B \to \mathbb{C}$ se denota $H(B)$. Escribimos $H_b(B)$ para el espacio de funciones holomorfas en B de tipo acotado y $H^\infty(B)$ para el espacio de funciones holomorfas y acotadas en B. Vamos a considerar operadores de composición y de composición ponderados definidos en cada uno de estos espacios (tomando entonces $U = B$ en la definición). También consideramos operadores de composición definidos en el espacio vectorial de polinomios continuos y m-homogéneos (denotado $\mathscr{P}(^m X)$). En este caso tomamos $U = X$.

La tesis consta de cinco capítulos. En el Capítulo 1 damos las definiciones y resultados básicos necesarios para que el texto sea autocontenido. En el Capítulo 2 tratamos con operadores de composición ergódicos en media y acotados en potencias definidos en $\mathscr{P}(^m X)$. En el Capítulo 3 estudiamos operadores de composición ergódicos en media y acotados en potencias definidos en $H(B)$, $H_b(B)$ y $H^\infty(B)$; tratando también el caso particular en que B es la bola de un espacio de Hilbert. En el Capítulo 4 estudiamos la compacidad de operadores de composición ponderados definidos en $H^\infty(B)$, así como la acotación, reflexividad, cuándo es Montel y la compacidad (débil) en $H_b(B)$. Finalmente, en el Capítulo 5 obtenemos resultados sobre la acotación

en potencias y ergodicidad en media de operadores de composición ponderados actuando en $H(B)$, $H_b(B)$ y $H^\infty(B)$; así como sobre compacidad y ergodicidad en media del operador de multiplicación.

Summary

The main aim in this thesis is to study different properties (mostly ergodic) of composition and weighted composition operators acting on spaces of holomorphic functions defined on an infinite dimensional complex Banach space.

Let X be a Banach space and U some open subset. Given a mapping $\varphi\colon U \to U$ the action $f \mapsto C_\varphi(f) = f \circ \varphi$ defines an operator, called *composition operator* (and φ is called the *symbol* of the operator). We consider this operator acting on different spaces of functions. The general philosophy is to try to characterise in each case the properties of our interest in terms of conditions on φ. Also, given $\psi\colon U \to \mathbb{C}$ the *multiplication operator* is defined as $M_\psi(f) = \psi \cdot f$ and (with φ as above), the *weighted composition operator* as $C_{\psi,\varphi}(f) = \psi \cdot (f \circ \varphi)$ (here ψ is called the *weight* or *multiplier* of the operator). Again, the idea is to describe properties of these operators in terms of conditions on ψ and/or φ. Clearly $C_{\psi,\varphi} = M_\psi \circ C_\varphi$, and taking $\varphi = \mathrm{id}_U$ (the identity on U) or $\psi \equiv 1$ (the constant function 1) we recover M_ψ and C_φ.

We denote the open unit ball of X by B. The space of all holomorphic functions $f : B \to \mathbb{C}$ is denoted by $H(B)$. We write $H_b(B)$ for the space holomorphic functions of bounded type on B, and $H^\infty(B)$ for the space of bounded holomorphic functions on B. We are going to consider composition and weighted composition operators defined on each one of these spaces (taking then $U = B$ in the definition). We also consider composition operators defined on the vector space of all continuous m-homogeneous polynomials on X (which is denoted by $\mathscr{P}(^mX)$). In this case we take $U = X$.

The thesis consists of 5 chapters. In Chapter 1 we introduce definitions and basic results, needed to make the text self-contained. In Chapter 2 we deal with mean ergodic and power bounded composition operators defined on $\mathscr{P}(^mX)$. In Chapter 3 we study mean ergodic and power bounded composition operators defined on $H(B)$, $H_b(B)$ and $H^\infty(B)$; considering also the particular case when B is the ball of a Hilbert space. In Chapter 4 we study compactness of weighted composition operators defined on $H^\infty(B)$, as well as boundedness, reflexivity, being Montel and (weak) compactness on $H_b(B)$. Finally, in Chapter 5 we obtain different results about power boundedness and mean ergodicity of weighted composition operators acting on $H(B)$, $H_b(B)$ and $H^\infty(B)$, as well as about compactness and mean ergodicity of the multiplication operator.

Contents

Introduction

The aim of this thesis is to study different properties (mainly of ergodic nature) of composition and weighted composition operators acting on spaces of holomorphic functions defined on an infinite dimensional complex Banach space.

The beginning of the study of holomorphic functions with infinitely many variables (or, to be more precise, defined on infinite dimensional topological vector spaces) goes back to the end of the 19th century, in works of Volterra and von Koch. It was developed in the first half of the 20th century with the contribution (among others) of Dunford, Fréchet, Hilbert, Hille or Taylor. In the decade of the 1960s a more 'structural' research was started by Nachbin and his school: different spaces of holomorphic functions were considered, endowed with different topologies, and their properties were carefully studied. The interested reader is referred to the monograph [24] and its notes and remarks. Not only the structure of the spaces, but also the behaviour of the operators acting between them has been a central problem of operator theory from the last century. When dealing with spaces of functions (in particular of holomorphic functions), composition operators play a central role. When acting on spaces of holomorphic functions defined on finite dimensional spaces, these have been extensively studied, and there is a wide literature on the subject. The study of composition operators between spaces of holomorphic functions defined on infinite dimensional spaces, however, is much more scarce and, to our best knowledge, the dynamical or ergodic properties of such operators have not been object of study so far.

Let X be a complex Banach space and U some open subset. Given a function $\varphi\colon U \to U$ the action $f \mapsto C_\varphi(f) = f \circ \varphi$ defines an operator (called *composition operator* with *symbol* φ) on a suitable space of functions $f : U \to \mathbb{C}$. We take X to be infinite dimensional, and denote its open unit ball by B. We are going to consider such operators acting on five different spaces of holomorphic functions (see Section 1.3 below for the precise definitions, both for the spaces and the topologies): on the space of holomorphic functions $f : B \to \mathbb{C}$ (denoted $H(B)$ and endowed with the the topology τ_0 of uniform convergence on compact subsets of B), the space of holomorphic functions of bounded type on B (denoted $H_b(B)$, endowed with the topology τ_b of uniform convergence on B-bounded sets), the space of bounded holomorphic functions on B (denoted $H^\infty(B)$ and endowed with the topology induced by the sup-norm), and the space of continuous m-homogeneous polynomials (denoted $\mathscr{P}(^m X)$) which we endow with two possible topologies: the one of convergence on compact sets (in which case we write $\mathscr{P}(^mX)_{\tau_0}$), or the one defined by the norm $\|p\| = \sup_{\|x\|_X \leq 1} |p(x)|$ (in which case we write $\mathscr{P}(^mX)_{\|\cdot\|}$). Note that in the first three cases we take $U = B$ in the definition of the composition operator, and φ has to be respectively holomorphic, holomorphic of bounded type and holomorphic. For the space of polynomials, we take $U = X$, and conditions on φ for the composition operator to be well defined will be given. Several authors have studied different properties of composition operators on spaces of holomorphic functions on the unit ball of a Banach space. See, for instance, [3, 28, 29, 32] and the references therein. However, there seems to be no previous literature about the dynamics of such operators.

Also, if $\psi : B \to \mathbb{C}$ is holomorphic (and φ is as above), we consider the *weighted composition operator* (with *symbol* φ and *weight* or *multiplier* ψ), defined by $f \mapsto C_{\psi,\varphi}(f) = \psi \cdot (f \circ \varphi)$ acting on the spaces $H(B)$, $H_b(B)$ and $H^\infty(B)$. Also, the multiplication operator is defined as $M_\psi(f) = \psi \cdot f$. Clearly $C_{\psi,\varphi} = M_\psi \circ C_\varphi$, and taking $\varphi = \mathrm{id}_B$ (the identity on B) or $\psi \equiv 1$ (the constant function 1) we recover M_ψ and C_φ. While weighted composition and multiplication operators are already quite well understood in the finite dimensional setting, we were not able to find any previous work on this operators when X is infinite dimensional. So, apart from ergodic properties, we have also studied other properties of the operators, such as compactness, reflexivity or being Montel.

In all cases the general philosophy is the same. We try to describe as precisely as possible the properties that we are interested in terms of conditions involving the symbol φ and/or (whenever is the case) the weight ψ.

The ergodic operator theory has its origin in the 1930s, when von Neumann proved that the Cesàro means of every unitary operator on $L_2(0,1)$ converge at every point or, to put in modern terms, that every such operator is mean ergodic (see Section 1.4 for precise definitions). This was immediately extended in various ways by the work of (among others) Kakutani, Riesz and Yosida. The latter showed that if an operator on a Banach space is power bounded (i.e. the set of the compositions with itself is bounded in norm) and the Cesàro means at each point are weakly compact, then it is mean ergodic. This is the starting point of a rich and vast theory that has been developed in various ways. These and other concepts, originally for Banach spaces were extended to more general situations, and other interesting ones have been defined. Here we are mainly (but not only) interested in mean ergodicity and power boundedness, and their interplay, for composition and weighted composition operators.

The motivation and inspiration of our investigation comes from several previous works, as [9], where mean ergodicity of $C_\varphi : H(U) \to H(U)$ (here U is a connected domain of holomorphy in $\mathbb{C}^d$) was characterised. More precisely, it was proved that C_φ is power bounded if and only if it is (uniformly) mean ergodic, and this happens if and only if the symbol φ has stable orbits. On the other hand, if the domain is the unit disc, it was characterised in [5] when C_φ is mean ergodic or uniformly mean ergodic on the disc algebra or on the space of bounded holomorphic functions in terms of the asymptotic behaviour of the symbol. Power boundedness and (uniform) mean ergodicity of weighted composition operators on the space of holomorphic functions on the unit disc was analysed in [6] in terms of the symbol and the multiplier (see Section 1.5 below for more details). In [39] power boundedness and mean ergodicity for (weighted) composition operators on function spaces defined by local properties was studied in a very general framework which extends previous work. In particular, it permits to characterise (uniform) mean ergodicity for composition operators on a large class of function spaces which are Fréchet-Montel spaces when equipped with the compact-open topology. The results of [39] do not apply to our setting (and cannot be adapted) because $H(B)$ and $\mathcal{P}(^mX)_{\tau_0}$ are Montel but not Fréchet, $H_b(B)$ is Fréchet but not Montel, and $H^\infty(B)$ and $\mathcal{P}(^mX)_{\|\cdot\|}$ are Banach (hence not Montel). Other recent contributions to this topic can be found in [40], where mean ergodicity of composition operators on the space of bounded holomorphic functions on the n-dimensional Euclidean ball is studied, and in [36], where the authors consider composition operators on weighted spaces of holomorphic functions on the disc.

Compactness of weighted composition operators defined on spaces of functions in $\mathbb{C}$ have been extensively studied, and there is a huge related literature (see, for example, [10, 17, 18, 19, 30, 52] and the references therein). However, for spaces of holomorphic functions on infinite dimensional spaces there are only a few references; for instance, the compactness of the composition operator C_φ defined on $H^\infty(B)$ was studied in [3] and on $H_b(B)$ in [31]. See Section 1.5 below for more detailed information about previous results.

The Thesis consists of 5 chapters. Chapter 1 collects the necessary definitions and prerequisites to follow the text in a self-contained way. Moreover, we give a detailed introduction of the results we are going to study in the last section of this chapter. Chapter 2 is based on the already published paper [37] and treats mean ergodic composition operators when acting on spaces of m-homogeneous polynomials. The situation is quite different for the two topologies considered in this space: while in the case of uniform convergence on compact sets every power bounded composition operator is uniformly mean ergodic, for the topology of the norm there is no relation between the latter properties. Chapter 3 is based on the already published paper [38] and here we study mean ergodic composition operators on the different spaces of holomorphic functions considered above ($H(B)$, $H_b(B)$ and $H^\infty(B)$) when defined on the unit ball of a Banach or a Hilbert space. Chapter 4 is based on the preprint [11] which has been recently accepted for publication in the Journal of Operator Theory. In this chapter, we study when the weighted composition operator $C_{\psi,\varphi}$ is compact in the space of all bounded analytic functions $H^\infty(B)$, and when it is bounded, reflexive, Montel and (weakly) compact in the space of analytic functions of bounded type $H_b(B)$. The study is given in terms of properties of the weight ψ and the symbol φ. Finally, in Chapter 5 we obtain different results about the ergodicity of weighted composition operators when acting in each of the spaces introduced above, as well as about the compactness and the ergodicity of the multiplication operator in terms of the weight.

Chapter 1

Preliminaries

This chapter will serve as a toolbox for the entire work. We give some definitions related with locally convex spaces, operator theory and holomorphic functions. We also recall some useful results for the study of mean ergodic and power bounded operators. In the last part of this chapter we revise the literature of (weighted) composition operators on spaces of holomorphic functions.

1.1 Basics of functional analysis

We collect here the basic definitions and results that will be used all along the text. We follow [41, 42, 45, 49].

Let E be a vector space over $\mathbb{R}$ or $\mathbb{C}$. We say that a map $p\colon E \to \mathbb{R}$ is a *seminorm* when it satisfies the following

- $p(x) \geq 0$,
- $p(\lambda x) = |\lambda| p(x)$ and,
- $p(x + y) \leq p(x) + p(y)$

for every $x, y \in E$ and $\lambda \in \mathbb{K}$. The map p is a *norm* if additionally $p(x) = 0$ implies that $x = 0$. Let Γ_E denote a collection of seminorms on E determining its topology. Then, we say that (E, Γ_E) is a *locally convex Hausdorff space* (briefly lcHs) if the family Γ_E satisfies the following

- For each $p, q \in \Gamma_E$ there is $r \in \Gamma_E$ such that $\max(p(x), q(x)) \leq r(x)$ for every $x \in E$.
- For each $x \in E$ with $x \neq 0$ there is $p \in \Gamma_E$ such that $p(x) > 0$.

A lcHs (E, Γ_E) is a *Fréchet space* if it is complete and metrizable. In such a case Γ_E may be chosen countable. If (E, Γ_E) is a Banach space then Γ_E is even a singleton.

Let (E, Γ_E) be a lcHs. We denote by E' the *dual space* of E that is, the collection of all functions $u: E \to \mathbb{C}$ that are linear and continuous. The dual space of E' is denoted by E'', this vector space is also known as the *bidual* of E. Given two nonempty sets $M \subset E$ and $N \subset E'$, following [45, p. 255], we denote the *polar* sets of M and N as

$$M^\circ := \{u \in E' : \sup_{x \in M} |u(x)| \leq 1\}$$
$$N^\circ := \{x \in E : \sup_{u \in N} |u(x)| \leq 1\}$$

(see also [35, p. 190-192]). Now we recall [45, Theorem 22.13].

Theorem 1.1 (Bipolar Theorem). *Let E be a lcHs and A be an absolutely convex subset of E. Then, $\overline{A} = (A^\circ)^\circ =: A^{\circ\circ}$.*

We denote by $\sigma(E, E')$ the topology of weak convergence on E, where each $u \in E'$ defines a seminorm $p_{\{u\}}: E \to \mathbb{R}$ given by $p_{\{u\}}(x) = |u(x)|$ for every $x \in E$. On the other hand, the topology $\sigma(E', E)$ of pointwise convergence on E' is usually known as the weak* topology. That is, each $x \in E$ defines a seminorm $p_{\{x\}}: E' \to \mathbb{R}$ given by $p_{\{x\}}(u) = |u(x)|$ for every $u \in E'$. We write this dual space as $(E', \sigma(E', E))$ or shortly, E'_σ. We denote by $\beta(E', E)$ the topology of uniform convergence on the bounded sets of E in E', and we write this space as $(E', \beta(E', E))$ or shortly, E'_β and also E'. In this case, each bounded set $V \subset E$ defines a seminorm $p_V: E' \to \mathbb{R}$ given by $p_V(u) = \sup_{x \in V} |u(x)|$ for every $u \in E'$.

Let X be a Banach space. We denote by B the open unit ball of X (or B_X if we need to stress the space X). For $x_0 \in X$ and $\varepsilon > 0$, the open ball centred at x_0 and radius ε is denoted by $B(x_0, \varepsilon)$. For the special case of $X = \mathbb{C}$ we denote by $\mathbb{D}$ the open unit disc and by $D(z_0, \varepsilon)$ the open disc centred at $z_0 \in \mathbb{C}$ and radius $\varepsilon > 0$. The dual space X' is a Banach space when it is endowed with the dual norm $\|u\|_{X'} = \sup_{\|x\| \leq 1} |u(x)|$. We also have that $(X', \|\cdot\|_{X'})$ coincides with $(X', \beta(X', X))$.

We recall that ℓ_p, with $1 \leq p < \infty$, is the following space

$$\ell_p := \left\{ x \in \mathbb{C}^{\mathbb{N}} : \|x\|_p := \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} < \infty \right\}.$$

The spaces ℓ_∞ and c_0 are given, respectively, by

$$\ell_\infty := \left\{ x \in \mathbb{C}^{\mathbb{N}} : \|x\|_\infty := \sup_{j \in \mathbb{N}} |x_j| < \infty \right\},$$
$$c_0 := \left\{ x \in \ell_\infty : \lim_{j \to \infty} x_j = 0 \right\}.$$

We denote by e_n the n-th element of the canonical basis of these spaces.

A lcHs E is said to be *quasinormable* if given a 0-neighbourhood V in E there is a 0-neighbourhood U in E such that for any $\varepsilon > 0$ there is a bounded set B in E satisfying $U \subset B + \varepsilon V$.

A subset M of a vector space E is said to be *absorbing* if $E = \bigcup_{n\in\mathbb{N}} nM$. A *barrel* in a lcHs E is a subset which is closed, absolutely convex and absorbing in E. The space E is *barrelled* if each barrel in E is a 0-neighbourhood. Banach spaces and Fréchet spaces are examples of barrelled spaces [45, Proposition 23.23].

A lcHs (E, Γ_E) is *semi-reflexive* if $E = E''$ as vector spaces. Moreover, if $(E, \Gamma_E) = (E'', \beta(E'', E'))$ as lcHs we say that E is *reflexive*. A lcHs E is reflexive if and only if E is semi-reflexive and barrelled (see [45, Proposition 23.22]). The following gives a characterization for semi-reflexive spaces

Proposition 1.2. [45, Proposition 23.18]. *A lcHs E is semi-reflexive if and only if every bounded set in E is relatively weakly compact.*

Now, we say that a lcHs E is *semi-Montel* if every bounded set in E is relatively compact. If additionally E is barrelled we say that E is *Montel*.

1.2 Linear operators in lcHs

Here we recall the basic definitions for linear operators in lcHs and a characterization for barreled spaces.

Given two lcHs E, F, we denote by $\mathscr{L}(E, F)$ the space of continuous linear operators $T \colon E \to F$. For the space $\mathscr{L}(E, E)$ we simply write $\mathscr{L}(E)$. The identity operator on $\mathscr{L}(E)$ is denoted by id (or id_E if we need to stress the space). The transposed operator of $T \in \mathscr{L}(E, F)$ is denoted by $T^t \in \mathscr{L}(F', E')$.

We say that an operator $T \colon E \to F$ is

- *bounded* if there is a 0-neighbourhood U in E such that $T(U)$ is bounded in F;
- *compact* if there is a 0-neighborhood U in E such that $T(U)$ is relatively compact in F;
- *weakly compact* it there is a 0-neighbourhood U in E such that $T(U)$ is weakly relatively compact in F;
- *Montel* if it maps every bounded set in E into a relatively compact set in F;
- *reflexive* if it maps every bounded set in E into a weakly relatively compact set in F.

In the space $\mathscr{L}(E, F)$ we consider different topologies:

- τ_s is the topology of pointwise convergence. That is, each seminorm $p \in \Gamma_F$ and each $x \in E$ define a seminorm $q_{p,\{x\}}$ in $\mathscr{L}(E,F)$ as

$$q_{p,\{x\}}(T) = p(Tx) \quad \text{for every } T \in \mathscr{L}(E,F).$$

- τ_0 is the topology of uniform convergence on the compact sets of E. That is, each seminorm $p \in \Gamma_F$ and each compact set $K \subset E$ define a seminorm $q_{p,K}$ in $\mathscr{L}(E,F)$ as

$$q_{p,K}(T) = \sup_{x \in K} p(Tx) \quad \text{for every } T \in \mathscr{L}(E,F).$$

- τ_b is the topology of uniform convergence on the bounded sets of E. That is, each seminorm $p \in \Gamma_F$ and each bounded set $V \subset E$ define a seminorm $q_{p,V}$ in $\mathscr{L}(E,F)$ as

$$q_{p,V}(T) = \sup_{x \in V} p(Tx) \quad \text{for every } T \in \mathscr{L}(E,F).$$

Let E and F be two lcHs. A subset H of $\mathscr{L}(E,F)$ is said to be *pointwise bounded* if for each $x \in E$ the set $\{Tx : T \in H\}$ is bounded in F. A subset H of $\mathscr{L}(E,F)$ is *equicontinuous* if for each $p \in \Gamma_F$ there is $c > 0$ and $q \in \Gamma_E$ so that

$$\sup_{T \in H} p(Tx) \leq cq(x) \quad \text{for every } x \in E.$$

The Uniform Boundedness Principle gives a characterization for barrelled spaces. Here we include the version of [49, Proposition 4.1.3]

Proposition 1.3 (Uniform Boundedness Principle). *Let E be a lcHs. Then E is barrelled if and only if for every lcHs F we have that every pointwise bounded set $U \subset \mathscr{L}(E,F)$ is equicontinuous.*

Let X be a Banach space. Then for any operator $T \in \mathscr{L}(X)$ we denote by

$$\|T\| := \sup_{\|x\| \leq 1} \|Tx\|$$

the *operator norm*. With this norm we have that $(\mathscr{L}(X), \|\cdot\|) = (\mathscr{L}(X), \tau_b)$. We clearly have that every operator $T \in \mathscr{L}(X)$ is bounded. We also have that an operator $T \in \mathscr{L}(X)$ is (weakly) compact if and only if it is (reflexive) Montel.

1.3 Spaces of holomorphic functions

In this section we present the spaces of holomorphic functions and homogeneous polynomials on which we study the dynamics of (weighted) composition operators. We follow [24, 47].

Given two Banach spaces X and Y over $\mathbb{C}$, a mapping $p\colon X \to Y$ is a continuous *m-homogeneous polynomial* if there exists a continuous m-linear form $L\colon X \times \overset{m}{\dots} \times X \to Y$ such that $p(x) = L(x, \dots, x)$ for every $x \in X$. For the rest of the work we will refer to continuous m-homogeneous polynomials briefly as m-homogeneous polynomials. The vector space of all such m-homogeneous polynomials is denoted by $\mathscr{P}(^m X, Y)$, and by $\mathscr{P}(^m X)$ whenever $Y = \mathbb{C}$. Note that $\mathscr{P}(^1 X)$ is the dual of X.

Let $U \subset X$ be an open subset. A function $f : U \to Y$ is *holomorphic* if for each $a \in U$ there exists a ball $B(a, r) \subset U$ and a sequence $(p_m)_{m\geq 0}$, where each $p_m\colon X \to Y$ is an m-homogeneous polynomial for $m > 0$ and $p_0\colon X \to Y$ is a constant function, such that

$$f(x) = \sum_{m=0}^{\infty} p_m(x - a) \tag{1.1}$$

converges uniformly on $B(a, r)$. Moreover, we can see in [47, Remark 5.2] that the sequence is unique. We denote by df_a the differential of f at a, that is $df_a(x) = p_1(x - a)$ for all $x \in U$. Also, we have that a mapping $p\colon X \to Y$ is a continuous m-homogeneous polynomial if and only if p is holomorphic and

$$p(\lambda x) = \lambda^m p(x) \tag{1.2}$$

for every $x \in X$ and $\lambda \in \mathbb{C}$ (see [20, Corollary 15.34]).

We say that $f : U \to Y$ is *G-holomorphic* if for each $x \in U$ and $y \in X$ the mapping $\lambda \mapsto f(x + \lambda y)$ is holomorphic (as a vector-valued function of one complex variable) on the open set $\{\lambda \in \mathbb{C}\colon x + \lambda y \in U\}$. We say that $f : U \to Y$ is *weakly (G-)holomorphic* if $u \circ f$ is (G-)holomorphic for every $u \in Y'$.

All these notions of holomorphy are closely related. We collect this in the following result. It is classical within the theory of holomorphic functions on infinitely dimensional spaces and its proof can be found in [47, Theorems 8.7 and 8.12].

Theorem 1.4. *Let X, Y be Banach spaces and $U \subset X$ open. For a function $f : U \to Y$ the following are equivalent.*

a) f is holomorphic.

b) f is weakly holomorphic.

c) f is continuous and G-holomorphic.

d) f is continuous and weakly G-holomorphic.

The space of all holomorphic functions $f : U \to Y$ is denoted by $H(U, Y)$. We mostly focus on the case of $U = B$. We write $H(B)$ for $H(B, \mathbb{C})$. Let τ_0 denote the topology of uniform convergence on compact sets of B, then $(H(B), \tau_0)$ is denoted $H(B)_{\tau_0}$ or simply $H(B)$. With this topology $H(B)$ becomes a locally convex Hausdorff space. We also denote this topology by τ_0 since there is no possibility of confusion with the topology of uniform convergence on compact subsets for $\mathscr{L}(E,F)$ defined above. We have that each compact subset $K \subset B$ defines a seminorm on $H(B)$ as follows

$$\|f\|_K := \sup_{x \in K} |f(x)|.$$

For the open unit ball $B \subset X$ we say that a set $A \subseteq B$ is *B-bounded* if there is $0 < r < 1$ so that $A \subset rB$. Then, we say that a mapping $F : B \to B$ is of *bounded type* if it maps B-bounded sets into B-bounded sets. We also say that a mapping $f : B \to \mathbb{C}$ is of bounded type if it maps B-bounded sets into bounded sets. Equivalently, the following are satisfied:

(BTa) A map $F : B \to B$ is of bounded type if and only if for each $0 < r < 1$ there is $0 < s < 1$ such that $F(rB) \subseteq sB$.

(BTb) A map $f : B \to \mathbb{C}$ is of bounded type if and only if for each $0 < r < 1$ we have $\sup_{\|x\|<r} |f(x)| < \infty$.

We denote by $H_b(B)$ the space of all functions in $H(B)$ which are of bounded type. The space $H_b(B)$ is Fréchet when endowed with the topology τ_b induced by the seminorms

$$\|f\|_r := \sup_{\|x\|<r} |f(x)|, \quad \text{where } 0 < r < 1,$$

i.e. of uniform convergence on B-bounded sets (as before there is no confusion with the analogous topology defined on $\mathscr{L}(E,F)$).

We denote by $H^\infty(B)$ the subspace of $H(B)$ which consists of all bounded holomorphic functions on B. Given $f \in H^\infty(B)$ we define the natural norm

$$\|f\|_\infty := \sup_{\|x\|<1} |f(x)|,$$

which turns $H^\infty(B)$ into a Banach space (which in fact is a Banach algebra).

For these spaces we have the following continuous inclusions:

$$H^\infty(B) \hookrightarrow H_b(B) \hookrightarrow H(B).$$

The space $\mathscr{P}(^m X)$ can be endowed with several different topologies. We focus only on two of them. On one hand we consider τ_0, the topology of uniform convergence on the compact sets of X. We denote $(\mathscr{P}(^m X), \tau_0)$ by $\mathscr{P}(^m X)_{\tau_0}$. On the other hand, see [47, Corollary 2.3], the norm

$$\|p\| := \sup_{\|x\|_X \leq 1} |p(x)| \tag{1.3}$$

turns $\mathscr{P}(^m X)$ into a Banach space. We denote it by $\mathscr{P}(^m X)_{\|\cdot\|}$.

Let us review the basic properties of these spaces:

- $H(B)$ is a semi-Montel lcHs (see [47, Proposition 9.16]) and it is barrelled if and only if X is finite-dimensional (in this case $H(B) = H_b(B)$, see Proposition 1.5 below).
- $H_b(B)$ is Fréchet (hence barrelled) and quasinormable [2]. It is (semi-)Montel if and only if X is finite-dimensional (see Proposition 1.6 below).
- $H^\infty(B)$ is a Banach space (hence barrelled).
- $\mathscr{P}(^mX)_{\tau_0}$ is semi-Montel (see [46, Theorem 2.5] and [49, Definition 8.3.49]).
- $\mathscr{P}(^mX)_{\|\cdot\|}$ is a Banach space (hence barrelled).

Proposition 1.5. *The space $H(B)$ is barrelled if and only if X is finite dimensional. In this case $H(B) = H_b(B)$.*

Proof. Suppose $H(B)$ is barrelled, then every complemented subspace must be barrelled (see [49, Corollary 4.2.2 (i)]). We denote by X'_{τ_0} the dual space X' endowed with the topology of uniform convergence on the compact sets of X. We claim that $H(B)$ has a complemented copy of $X'_{\tau_0} \subset H(B)$. That is, there exist $J : X'_{\tau_0} \to H(B)$ and $P : H(B) \to X'_{\tau_0}$ continuous linear maps such that $P \circ J : X'_{\tau_0} \to X'_{\tau_0}$ is the identity. We consider J as the restriction map given by $u \mapsto u|_B$, for each $u \in X'$, and P as the map $f \mapsto df_0$, for each $f \in H(B)$. The maps J and P are clearly continuous. Since $u \in X'$ we have $du_0 = u$. Then

$$(P \circ J)(u) = P(J(u)) = P(u|_B) = d(u|_B)_0 = du_0 = u,$$

and we obtain the claim.

By Mackey-Arens theorem (see [45, Theorem 23.8]) we have that $(X'_{\tau_0})' = X$. Since B is a pointwise bounded set in $(X'_{\tau_0})'$, B is an equicontinuous set because X'_{τ_0} is barrelled. By [48, Theorem 8.6.4] there exists an absolutely convex compact set $K \subset X$ such that $B \subset K$. Then, B is open and relatively compact. X is finite dimensional.

Conversely, if X is finite-dimensional every bounded set of X is relatively compact in X. In particular, every B-bounded set is relatively compact. Thus the topology τ_0 is finer than the topology of uniform convergence on B-bounded sets, so this two topologies coincide. Therefore $H(B) = H_b(B)$, which is a Fréchet space, and so barrelled. □

Proposition 1.6. *The space $H_b(B)$ is semi-Montel if and only if X is finite dimensional.*

Proof. Assume $H_b(B)$ is semi-Montel. Then every bounded subset is relatively compact. In particular, the set $B_{X'}$ is relatively compact in $X' = X'_\beta$. So, necessarily, X' is finite dimensional, and so is X.

Conversely, as we have seen previously, when X is finite dimensional we have $H(B) = H_b(B)$ and then the space is semi-Montel. □

Remark 1.7. Let $f : B \to \mathbb{C}$ be a holomorphic function. Consider the Taylor series expansion of f at 0 given by the sequence $(p_m)_{m\geq 0}$ (see (1.1)). By [20, Proposition 15.33] we have that this series converges for every $x \in B$ and the following is satisfied:

$$\sup_{\|x\|<r} |p_m(x)| \leq \sup_{\|x\|<r} |f(x)| \quad \text{for every } 0 < r < 1 \text{ and } m \in \mathbb{N}_0. \tag{1.4}$$

Now, take $f \in H_b(B)$ and fix $0 < r < 1$. Choosing any $r < s < 1$, for each $\|x\| < r$ we have using (1.4)

$$\begin{aligned}\sum_{m=0}^{\infty} |p_m(x)| &= \sum_{m=0}^{\infty} \left| p_m\left(\frac{r}{s}\frac{s}{r}x\right)\right| = \sum_{m=0}^{\infty} \left(\frac{r}{s}\right)^m \left| p_m\left(\frac{s}{r}x\right)\right| \\ &\leq \sum_{m=0}^{\infty} \left(\frac{r}{s}\right)^m \sup_{\|y\|<s} |p_m(y)| \leq \sum_{m=0}^{\infty} \left(\frac{r}{s}\right)^m \sup_{\|y\|<s} |f(y)|.\end{aligned}$$

Observe that $\sup_{\|y\|<s} |f(y)|$ is bounded and, by the Weierstrass M-test, the series $\sum_{m=0}^{\infty} p_m$ converges uniformly in rB to f. Since $0 < r < 1$ was arbitrary we have that the Taylor series converges uniformly to f in the B-bounded sets i.e., in the topology of $H_b(B)$.

1.4 Mean ergodic operators on locally convex Hausdorff spaces

We begin this section by fixing some notation.

Let (E, Γ_E) be a lcHs and take $T \in \mathscr{L}(E)$. The iterates of T are constructed as follows:

- $T^0 = \text{id}$,
- $T^1 = T$ and
- $T^n = T^{n-1} \circ T$ for $n \in \mathbb{N}$.

We denote by $T_{[n]} : E \to E$, with $n \in \mathbb{N}$, the n-th *Cesàro mean* of the operator $T \in \mathscr{L}(E)$. This continuous linear operator is defined as

$$T_{[n]} := \frac{1}{n} \sum_{k=0}^{n-1} T^k.$$

Our aim is to study properties of T^n and $T_{[n]}$ as $n \to \infty$. For this purpose we need the following definitions:

- An operator $T: E \to E$ is said to be *topologizable* if for every $p \in \Gamma_E$ there is $q \in \Gamma_E$ such that for every $n \in \mathbb{N}$ there is $a_n > 0$ with

$$p(T^n x) \leq a_n q(x) \text{ for every } x \in E. \tag{1.5}$$

- An operator $T: E \to E$ is said to be *power bounded* if the sequence $(T^n)_n$ is equicontinuous in $\mathscr{L}(E)$. That is, for every $p \in \Gamma_E$ there are $q \in \Gamma_E$ and $c > 0$ such that

$$p(T^n x) \leq c q(x) \text{ for every } x \in E \text{ and } n \in \mathbb{N}. \tag{1.6}$$

- An operator $T: E \to E$ is said to be *Cesàro bounded* if the set $(T_{[n]})_n$ is equicontinuous in $\mathscr{L}(E)$. That is, for every $p \in \Gamma_E$ there are $q \in \Gamma_E$ and $c > 0$ such that

$$p(T_{[n]} x) \leq c q(x) \text{ for every } x \in E \text{ and } n \in \mathbb{N}. \tag{1.7}$$

- An operator $T: E \to E$ is said to be *mean ergodic* if there is $P \in \mathscr{L}(E)$ such that $T_{[n]} \to P$ in τ_s. That is, the limit

$$Px := \lim_{n\to\infty} T_{[n]} x \text{ exists for every } x \in E. \tag{1.8}$$

- An operator $T: E \to E$ is said to be *uniformly mean ergodic* if there is $P \in \mathscr{L}(E)$ such that $T_{[n]} \to P$ in τ_b.

Observe that if T is power bounded then, T is topologizable. Conversely, if T is topologizable and $a_n = c > 0$ for every $n \in \mathbb{N}$ we have that T is power bounded.

Assume E is a barrelled lcHs and $(T_{[n]})_n$ converges in τ_s to some linear map $P: E \to E$. Then, the map P is also continuous and T is mean ergodic. Indeed, we have $(T_{[n]})_n$ is a pointwise bounded set in $\mathscr{L}(E)$. By the Uniform Boundedness Principle the set $(T_{[n]})_n$ is equicontinuous and $P \in \mathscr{L}(E)$.

Proposition 1.8. *Let E be a barrelled lcHs. If $T \in \mathscr{L}(E)$ is mean ergodic, it is Cesàro bounded.*

Proof. Assume $T_{[n]}x$ is convergent in E for every $x \in E$. Then, the set $(T_{[n]}x)_n$ is bounded in E for every $x \in E$. That is, $(T_{[n]})_n$ is pointwise bounded in $\mathscr{L}(E)$. By the Uniform Boundedness Principle the family of operators $(T_{[n]})_n \subset \mathscr{L}(E)$ is equicontinuous. □

Remark 1.9. If $T \in \mathscr{L}(E)$ is power bounded, it is also Cesàro bounded. In fact, since for every $p \in \Gamma_E$ we can find a seminorm $q \in \Gamma_E$ and $c > 0$ as in (1.6), we obtain

$$p(T_{[n]}(x)) \leq \frac{1}{n} \sum_{k=0}^{n-1} p(T^k x) \leq c q(x),$$

for every $x \in E$ and $n \in \mathbb{N}$. Thus, the set $(T_{[n]})_{n\in\mathbb{N}}$ is equicontinuous.

It is easy to check that, for $T \in \mathscr{L}(E)$ and $n \in \mathbb{N}$, the following identities hold

$$(T - \mathrm{id})T_{[n]} = T_{[n]}(T - \mathrm{id}) = \frac{1}{n}(T^n - \mathrm{id}), \tag{1.9}$$

$$\frac{1}{n}T^n = \frac{n+1}{n}T_{[n+1]} - T_{[n]}. \tag{1.10}$$

Using (1.10) we have that if T is mean ergodic then

$$\lim_{n\to\infty} \frac{T^n x}{n} = 0 \tag{1.11}$$

holds for all $x \in E$.

The following result is a special case of Eberlein's mean ergodic theorem. It can be found in [43, Chapter 2, § 2.1, Theorem 1.1] and we include a proof for the sake of completeness. Before we state it, let us recall that:

- If (E,τ) is a Hausdorff topological space and $(x_n)_n$ is a sequence in E, then a point $y \in E$ is called a *cluster point* of the sequence whenever any neighbourhood of y contains infinitely many points of $(x_n)_n$. Evidently, every cluster point of the sequence belongs to the closure of the set $\{x_n : n \in \mathbb{N}\}$.

- If E is compact, then every sequence $(x_n)_n$ in E has a cluster point. To see it, suppose that for every $x \in E$ there exists an open neighbourhood U_x of x which contains x_n only for finite many values of n. Since the open cover $\{U_x : x \in E\}$ has a finite subcover, this yields a contradiction because a finite subcover of $\{U_x : x \in E\}$ cannot contain the whole sequence $(x_n)_n$.

- If E is any topological space and $Y \subseteq E$ is relatively compact, then every sequence in Y has a cluster point in E. Indeed, it is enough to apply the arguments above to the compact set $\overline{Y}$.

Theorem 1.10. *Let $T \in \mathscr{L}(E)$ be Cesàro bounded. Let $x \in E$ be such that $\lim_{n\to\infty} \frac{T^n x}{n} = 0$. The following conditions are equivalent for $y \in E$:*

a) $Ty = y$ and y belongs to the closed convex hull of the set $\{T^m x : m \in \mathbb{N}_0\}$,

b) $y = \lim_{n\to\infty} T_{[n]}x$,

c) $y = \sigma(E,E') - \lim_{n\to\infty} T_{[n]}x$,

d) y is a $\sigma(E,E')$-cluster point of $(T_{[n]}x)_n$, that is, for each 0-neighbourhood U in $(E,\sigma(E,E'))$ and each $m \in \mathbb{N}$ there is $n > m$ such that $y - T_{[n]}x \in U$.

Proof. The implications b)⇒c)⇒d) are trivial.

a)⇒b) Fix a seminorm $p \in \Gamma_E$ and $\varepsilon > 0$. By the assumption there is $q \in \Gamma_E$ and $c > 0$ such that

$$p(T_{[n]}z) \leq cq(z),$$

for all $z \in E$ and all $n \in \mathbb{N}$. We can find $m \in \mathbb{N}_0$ and $\alpha_0, \dots, \alpha_m \geq 0$, with $\sum_{i=0}^m \alpha_i = 1$, such that the expression $Sx := \sum_{i=0}^m \alpha_i T^i x$ satisfies $q(y - Sx) < \varepsilon/(2c)$. Observe that for each $k \leq m$ and $n \in \mathbb{N}$ we have

$$T_{[n]}T^k x - T_{[n]}x = \frac{1}{n}\sum_{i=0}^{n-1} T^{k+i}x - \frac{1}{n}\sum_{i=0}^{n-1} T^i x = \frac{1}{n}\sum_{i=0}^{k-1} T^{n+i}x - \frac{1}{n}\sum_{i=0}^{k-1} T^i x.$$

On the one hand, since $k \leq m$ is fixed we can find $n_1(k) \in \mathbb{N}$ with

$$p\left(\frac{1}{n}\sum_{i=0}^{k-1} T^i x\right) < \frac{\varepsilon}{4} \text{ for every } n \geq n_1(k). \tag{1.12}$$

On the other hand, since T is Cesàro bounded we get

$$p\left(\frac{1}{n}\sum_{i=0}^{k-1} T^{n+i}x\right) = p\left(\frac{k}{n}T_{[k]}T^n x\right) \leq kc \cdot q\left(\frac{1}{n}T^n x\right).$$

Now, since $\lim_{n\to\infty} \frac{T^n x}{n} = 0$ we can find $n_2(k) \in \mathbb{N}$ such that

$$p\left(\frac{1}{n}\sum_{i=0}^{k-1} T^{n+i}x\right) < \frac{\varepsilon}{4} \text{ for every } n \geq n_2(k). \tag{1.13}$$

We choose $N = \max\{n_1(k), n_2(k) : 0 \leq k \leq m\}$. Now, applying (1.12) and (1.13) we obtain

$$p(T_{[n]}T^k x - T_{[n]}x) \leq p\left(\frac{1}{n}\sum_{i=0}^{k-1} T^{n+i}x\right) + p\left(\frac{1}{n}\sum_{i=0}^{k-1} T^i x\right) < \frac{\varepsilon}{2},$$

for every $k \leq m$ and $n \geq N$. Observe that $T_{[n]}y = y$ for every $n \in \mathbb{N}$. Using again that T is Cesàro bounded we have, for every $n \geq N$,

$$\begin{aligned}
p(y - T_{[n]}x) &= p(T_{[n]}y - T_{[n]}Sx + T_{[n]}Sx - T_{[n]}x) \\
&\leq p\left(T_{[n]}y - T_{[n]}Sx\right) + p\left(T_{[n]}Sx - T_{[n]}x\right) \\
&= p\left(T_{[n]}(y - Sx)\right) + p\left(\sum_{i=0}^m \alpha_i T_{[n]}T^i x - T_{[n]}x\right) \\
&\leq cq(y - Sx) + \sum_{i=0}^m \alpha_i p(T_{[n]}T^i x - T_{[n]}x) < \varepsilon.
\end{aligned}$$

Therefore $\lim_{n\to\infty} T_{[n]}x = y$.

d)⇒a) Denote by C the convex hull of the set $\{T^m x : m \in \mathbb{N}_0\}$. Observe that $\{T_{[m]}x : m \in \mathbb{N}\} \subseteq C$. By the Hahn-Banach Theorem the closure of a convex set is the same for the topology of E and for $\sigma(E, E')$. We conclude that $y \in \overline{C}$. It remains to show that $y = Ty$.

Since $\lim_{n\to\infty} \frac{T^n x}{n} = 0$, (1.9) implies that $TT_{[n]}x - T_{[n]}x \to 0$ as $n \to \infty$. Therefore, for each $v \in E'$,

$$\lim_{n\to\infty} v(TT_{[n]}x - T_{[n]}x) = 0.$$

Given $\varepsilon > 0$ and $v \in E'$, there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$\left|v(TT_{[n]}x - T_{[n]}x)\right| < \frac{\varepsilon}{3}.$$

The set $U := \left\{z \in E : |v(z-y)| < \varepsilon/3, \left|T^t(v)(z-y)\right| < \varepsilon/3\right\}$ is a $\sigma(E, E')$-neighbourhood of y in E. Since y is a $\sigma(E, E')$-cluster point of $(T_{[n]}x)_n$, given n_0 there is $n > n_0$ such that $\left|v(y - T_{[n]}x)\right| < \varepsilon/3$ and $\left|T^t(v)(y - T_{[n]}x)\right| < \varepsilon/3$. Now, we have

$$|v(y - Ty)| \leq \left|v(y - T_{[n]}x)\right| + \left|v(T_{[n]}x - TT_{[n]}x)\right| + \left|v(TT_{[n]}x - Ty)\right| < \varepsilon.$$

Since $v \in E'$ and $\varepsilon > 0$ were arbitrary, the Hahn-Banach Theorem yields $y = Ty$. □

As a consequence of the previous result we are able to characterise mean ergodicity for Cesàro bounded operators (cf. [1]).

Proposition 1.11. *Let $T \in \mathscr{L}(E)$ be Cesàro bounded. Then T is mean ergodic if and only if $\lim_{n\to\infty} \frac{T^n x}{n} = 0$ for all $x \in E$ and $(T_{[n]}x)_n$ is $\sigma(E, E')$-relatively compact for each $x \in E$.*

Proof. It T is mean ergodic, $\lim_{n\to\infty} \frac{T^n x}{n} = 0$ holds for all $x \in E$ by (1.11). The sequence $(T_{[n]}x)_n$ is convergent in E for every $x \in E$. In particular, it is a $\sigma(E, E')$-relatively compact set for each $x \in E$.

Conversely, we have that given $x \in E$ the set $(T_{[n]}x)_n$ is $\sigma(E, E')$-relatively compact, and therefore this sequence has a $\sigma(E, E')$-cluster point $y \in E$. By Theorem 1.10 necessarily $y = \lim_{n\to\infty} T_{[n]}x$. We can define

$$Px := \lim_{n\to\infty} T_{[n]}x$$

for each $x \in E$. Since $(T_{[n]})_n$ is equicontinuous we have that $P \in \mathscr{L}(E)$. T is mean ergodic. □

The following results form a collection of interesting consequences derived from Proposition 1.11. In [1, Theorem 2.4 and Corollary 2.7] the same results are given, assuming that $(T_{[n]}x)_n$ is $\sigma(E, E')$-sequentially relatively compact for every $x \in E$. Here we replace this condition by simply $\sigma(E, E')$-relatively compact.

Proposition 1.12. *Let E be a barrelled locally convex Hausdorff space. Then $T \in \mathscr{L}(E)$ is mean ergodic if and only if $\lim_{n\to\infty} \frac{T^n x}{n} = 0$ for every $x \in E$ and $(T_{[n]}x)_n$ is $\sigma(E,E')$-relatively compact for each $x \in E$.*

Proof. Assume T is mean ergodic. By Remark 1.8 the operator T is Cesàro bounded. Now Proposition 1.11 yields the conclusion.

Conversely, if the set $(T_{[n]}x)_n$ is $\sigma(E,E')$-relatively compact, it is $\sigma(E,E')$-bounded. Therefore $(T_{[n]}x)_n$ is a bounded set for all $x \in E$ (see [35, Chapter 3 §5 Theorem 3]). Again, the Uniform Boundedness Principle gives that T is Cesàro bounded and Proposition 1.11 completes the proof. □

Proposition 1.13. *Assume $T \in \mathscr{L}(E)$ is a power bounded operator. Then T is mean ergodic if and only if $(T_{[n]}x)_n$ is $\sigma(E,E')$-relatively compact for each $x \in E$.*

Proof. By Remark 1.9 we have that T is Cesàro bounded.

Now, if T is mean ergodic, Proposition 1.11 gives the conclusion.

Conversely, assume that $(T_{[n]}x)_n$ is $\sigma(E,E')$-relatively compact for each $x \in E$. Observe that since T is power bounded, for each $p \in \Gamma_E$ there is $q \in \Gamma_E$ and $c > 0$ such that

$$p\left(\frac{T^n x}{n}\right) = \frac{1}{n} p(T^n x) \leq \frac{cq(x)}{n} \xrightarrow{n} 0$$

for each $x \in E$. Thus, we have $\lim_{n\to\infty} \frac{T^n x}{n} = 0$ for all $x \in E$. By Proposition 1.11 we obtain that T is mean ergodic. □

Proposition 1.14. [8, Proposition 3.3]. *Let E be a semi-reflexive locally convex Hausdorff space. Then every power bounded operator in E is mean ergodic.*

Proof. By Remark 1.9 we have that T is Cesàro bounded. Now, since $(T_{[n]}x)_n$ is a bounded set in E for each $x \in E$, it is $\sigma(E,E')$-relatively compact for each $x \in E$. Applying Proposition 1.13 we finish the proof. □

Every precompact set in a lcHs is bounded, but the converse does not hold in general. We also have that every relatively compact set is precompact but the converse does not hold in general.

Proposition 1.15. [8, p. 917]. *Let E be a semi-Montel locally convex Hausdorff space. Then every power bounded operator in E is uniformly mean ergodic.*

Proof. If the space E is semi-Montel, in particular, it is semi-reflexive and by Proposition 1.14, T is mean ergodic. This means that the sequence $(T_{[n]})_n$ converges pointwise. On the other hand, by Remark 1.9, $(T_{[n]})_n$ is equicontinuous. Hence, $Px := \lim_n T_{[n]}x$, for $x \in E$, defines an operator $P \in \mathscr{L}(E)$. Now, by [42, (2), p. 139], the topology of pointwise convergence and of uniform convergence on precompact sets coincide in $(T_{[n]})_n$. Since E is semi-Montel bounded sets and relatively compact sets coincide. So $(T_{[n]})_n$ converges to P uniformly on every bounded set of E i.e., T is uniformly mean ergodic. □

Corollary 1.16. *Assume E is a reflexive locally convex Hausdorff space. Then $T \in \mathscr{L}(E)$ is mean ergodic if and only if $\lim_{n\to\infty} \frac{T^n x}{n} = 0$ for all $x \in E$ and T is Cesàro bounded.*

Proof. Assume T is mean ergodic. Since E is barrelled, by Remark 1.8 we have that T is Cesàro bounded. By (1.11) we obtain that $\lim_{n\to\infty} \frac{T^n x}{n} = 0$ for all $x \in E$.

Conversely, if T is Cesàro bounded, the set $(T_{[n]}x)_n$ is bounded for all $x \in E$. Since E is reflexive $(T_{[n]}x)_n$ is $\sigma(E, E')$-relatively compact for all $x \in E$. We finish applying Proposition 1.12. □

Corollary 1.17. *Assume E is a Montel locally convex Hausdorff space. Then $T \in \mathscr{L}(E)$ is uniformly mean ergodic if and only if $\lim_{n\to\infty} \frac{T^n x}{n} = 0$ is satisfied for all $x \in E$ and T is Cesàro bounded.*

Proof. Assume T is uniformly mean ergodic. In particular, it is mean ergodic. Since E is barrelled, by Remark 1.8 we have that T is Cesàro bounded. By (1.11) we obtain that $\lim_{n\to\infty} \frac{T^n x}{n} = 0$ for all $x \in E$.

Conversely, since E is Montel it is also reflexive. Corollary 1.16 gives that there is $P \in \mathscr{L}(E)$ such that $T_{[n]} \to P$ in τ_s. Again by [42, (2), p. 139], the topology of pointwise convergence and of uniform convergence on precompact sets coincide in $(T_{[n]})_n$, which is an equicontinuous set. In E bounded sets and relatively compact sets coincide. So $T_{[n]} \to P$ in τ_b and T is uniformly mean ergodic. □

Remark 1.18. It is worth noting that if X is a Banach space, then for an operator $T : X \to X$ the following hold:

- T is always topologizable.
- T is power bounded if and only if $\sup_{n\in\mathbb{N}} \|T^n\| < \infty$.
- T is Cesàro bounded if and only if $\sup_{n\in\mathbb{N}} \|T_{[n]}\| < \infty$.
- If T is mean ergodic, it is also Cesàro bounded.
- T is mean ergodic if and only if $T_{[n]}x$ converges for every $x \in X$.
- T is uniformly mean ergodic if and only if there is $P \in \mathscr{L}(X)$ such that $T_{[n]} \to P$ in the operator norm.

Indeed, for an operator $T \in \mathscr{L}(X)$ we have that T^n is continuous for every $n \in \mathbb{N}$. We can take $a_n = \|T^n\| + 1 > 0$ to obtain

$$\|T^n x\| \leq a_n \|x\|,$$

for every $x \in X$ and every $n \in \mathbb{N}$. This clearly gives that T is topologizable.
Now, as we have seen earlier, if T is mean ergodic (since X is barrelled) by the Uniform Boundedness Principle $(T_{[n]})_n$ is equicontinuous and converges in τ_s to some $P \in \mathscr{L}(X)$.

1.5 General aim and framework of the thesis

Our main aim in this work is to study some of the properties that we have introduced (mostly ergodic properties, but also continuity or compactness) for some concrete (by now classical) operators acting on the spaces of holomorphic functions defined in Section 1.3. Our main interest are composition operators, but we will also pay attention to other operators, such as multiplication and weighted composition operators.

Let X be a Banach space and B its open unit ball and let $\varphi : B \to B$ be a holomorphic mapping. Then, we denote by $C_\varphi : H(B) \to H(B)$ the *composition operator* given by

$$C_\varphi(f) = f \circ \varphi,$$

for every $f \in H(B)$, which is clearly well defined. The function φ is called the *symbol* of the composition operator. We take X an infinite dimensional Banach space, and the composition operator defined on the spaces of holomorphic functions considered above:

- $C_\varphi : H_b(B) \to H_b(B)$
- $C_\varphi : H^\infty(B) \to H^\infty(B)$
- $C_\varphi : \mathcal{P}(^m X) \to \mathcal{P}(^m X)$ (note that in this case we take $\varphi : X \to X$)

Then, our aim is to find conditions that on φ ensure that C_φ satisfies the ergodic properties mentioned before. These conditions will be different, depending on the space where C_φ is defined (in the case of $\mathcal{P}(^m X)$, also on the topology, since in this case we will consider both the compact-open and the norm topologies). Composition operators on spaces of holomorphic functions have been extensively studied by a number of authors, especially for the finite-dimensional case. Some classical references are [19, 52].

As we said, we are mainly interested in composition operators, but we are interested also in other operators. If $\psi : B \to \mathbb{C}$ is a holomorphic mapping. Then,

$$M_\psi(f) = \psi \cdot f,$$

for $f \in H(B)$ defines an operator (on $H(B)$), called *multiplication operator*. The mapping ψ is called the *weight* of the multiplication operator.

Finally, if $\psi : B \to \mathbb{C}$ and $\varphi : B \to B$ are holomorphic mappings then, by doing

$$C_{\psi,\varphi}(f) = \psi \cdot (f \circ \varphi),$$

we define an operator on $H(B)$ called *weighted composition operator*. Note that we have $C_{\psi,\varphi} = M_\psi \circ C_\varphi$. As before, we consider these operators acting on $H_b(B)$, $H^\infty(B)$ and $\mathcal{P}(^m X)$ (in the this case, taking $\psi : X \to \mathbb{C}$ and $\varphi : X \to X$), and our aim is to study in each case the properties of these operators in terms of φ and ψ. In this case we look not only at ergodic properties (such as power boundedness or mean ergodicity), but also at continuity and compactness.

Let us briefly collect some of the known results that motivate our research. We start with the finite dimensional case (that is, when the space X is finite dimensional). Let us note that in this case the three spaces of holomorphic functions that we are interested in reduce to just two, since $H(B) = H_b(B)$ (recall Proposition 1.5). In this case the space $\mathscr{P}(^m X)$ is finite dimensional and, then it is not interesting.

Our starting point is the following result due to Bonet and Domański, in which power boundedness and mean ergodicity of C_φ on $H(B)$ is completely described in the finite dimensional setting. The definition of φ having stable orbits can be found in Section 3.3.

Theorem 1.19. [9, Proposition 1]. *Let $B \subset \mathbb{C}^d$ be the open unit ball for some norm and let $\varphi : B \to B$ be a holomorphic function. The following are equivalent:*

a) $C_\varphi : H(B) \to H(B)$ *is power bounded.*

b) $C_\varphi : H(B) \to H(B)$ *is uniformly mean ergodic.*

c) $C_\varphi : H(B) \to H(B)$ *is mean ergodic.*

d) The map φ has stable orbits on B.

Ergodic properties of composition operators defined on $H^\infty(\mathbb{D})$ were studied by Beltrán-Meneu, Gómez-Collado, Jordá and Jornet in [5]. Note that in this case $\|C_\varphi\| \leq 1$ and every composition operator is power bounded. As for mean ergodicity, there the authors show the following

Theorem 1.20. [5, Theorem A]. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic. Then the following are equivalent*

a) $C_\varphi : H^\infty(\mathbb{D}) \to H^\infty(\mathbb{D})$ *is mean ergodic.*

b) $C_\varphi : H^\infty(\mathbb{D}) \to H^\infty(\mathbb{D})$ *is uniformly mean ergodic.*

c) $(\varphi^n)_n$ *converges uniformly to an interior Denjoy–Wolff point $z_0 \in \mathbb{D}$, or φ is a periodic elliptic automorphism.*

We will not get into details about the precise meaning of the third statement, since it plays no role in our forthcoming research. We just mention that the Möbius transformations on $\mathbb{D}$ are a basic tool within the proof (sending the Denjoy-Wolff point z_0 to 0). In an infinite dimensional Banach space does not exist, in general, such a family of functions, so our approach is necessarily different, and our results somewhat less far-reaching. In some cases (such as, for example, Hilbert spaces) we do have analogues for the Möbius transformations, and we can get sharper results.

So, Theorems 1.19 and 1.20 are our starting point and in some sense the guiding line of our research. What we want to do is to find versions of these results when infinite dimensional Banach spaces are considered. Composition operators in this setting have been much less studied, and the literature is more scarce. In [3, 31] continuity, compactness and other related properties of these operators were studied. Let us collect the two results of those papers that are more related to our work.

Proposition 1.21. [3, Proposition 3]. *Let X be a Banach space and let $\varphi\colon B \to B$ be a holomorphic function. The following are equivalent:*

a) $C_\varphi\colon H^\infty(B) \to H^\infty(B)$ *is compact.*

b) $C_\varphi\colon H^\infty(B) \to H^\infty(B)$ *is weakly compact and* $\varphi(B)$ *is relatively compact in* X.

c) There is $0 < r < 1$ *so that* $\varphi(B) \subseteq rB$ *and* $\varphi(B)$ *is relatively compact in* X.

Proposition 1.22. [31, Proposition 2.12]. *Let X be a Banach space and let $\varphi\colon B \to B$ be a holomorphic function of bounded type. Then $C_\varphi\colon H_b(B) \to H_b(B)$ is a Montel operator if and only if φ maps B-bounded sets into relatively compact sets in X.*

To the best of our knowledge, there is no previous work on ergodic properties for composition operators in the infinite dimensional case, and our research here seems to be the first in this line. In Chapter 2 we study power boundedness and mean ergodicity of C_φ on $\mathscr{P}(^m X)$, endowed with the τ_0 topology or the norm topology. In Chapter 3 we move to the spaces of holomorphic functions on the ball, trying to extend Theorems 1.19 and 1.20. We will see how Theorem 1.19 splits into two results, one for $H(B)$ and one for $H_b(B)$ that, for finite dimensional Banach spaces coincide.

The research on weighted composition operators is much more recent. Our starting point here are the work of Contreras and Díaz-Madrigal [17] (for compactness of $C_{\psi,\varphi}$ on $H^\infty(\mathbb{D})$) and Beltrán-Meneu, Gómez-Collado, Jordá and Jornet [6] (for the interplay between power boundedness and mean ergodicity on $H(\mathbb{D})$). Once again, we collect here some of the results that we aim at extending to infinite dimensional spaces. To begin with, regarding when a weighted composition operator on $H^\infty(\mathbb{D})$ is compact, we have the following.

Theorem 1.23. [17, Proposition 2.3]. *Given $\psi, \varphi \in H^\infty(\mathbb{D})$ such that $\varphi\colon \mathbb{D} \to \mathbb{D}$, consider $C_{\psi,\varphi}\colon H^\infty(\mathbb{D}) \to H^\infty(\mathbb{D})$. Then the following are equivalent:*

a) $C_{\psi,\varphi}$ *is compact.*

b) $C_{\psi,\varphi}$ *is weakly compact.*

c) One of the following properties hold:

(i) There is $0 < s < 1$ *such that* $\varphi(\mathbb{D}) \subseteq s\mathbb{D}$,

(ii) $\lim_{r\to 1^-} \sup_{|\varphi(z)|>r} |\psi(z)| = 0.$

On the other hand, [6] focuses on ergodic properties of weighted composition operators on $H(\mathbb{D})$ (in fact, for $H(U)$, where U is an arbitrary open, bounded and connected subset of $\mathbb{C}$; for the sake of clarity we state here the results for $\mathbb{D}$). First of all, the interaction between power boundedness and mean ergodicity is analysed.

Theorem 1.24. [6, Proposition 3.1]. *Consider $C_{\psi,\varphi}\colon H(\mathbb{D}) \to H(\mathbb{D})$.*

a) If $C_{\psi,\varphi}$ is power bounded, then it is uniformly mean ergodic and $\left(\prod_{m=0}^{n}(\psi \circ \varphi^n)\right)_n$ is a bounded sequence in $H(\mathbb{D})$.

b) If $C_{\psi,\varphi}$ is mean ergodic, then it is topologizable and $\lim_n \frac{1}{n}\left(\prod_{m=0}^{n}(\psi \circ \varphi^n)\right) = 0$ on τ_0.

Also, power boundedness of $C_{\psi,\varphi}$ is characterised in terms of the two functions ψ and φ.

Theorem 1.25. [6, Theorem 3.3]. *Consider $C_{\psi,\varphi}\colon H(\mathbb{D}) \to H(\mathbb{D})$. The following are equivalent:*

a) $C_{\psi,\varphi}$ is power bounded.

b) φ has stable orbits and $\left(\prod_{m=0}^{n}(\psi \circ \varphi^n)\right)_n$ is a bounded sequence in $H(\mathbb{D})$.

As before, our aim here is to perform an analogous study for weighted composition operators (and, as a particular case, for multiplication operators) acting on spaces of holomorphic functions defined on the open unit ball of an infinite dimensional Banach space. In Chapter 4 we focus on the study of compactness. Note that, since $C_\varphi = C_{1,\varphi}$, the results that we obtain should in one way or another be connected to Proposition 1.21. Finally, the study of ergodic properties of weighted composition operators is done in Chapter 5.

Chapter 2

Composition operators on spaces of homogeneous polynomials

2.1 Introduction

If X is a complex Banach space and $\varphi : X \to X$ is holomorphic, then $C_\varphi : H(X) \to H(X)$, the *composition operator of symbol* φ, is defined as $C_\varphi(f) = f \circ \varphi$. In this chapter we deal with the restriction of such an operator to the space $\mathscr{P}(^mX)$ of m-homogeneous polynomials, and we ask different questions. In the first place, for which φ's does this restriction take values again in $\mathscr{P}(^mX)$? (in other words, when do we have $C_\varphi : \mathscr{P}(^mX) \to \mathscr{P}(^mX)$ is well defined?). Once we have settled this question (see Proposition 2.3) we endow $\mathscr{P}(^mX)$ with two different topologies (the τ_0 and the norm topology) and move on to the study of power boundedness and mean ergodicity for composition operators on $\mathscr{P}(^mX)$.

If $p \circ \varphi \in \mathscr{P}(^mX)$ for every m-homogeneous polynomial p, the iterates of the corresponding composition operator are given by

$$C_\varphi^n(p)(x) = p(\varphi^n(x)), \tag{2.1}$$

for every p and all $x \in X$. This enables us to characterise the power boundedness of the composition operator in terms of conditions on the symbol (see Propositions 2.7 and 2.9).

The main results of this chapter are based on [37].

2.2 Well-defined and continuous composition operators

If we want to iterate the composition of a composition operator with itself we obviously need it to take values in $\mathscr{P}(^mX)$. This is the first thing that we have to settle, and we start with a simple observation.

Remark 2.1. Suppose X is a Banach space. If $x, y \in X$ satisfy that there are $\gamma_0 \in X'$ and $r > 0$ such that $\gamma(x) = \gamma(y)$ for every $\gamma \in X'$ with $\|\gamma - \gamma_0\| < r$, then $x = y$. Indeed, take any $\phi \in X'$, fix $c > \|\phi\|$ and consider $\gamma := \frac{r}{c}\phi + \gamma_0$. Then $\|\gamma - \gamma_0\| < r$ and

$$\frac{r}{c}\phi(x) + \gamma_0(x) = \gamma(x) = \gamma(y) = \frac{r}{c}\phi(y) + \gamma_0(y).$$

The fact that $\gamma_0(x) = \gamma_0(y)$ immediately gives $\phi(x) = \phi(y)$ and, since ϕ was arbitrary, $x = y$.

The following result was communicated by Prof. Manuel Maestre.

Lemma 2.2. *Let $\varphi \colon X \to X$ be a continuous mapping. If there exists $m \in \mathbb{N}$ such that $\gamma^m \circ \varphi$ is holomorphic for every $\gamma \in X'$, then φ is holomorphic.*

Proof. Since φ is continuous it is enough to check that φ is weakly G-holomorphic (see Theorem 1.4). That is, for any arbitrary $z_0, a_0 \in X$ the function

$$\lambda \mapsto \gamma(\varphi(\lambda z_0 + a_0))$$

is holomorphic for every $\gamma \in X'$.

By hypothesis the map $\lambda \mapsto \gamma(\varphi(\lambda z_0 + a_0))^m$ is holomorphic for every $\gamma \in X'$. Define $h \colon \mathbb{C} \to \mathbb{C}$ by $h(\lambda) = \gamma(\varphi(\lambda z_0 + a_0))^m$ and define $f : \mathbb{C} \to \mathbb{C}$ by $f(\lambda) = \gamma(\varphi(\lambda z_0 + a_0))$. If $f \equiv 0$ we are done. Suppose this is not the case. Since h is holomorphic, the set

$$A = \{0\} \cup h^{-1}(\{0\})$$

is discrete and, by the Identity Principle, has no accumulation point. Then $\mathbb{C} \setminus A$ is an open non-empty set. Take now $\lambda_0 \in \mathbb{C} \setminus A$. Then there is $r = r(\lambda_0) > 0$ such that $h(\lambda) \neq 0$ for all $\lambda \in D(\lambda_0, r)$. Since h is holomorphic on $D(\lambda_0, r)$ there exists $L \colon D(\lambda_0, r) \to \mathbb{C}$, a holomorphic logarithm such that

$$h(\lambda) = e^{L(\lambda)}, \tag{2.2}$$

for all $\lambda \in D(\lambda_0, r)$ (see [51, Chapter 10, § X.5]).

On the other hand, by the continuity of f, there is $0 < r_1 \leq r$ such that $f(D(\lambda_0, r_1)) \subseteq D(f(\lambda_0), |f(\lambda_0)|)$. Then there is a suitable branch of the logarithm, which we denote $\log_\alpha \colon D(f(\lambda_0), |f(\lambda_0)|) \to \mathbb{C}$, and we can define $L_1 \colon D(\lambda_0, r_1) \to \mathbb{C}$ as $L_1(\lambda) := \log_\alpha(f(\lambda))$. Therefore L_1 is continuous and $f(\lambda) = e^{L_1(\lambda)}$ for all $\lambda \in D(\lambda_0, r_1)$. Then

$$f(\lambda)^m = e^{m \cdot L_1(\lambda)}, \tag{2.3}$$

for all $\lambda \in D(\lambda_0, r_1)$. Now using (2.2) and (2.3) we obtain

$$e^{m \cdot L_1(\lambda)} = e^{L(\lambda)},$$

for all $\lambda \in D(\lambda_0, r_1)$. This means that there exists $k(\lambda) \in \mathbb{Z}$ such that $m \cdot L_1(\lambda) = L(\lambda) + 2\pi i k(\lambda)$ on $D(\lambda_0, r_1)$, which is connected and where both logarithms are continuous, then $k(\lambda) = k_0$ is constant. This implies that

$$L_1(\lambda) = \frac{L(\lambda) + 2\pi i k_0}{m}$$

is holomorphic on $D(\lambda_0, r_1)$. Therefore $f(\lambda) = e^{L_1(\lambda)}$ is holomorphic on $D(\lambda_0, r_1)$ for every $\lambda_0 \in \mathbb{C} \setminus A$. Since f is continuous on all $\mathbb{C}$, it is entire. Moreover, since $\gamma \in X'$ and $z_0, a_0 \in X$ were arbitrary we obtain the result. □

Proposition 2.3. *Let $\varphi : X \to X$ be a continuous mapping. The composition operator $C_\varphi : \mathscr{P}(^m X) \to \mathscr{P}(^m X)$ is well defined if and only if φ is linear.*

Proof. First, we assume that $\varphi : X \to X$ is linear and continuous. If $p \in \mathscr{P}(^m X)$, we have

$$C_\varphi(p)(\lambda x) = p(\varphi(\lambda x)) = p(\lambda \varphi(x)) = \lambda^m p(\varphi(x)) = \lambda^m C_\varphi(p)(x),$$

for all $x \in X$ and $\lambda \in \mathbb{C}$. Since $C_\varphi(p)$ is holomorphic, (1.2) gives that it is an m-homogeneous polynomial and, therefore $C_\varphi : \mathscr{P}(^m X) \to \mathscr{P}(^m X)$ is well defined.

Suppose now that $C_\varphi : \mathscr{P}(^m X) \to \mathscr{P}(^m X)$ is well defined. In particular, we have that for every $\gamma \in X'$ the function $\gamma^m \circ \varphi$ is holomorphic. By Lemma 2.2 the mapping φ is holomorphic. On the other hand, we have

$$p(\varphi(\lambda x)) = \lambda^m p(\varphi(x)) = p(\lambda \varphi(x)),$$

for all $p \in \mathscr{P}(^m X)$, $\lambda \in \mathbb{C}$ and $x \in X$. Given $\gamma \in X'$ we have that γ^m, defined by $\gamma^m(x) = (\gamma(x))^m$, belongs to $\mathscr{P}(^m X)$. So

$$\gamma(\varphi(\lambda x))^m = \lambda^m \gamma(\varphi(x))^m, \tag{2.4}$$

for all $\lambda \in \mathbb{C}$ and $x \in X$. Then, for each γ, λ, x there is some $\mu = \mu(\gamma, \lambda, x) \in \mathbb{C}$ with $\mu^m = 1$ such that

$$\gamma(\varphi(\lambda x)) = \mu \lambda \gamma(\varphi(x)). \tag{2.5}$$

Note that if $\gamma(\varphi(x)) = 0$, then by (2.4), $\gamma(\lambda \varphi(x)) = 0$, and we can take $\mu(\gamma, \lambda, x) = 1$ for every λ (in fact, in this case the equality holds for any value of μ we choose). Our aim is to show that we can also take $\mu(\gamma, \lambda, x) = 1$ for every γ, λ, x. To begin with, we show that μ does not depend on γ (i.e. $\mu = \mu(\lambda, x)$). Fix $x_0 \in X$ and $\gamma_0 \in X'$ such that $\gamma_0(\varphi(x_0)) \neq 0$. Since $T : X' \to \mathbb{C}$ defined as

$$T(\gamma) := \gamma(\varphi(x_0))$$

is a well defined continuous linear operator, given any $\varepsilon > 0$ we find $r > 0$ such that

$$|T(\gamma)| = |\gamma(\varphi(x_0))| > \varepsilon,$$

for every $\gamma \in B(\gamma_0, r) \subset X'$. We now fix $\lambda_0 \in \mathbb{C} \setminus \{0\}$ and consider the function $f : B(\gamma_0, r) \to \mathbb{C}$ given by

$$f(\gamma) = \frac{\gamma(\varphi(\lambda_0 x_0))}{\lambda_0 \gamma(\varphi(x_0))}.$$

This is continuous and $f(\gamma) = \mu = \mu(\gamma, \lambda_0, x_0)$ for every $\gamma \in B(\gamma_0, r)$. But μ is an m-th root of 1, so f takes values in a finite set and therefore has to be constant. In other words, there is some $\mu_0 = \mu_0(\lambda_0, x_0)$ so that $f(\gamma) = \mu_0$ for all $\gamma \in B(\gamma_0, r)$, that is

$$\gamma(\varphi(\lambda_0 x_0)) = \gamma(\lambda_0 \mu_0 \varphi(x_0))$$

for every γ with $\|\gamma - \gamma_0\| < r$. Remark 2.1 yields

$$\varphi(\lambda_0 x_0) = \mu_0 \lambda_0 \varphi(x_0).$$

This shows that for each λ and x there is some $\mu = \mu(\lambda, x)$ such that (2.5) holds for every γ.

Our next step is to see that μ can also be taken independently from λ. To do so, first we observe that the mapping $\lambda \in \mathbb{C} \mapsto \lambda \gamma_0(\varphi(x_0)) \in \mathbb{C}$ is continuous (recall that γ_0 and x_0 are chosen so that $\gamma_0(\varphi(x_0)) \neq 0$). Then the function $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by

$$g(\lambda) = \frac{\gamma_0(\varphi(\lambda x_0))}{\lambda \gamma_0(\varphi(x_0))}$$

is continuous and $g(\lambda) = \mu = \mu(\lambda, x_0)$. As before, g takes values on a finite set. Hence it is constant and we can find $\mu_0(x_0)$ so that $\mu_0(\lambda, x_0) = \mu_0(x_0)$ for every $\lambda \in \mathbb{C}$ (note that taking $\lambda = 0$ in (2.4), the equality in (2.5) holds for any μ). Then for each fixed x there is $\mu = \mu(x)$ such that (2.5) holds for every γ, λ. In other words, given λ and x we have that $\gamma(\varphi(\lambda x)) = \gamma(\mu(x)\lambda\varphi(x))$ for every $\gamma \in X'$ and, then

$$\varphi(\lambda x) = \mu(x) \lambda \varphi(x),$$

for every $\lambda \in \mathbb{C}$. Taking $\lambda = 1$ shows that in (2.5) we may take $\mu(x) = 1$ for every x. This gives our claim and shows that

$$\gamma(\varphi(\lambda x)) = \lambda \gamma(\varphi(x)),$$

for every γ, λ, x. Therefore $\lambda\varphi(x) = \varphi(\lambda x)$ for every λ, x, as we have $\gamma(\varphi(\lambda x)) = \gamma(\lambda\varphi(x))$ for all γ.

Since φ is holomorphic we can take the Taylor expansion at $a = 0$ (see (1.1)) to have

$$\lambda \sum_{m=0}^{\infty} p_m(x) = \lambda \varphi(x) = \varphi(\lambda x) = \sum_{m=0}^{\infty} p_m(\lambda x) = \sum_{m=0}^{\infty} \lambda^m p_m(x),$$

for every $\lambda \in \mathbb{C}$ and $x \in X$. The uniqueness of the sequence of polynomials yields $(\lambda^m - \lambda) p_m(x) = 0$ for every $m \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$ and $x \in X$. Taking any $\lambda^{m-1} \neq 1$ shows that $p_m \equiv 0$ for every $m \neq 1$ and therefore $\varphi = p_1$ is linear. □

In view of Proposition 2.3, without loss of generality we may assume for the rest of the chapter that φ is a continuous linear mapping. We see that in both spaces, $\mathscr{P}(^mX)_{\tau_0}$ and $\mathscr{P}(^mX)_{\|\cdot\|}$, if the operator is well defined, then it is continuous.

Proposition 2.4. *Let $\varphi : X \to X$ be a continuous linear operator. If $\tau = \tau_0$ or $\|\cdot\|$, the composition operator $C_\varphi : \mathscr{P}(^mX)_\tau \to \mathscr{P}(^mX)_\tau$ is continuous.*

Proof. If $\tau = \tau_0$, given any arbitrary compact subset $K \subset X$, the set $L := \varphi(K)$ is also compact and

$$\sup_{x\in K} |C_\varphi(p)(x)| = \sup_{x\in K} |p(\varphi(x))| = \sup_{x\in L} |p(x)|,$$

for all $p \in \mathscr{P}(^mX)$. If $\tau = \|\cdot\|$, we observe

$$\|C_\varphi(p)\|_{\mathscr{P}(^mX)} = \sup_{\|x\|_X\le 1} |p(\varphi(x))| \le \|p\|_{\mathscr{P}(^mX)} \sup_{\|x\|_X\le 1} \|\varphi(x)\|_X^m \le \|\varphi\|^m_{\mathscr{L}(X)} \|p\|_{\mathscr{P}(^mX)},$$

for all $p \in \mathscr{P}(^mX)$. □

2.3 Dynamics with the compact-open topology

Our aim in this section is to show that if the composition operator $C_\varphi : \mathscr{P}(^mX)_{\tau_0} \to \mathscr{P}(^mX)_{\tau_0}$ is power bounded, then it is uniformly mean ergodic (see Proposition 2.5), in particular mean ergodic, and that the converse implication does not hold in general (see Example 2.8). Since $\mathscr{P}(^mX)_{\tau_0}$ is semi-Montel (recall Section 1.3), the first assertion is a straightforward consequence of Proposition 1.15.

Proposition 2.5. *Let $\varphi : X \to X$ be a continuous linear mapping. If $C_\varphi : \mathscr{P}(^mX)_{\tau_0} \to \mathscr{P}(^mX)_{\tau_0}$ is power bounded, then it is uniformly mean ergodic.*

Now, we characterise the power boundedness of the composition operator in terms of properties of the symbol φ. We begin with the following

Lemma 2.6. *Let $K \subseteq X$ be a compact set and $m \ge 1$. Then the set*

$$\widehat{K}_{\mathscr{P}(^mX)} := \{x \in X : |p(x)| \le \sup_{y\in K} |p(y)|, \textit{ for all } p \in \mathscr{P}(^mX)\}$$

is compact.

Proof. Note first that, since each $p \in \mathscr{P}(^mX)$ is continuous, the set $\{x \in X : |p(x)| \le \sup_{y\in K} |p(y)|\}$ is closed. Then $\widehat{K}_{\mathscr{P}(^mX)}$ (being the intersection of all those sets) is also closed. Now, for every $\gamma \in X'$ and $x \in \widehat{K}_{\mathscr{P}(^mX)}$, we have $|\gamma^m(x)| \le \sup_{y\in K} |\gamma^m(y)|$ (because $\gamma^m \in \mathscr{P}(^mX)$) and, consequently,

$$\left(\sup_{x\in \widehat{K}_{\mathscr{P}(^mX)}} |\gamma(x)| \right)^m = \sup_{x\in \widehat{K}_{\mathscr{P}(^mX)}} |\gamma^m(x)| \le \sup_{x\in K} |\gamma^m(x)| = \left(\sup_{x\in K} |\gamma(x)| \right)^m .$$

This implies

$$K^\circ = \big\{\gamma \in X' : \sup_{x\in K} |\gamma(x)| \le 1\big\} \subseteq \big\{\gamma \in X' : \sup_{x\in \widehat{K}_{\mathscr{P}(^mX)}} |\gamma(x)| \le 1\big\} = (\widehat{K}_{\mathscr{P}(^mX)})^\circ.$$

An application of Krein's theorem [41, (4), pg. 325] gives that the closure of the absolutely convex hull $\overline{\mathrm{co}}(K)$ of K is compact. Finally, the Bipolar Theorem gives

$$\widehat{K}_{\mathscr{P}(^mX)} \subseteq (\widehat{K}_{\mathscr{P}(^mX)})^{\circ\circ} \subseteq K^{\circ\circ} \subseteq \big(\overline{\mathrm{co}}(K)\big)^{\circ\circ} = \overline{\mathrm{co}}(K),$$

and this yields our claim. □

We say that a continuous mapping $\varphi : X \to X$ has *stable orbits* if for every compact set $K \subseteq X$ there is some compact set $L \subseteq X$ such that $\varphi^n(K) \subseteq L$ for every $n \in \mathbb{N}$.

Proposition 2.7. *Let $\varphi : X \to X$ be a continuous linear map. Then $C_\varphi : \mathscr{P}(^mX)_{\tau_0} \to \mathscr{P}(^mX)_{\tau_0}$ is power bounded if and only if φ has stable orbits.*

Proof. Let us suppose first that φ has stable orbits and fix $K \subseteq X$ compact. Then we can find a compact set $L \subseteq X$ such that $\varphi^n(x) \in L$ for every $x \in K$ and $n \in \mathbb{N}$. This gives

$$\sup_{x\in K} |C_\varphi^n(p)(x)| = \sup_{x\in K} |p(\varphi^n(x))| \le \sup_{x\in L} |p(x)|,$$

for every $p \in \mathscr{P}(^mX)$ and $n \in \mathbb{N}$. Hence, C_φ is power bounded.

Assume conversely that C_φ is power bounded and supose that φ does not have stable orbits. Then there is a compact set $K \subset X$ such that $\bigcup_{n=0}^\infty \varphi^n(K)$ is not relatively compact. Since $(C_\varphi^n)_n = (C_{\varphi^n})_n$ is equicontinuous in $\mathscr{L}(\mathscr{P}(^mX))$, given the compact set K we can find another compact set $W \subseteq X$ and $c > 0$ such that

$$\sup_{x\in K} |p(\varphi^n(x))| \le c \sup_{x\in W} |p(x)| = \sup_{x\in W} |p(c^{1/m}x)| = \sup_{x\in c^{1/m}W} |p(x)|, \tag{2.6}$$

for all $p \in \mathscr{P}(^mX)$ and $n \in \mathbb{N}$. The set $V := c^{1/m}W$ is compact and, by Lemma 2.6, so also is $L := \widehat{V}_{\mathscr{P}(^mX)}$. If there are $n_0 \in \mathbb{N}$ and $x_0 \in K$ such that $\varphi^{n_0}(x_0) \notin L$, then we can find $p \in \mathscr{P}(^mX)$ such that $|p(\varphi^{n_0}(x_0))| > \sup_{y\in V} |p(y)|$, by the definition of $\widehat{V}_{\mathscr{P}(^mX)}$. But this is not possible by (2.6), which shows that $\varphi^n(K) \subseteq L$ for all $n \in \mathbb{N}$, contradicting the fact that $\bigcup_{n=0}^\infty \varphi^n(K)$ is not relatively compact. This completes the proof. □

We give an example of a composition operator showing that the converse implication in Proposition 2.5 does not hold in general. This example and others, that will be given later for $\mathscr{P}(^mX)_{\|\cdot\|}$, are based on some type of weighted backward shifts defined as follows. Fix $1 \le p < \infty$ and take $0 < \alpha < 1/p$. We consider the following *weighted backward shift* as the operator $\varphi_\alpha : \ell_p \to \ell_p$ defined by

$$\varphi_\alpha(e_1) = 0 \text{ and } \varphi_\alpha(e_k) = \Big(\frac{k}{k-1}\Big)^\alpha e_{k-1} \text{ for } k \ge 2, \tag{2.7}$$

that is $\varphi_\alpha(x_1, x_2, \ldots) = (w_2x_2, w_3x_3, \ldots)$, where $w_k = \big(\frac{k}{k-1}\big)^\alpha$.

Example 2.8. For each $1 < p < \infty$ and $0 < \alpha < 1/p$ the composition operator $C_{\varphi_\alpha} : \mathscr{P}(^1\ell_p)_{\tau_0} \to \mathscr{P}(^1\ell_p)_{\tau_0}$ is uniformly mean ergodic, but not power bounded.

Our argument relies on some known results on the behaviour of φ_α. First of all, combining [7, Corollary 2.3], the fact that ℓ_p is separable and [4, Theorem 1.2] we can find some $x_0 \in \ell_p$ such that $\big(\varphi_\alpha^n(x_0)\big)_n$ is dense in ℓ_p. Hence φ_α does not have stable orbits and, by Proposition 2.7, $C_{\varphi_\alpha} : \mathscr{P}(^1\ell_p)_{\tau_0} \to \mathscr{P}(^1\ell_p)_{\tau_0}$ cannot be power bounded.

Now we compute the n-th iterate of φ_α. Fixing $n \in \mathbb{N}$ and $x \in \ell_p$ we have

$$\begin{aligned}\varphi_\alpha^n(x) &= \varphi_\alpha^{n-1}(\varphi_\alpha(x)) = \varphi_\alpha^{n-1}(w_2x_2, w_3x_3, \ldots)\\ &= \varphi_\alpha^{n-2}(\varphi_\alpha(w_2x_2, w_3x_3, \ldots)) = \varphi_\alpha^{n-2}(w_2w_3x_3, w_3w_4x_4, \ldots)\\ &= \cdots = \left(x_{n+1}\prod_{i=1}^n w_{1+i},\ x_{n+2}\prod_{i=1}^n w_{2+i},\ x_{n+3}\prod_{i=1}^n w_{3+i}, \ldots\right).\end{aligned}$$

To see that $C_{\varphi_\alpha} : \mathscr{P}(^1\ell_p)_{\tau_0} \to \mathscr{P}(^1\ell_p)_{\tau_0}$ is mean ergodic we proceed in two steps. First of all, we want to see (using Corollary 1.16) that φ_α itself is mean ergodic.

On one hand, the previous calculations give

$$\varphi_\alpha^n(x) = \left((n+1)^\alpha x_{n+1}, \big(\frac{n+2}{2}\big)^\alpha x_{n+2}, \big(\frac{n+3}{3}\big)^\alpha x_{n+3}, \ldots\right), \tag{2.8}$$

for every $x \in \ell_p$ and $n \in \mathbb{N}$. Thus, the norm of $\frac{\varphi_\alpha^n(x)}{n}$ can be bounded as

$$\left\|\frac{\varphi_\alpha^n(x)}{n}\right\|_{\ell_p}^p = \frac{1}{n^p}\sum_{i=1}^\infty \big(\frac{n+i}{i}\big)^{\alpha p}|x_{n+i}|^p \le \frac{(n+1)^{\alpha p}}{n^p}\sum_{i=1}^\infty |x_{n+i}|^p \le \frac{(n+1)^{\alpha p}}{n^p}\|x\|_{\ell_p}^p.$$

Since $\alpha p < 1$, we obtain $\lim_{n\to\infty} \frac{\varphi_\alpha^n(x)}{n} = 0$ for every $x \in \ell_p$. On the other hand, [7, Theorem 2.1] shows that there is some $c > 0$ such that

$$\left\|\frac{1}{n}\sum_{k=0}^{n-1}\varphi_\alpha^k(x)\right\|_{\ell_p} \le c\|x\|_{\ell_p}, \tag{2.9}$$

for every $n \in \mathbb{N}$, $x \in \ell_p$. This gives that φ_α is Cesàro bounded and, since ℓ_p is reflexive, using Corollary 1.16 we obtain φ_α is mean ergodic. Then we can find $\varphi \in \mathscr{L}(\ell_p)$ such that $\lim_n(\varphi_\alpha)_{[n]}(x) = \varphi(x) \in \ell_p$ for every $x \in \ell_p$.

Now we deal with C_{φ_α}. Note first that, for each fixed $u \in \ell_p'$ continuity gives

$$\lim_{n\to\infty}(C_{\varphi_\alpha})_{[n]}(u)(x) = \lim_{n\to\infty} u(\varphi_{\alpha[n]}(x)) = u\big(\lim_{n\to\infty}\varphi_{\alpha[n]}(x)\big) = u(\varphi(x)),$$

for every $x \in \ell_p$. In other words, $\big((C_{\varphi_\alpha})_{[n]}(u)\big)_n$ converges pointwise to $C_\varphi(u)$ for every $u \in \ell_p'$. Now, by [42, (2), p. 139] the topologies of pointwise convergence and of uniform convergence on compact sets coincide on equicontinuous sets. But, for $u \in \ell_p'$ and $x \in \ell_p$, from (2.9), we immediately have

$$\left|\frac{1}{n}\sum_{k=0}^{n-1} C_{\varphi_\alpha}^k(u)(x)\right| = \left|\frac{1}{n}\sum_{k=0}^{n-1} u(\varphi_\alpha^k x)\right| \leq \|u\|_{\ell_p'} \left\|\frac{1}{n}\sum_{k=0}^{n-1} \varphi_\alpha^k(x)\right\|_p \leq c\,\|u\|_{\ell_p'}\|x\|_{\ell_p}.$$

This shows that, for each fixed $u \in \ell_p'$, the set $\big((C_{\varphi_\alpha})_{[n]}(u)\big)_n$ is equicontinuous and, therefore τ_0-converges to $C_\varphi(u)$. Hence, C_{φ_α} is mean ergodic.

We can say more, that C_{φ_α} is even uniformly mean ergodic. To check this first we observe that $\big((\varphi_\alpha)_{[n]} - \varphi\big)_n$ is pointwise convergent to 0 and so, equicontinuous in $\mathscr{L}(\ell_p)$. Therefore $\big((\varphi_\alpha)_{[n]} - \varphi\big)_n$ converges to 0 uniformly on the compact subsets of ℓ_p. Now, we take an arbitrary τ_0-bounded set $V \subset \ell_p'$, which is also norm-bounded in ℓ_p' (see, for instance, [35, Chapter 3 §5 Theorem 3]). Given any compact set $K \subset \ell_p$ and $n \in \mathbb{N}$, we can find some constant $c > 0$ such that

$$\begin{aligned}\sup_{u\in V}\sup_{x\in K}\left|\big((C_{\varphi_\alpha})_{[n]} - C_\varphi\big)(u)(x)\right| &= \sup_{u\in V}\sup_{x\in K}\left|u\big(((\varphi_\alpha)_{[n]} - \varphi)(x)\big)\right| \\ &\leq \sup_{u\in V}\sup_{x\in K}\|u\|_{p'}\left\|\big((\varphi_\alpha)_{[n]} - \varphi\big)(x)\right\|_p \leq c \cdot \sup_{x\in K}\left\|\big((\varphi_\alpha)_{[n]} - \varphi\big)(x)\right\|_p,\end{aligned}$$

which gives the conclusion.

2.4 Dynamics with the norm topology

We consider now the space $\mathscr{P}(^mX)$ endowed with the norm defined in (1.3). In contrast with what happens with the compact-open topology, now the properties of power boundedness and mean ergodicity are not related. We give examples of composition operators that are

- power bounded and not mean ergodic (Example 2.13);
- mean ergodic and not power bounded (Example 2.14).

We also ask about how being Cesàro bounded interacts with the two previous ones:

- Every power bounded operator is Cesàro bounded (Remark 1.9).
- Every mean ergodic operator in a Banach space is Cesàro bounded (Remark 1.18).
- There are Cesàro bounded composition operators that are neither power bounded, nor mean ergodic (Example 2.12).

As a first step we characterize, as we did in Proposition 2.7, the power boundedness of a composition operator by means of its symbol.

Proposition 2.9. *Let $\varphi: X \to X$ be a continuous linear map. Then $C_\varphi: \mathscr{P}(^mX)_{\|\cdot\|} \to \mathscr{P}(^mX)_{\|\cdot\|}$ is power bounded if and only if φ is power bounded.*

Proof. Suppose in first place that $\varphi: X \to X$ is power bounded. Then there is a constant $c > 0$ such that

$$\|\varphi^n(x)\|_X \le c\|x\|_X,$$

for all $n \in \mathbb{N}$ and for all $x \in X$. Using this we have, for $p \in \mathscr{P}(^mX)$ and $n \in \mathbb{N}$,

$$\|C_{\varphi^n}(p)\|_{\mathscr{P}(^mX)} \le \|p\|_{\mathscr{P}(^mX)} \sup_{\|x\|\le 1} \|\varphi^n(x)\|_X^m \le c^m \|p\|_{\mathscr{P}(^mX)},$$

and $C_\varphi: \mathscr{P}(^mX)_{\|\cdot\|} \to \mathscr{P}(^mX)_{\|\cdot\|}$ is power bounded.

Conversely, assume that $C_\varphi: \mathscr{P}(^mX)_{\|\cdot\|} \to \mathscr{P}(^mX)_{\|\cdot\|}$ is power bounded. We can find $c > 0$ such that $\|p \circ \varphi^n\|_{\mathscr{P}(^mX)} \le c\|p\|_{\mathscr{P}(^mX)}$, for every $p \in \mathscr{P}(^mX)$. In particular

$$\sup_{\|x\|_X \le 1} |\gamma(\varphi^n x)|^m \le c \sup_{\|z\|_X \le 1} |\gamma(z)|^m,$$

for every $n \in \mathbb{N}$ and every $\gamma \in X'$. Taking m-th roots yields $|\gamma(\varphi^n x)| \le c^{1/m}\|\gamma\|$ for every $\gamma \in X'$, every x with $\|x\|_X \le 1$ and all $n \in \mathbb{N}$. Since $\|\varphi^n x\|_X = \sup_{\|\gamma\|\le 1} |\gamma(\varphi^n x)|$ for every $n \in \mathbb{N}$, we have

$$\sup_{\|x\|_X \le 1} \|\varphi^n x\|_X \le c^{1/m},$$

for every $n \in \mathbb{N}$ and φ is power bounded. □

Proposition 2.10. *Let $\varphi: X \to X$ be a continuous linear map such that $C_\varphi: \mathscr{P}(^mX)_{\tau_0} \to \mathscr{P}(^mX)_{\tau_0}$ is power bounded. Then $C_\varphi: \mathscr{P}(^mX)_{\|\cdot\|} \to \mathscr{P}(^mX)_{\|\cdot\|}$ is power bounded.*

Proof. By Proposition 2.7, φ has stable orbits and for each $x \in X$ we can find a compact set $K_x \subseteq X$ such that $(\varphi^n(x))_n \subset K_x$. This gives that $\sup_{n\in\mathbb{N}} \|\varphi^n(x)\| < \infty$ for every $x \in X$ and, by the uniform boundedness principle, $\sup_{n\in\mathbb{N}} \|\varphi^n\| < \infty$. This shows that φ is power bounded and, by Proposition 2.9, so also is $C_\varphi: \mathscr{P}(^mX)_{\|\cdot\|} \to \mathscr{P}(^mX)_{\|\cdot\|}$. □

The converse implication is not true in general.

Example 2.11. Consider the composition operator $C_F: \mathscr{P}(^mc_0) \to \mathscr{P}(^mc_0)$ defined by the usual forward shift $F: c_0 \to c_0$ given by $F(x) = (0, x_1, x_2, \dots)$. Let us see that C_F is power bounded in $\mathscr{P}(^mX)_{\|\cdot\|}$ but it is not in $\mathscr{P}(^mX)_{\tau_0}$. On the one hand, we observe that $\|F^n(x)\| = \|x\|$ for every $x \in c_0$ and all $n \in \mathbb{N}$, but $(F^n(e_1))_n = (e_{n+1})_n$ is not relatively compact in c_0. This shows that F is power bounded but does not have stable orbits. As a consequence of Propositions 2.9 and 2.7, C_F is power bounded in $\mathscr{P}(^mc_0)_{\|\cdot\|}$ but not in $\mathscr{P}(^mc_0)_{\tau_0}$.

Example 2.12. Fix $m \geq 2$ and $0 < \alpha < 1/m$. The operator $\varphi_\alpha \colon \ell_m \to \ell_m$ defined in (2.7) satisfies that $C_{\varphi_\alpha} \colon \mathscr{P}({}^m\ell_m)_{\|\cdot\|} \to \mathscr{P}({}^m\ell_m)_{\|\cdot\|}$ is Cesàro bounded but neither power bounded nor mean ergodic.

From the proof of [7, Theorem 2.1] we obtain

$$\sum_{k=1}^{n} \|\varphi_\alpha^k(x)\|_{\ell_m}^m \leq 4n,$$

for every $\|x\|_{\ell_m} \leq 1$ and $n \in \mathbb{N}$. Hence given $p \in \mathscr{P}({}^m\ell_m)$ we have

$$\Big\| \sum_{k=1}^{n} p(\varphi_\alpha^k(x)) \Big\|_{\ell_m} \leq \sum_{k=1}^{n} \|p(\varphi_\alpha^k(x))\|_{\ell_m} \leq \sum_{k=1}^{n} \|p\|_{\mathscr{P}({}^m\ell_m)} \|\varphi_\alpha^k(x)\|_{\ell_m}^m \leq \|p\|_{\mathscr{P}({}^m\ell_m)} 4n\,,$$

for every $\|x\|_{\ell_m} \leq 1$ and $n \in \mathbb{N}$. This gives

$$\Big\| \frac{1}{n} \sum_{k=1}^{n} C_{\varphi_\alpha}^k(p) \Big\|_{\mathscr{P}({}^m\ell_m)} \leq 4\|p\|_{\mathscr{P}({}^m\ell_m)},$$

which shows that C_{φ_α} is Cesàro bounded.

We know that there exists $x_0 \in \ell_m$ such that $\big(\varphi_\alpha^n(x_0)\big)_n$ is dense in ℓ_m (recall Example 2.8). Hence, φ_α cannot be power bounded and so, by Proposition 2.9, neither is C_{φ_α}. To show that it is not mean ergodic we take the m-homogeneous polynomial given by

$$p(x) = \sum_{i \in \mathbb{N}} x_i^m, \tag{2.10}$$

and prove that $\big((C_{\varphi_\alpha})_{[n]}(p)\big)_{n \in \mathbb{N}}$ does not converge in $\mathscr{P}({}^mX)_{\|\cdot\|}$. First, recall (2.8) and observe that, for a fixed $n \in \mathbb{N}$ and $k \leq n$, we have

$$\varphi_\alpha^k(e_{n+1}) = \Bigg(0, \dots, 0, \underbrace{\left(\frac{n+1}{n+1-k}\right)^\alpha}_{(n+1-k)\text{-th coordinate}}, 0, \dots\Bigg).$$

From this we deduce that

$$\varphi_\alpha^k(e_{n+1}) = \begin{cases} \left(\frac{n+1}{n+1-k}\right)^\alpha e_{n+1-k}, & \text{if } n \geq k, \\ 0, & \text{if } n < k. \end{cases}$$

Then

$$\begin{aligned}
&|(C_{\varphi_\alpha})_{[n]}(p)(e_{n+1})-(C_{\varphi_\alpha})_{[2n]}(p)(e_{n+1})|\\
&\quad=\left|\frac{1}{n}\sum_{k=1}^{n}p(\varphi_\alpha^k(e_{n+1}))-\frac{1}{2n}\sum_{k=1}^{2n}p(\varphi_\alpha^k(e_{n+1}))\right|\\
&\quad=\left|\frac{1}{n}\sum_{k=1}^{n}p(\varphi_\alpha^k(e_{n+1}))-\frac{1}{2n}\sum_{k=1}^{n}p(\varphi_\alpha^k(e_{n+1}))\right|\\
&\quad=\frac{1}{2n}\left|\sum_{k=1}^{n}p(\varphi_\alpha^k(e_{n+1}))\right|=\frac{1}{2n}\left|\sum_{k=1}^{n}p\left(\left(\frac{n+1}{n+1-k}\right)^{\alpha}e_{n+1-k}\right)\right|\\
&\quad=\frac{1}{2n}\left|\sum_{k=1}^{n}\left(\frac{n+1}{n+1-k}\right)^{m\alpha}\right|=\frac{1}{2n}\left|\sum_{k=1}^{n}\left(1+\frac{k}{n+1-k}\right)^{m\alpha}\right|\\
&\quad\geq\frac{1}{2n}\left|\sum_{k=1}^{n}1\right|=\frac{1}{2}.
\end{aligned}$$

This implies

$$\|(C_{\varphi_\alpha})_{[n]}(p)-(C_{\varphi_\alpha})_{[2n]}(p)\|\geq\frac{1}{2},$$

for all $n\in\mathbb{N}$. Hence $\big((C_{\varphi_\alpha})_{[n]}(p)\big)_n$ is not Cauchy. So, C_{φ_α} is not mean ergodic.

This settles the relationship between Cesàro boundedness and power boundedness and mean ergodicity. We look now at the latter two. Unlike what we saw in Proposition 2.5 for the compact-open topology, when we consider the norm topology we may find composition operators that are power bounded but not mean ergodic.

Example 2.13. For $m\geq 1$ we consider the usual backward shift $\Sigma\colon \ell_m\to\ell_m$ defined as

$$\Sigma(x_1,x_2,x_3,\ldots)=(x_2,x_3,\ldots).$$

Then the composition operator $C_\Sigma\colon \mathscr{P}(^m\ell_m)_{\|\cdot\|}\to\mathscr{P}(^m\ell_m)_{\|\cdot\|}$ is power bounded but not mean ergodic.

Clearly $\|\Sigma\|\leq 1$, and Σ is power bounded. Applying Proposition 2.9 we obtain that C_Σ is power bounded.

To see that it is not mean ergodic we take the polynomial p defined in (2.10) and observe that

$$\Sigma^k e_{n+1}=\begin{cases}e_{n+1-k}, & \text{if } n\geq k,\\ 0, & \text{if } n<k.\end{cases}$$

Then

$$
\begin{aligned}
&|(C_\Sigma)_{[n]}(p)(e_{n+1})-(C_\Sigma)_{[2n]}(p)(e_{n+1})|\\
&=\left|\frac{1}{n}\sum_{k=1}^{n}p(\Sigma^k(e_{n+1}))-\frac{1}{2n}\sum_{k=1}^{2n}p(\Sigma^k(e_{n+1}))\right|=\left|\frac{1}{n}\sum_{k=1}^{n}p(\Sigma^k(e_{n+1}))-\frac{1}{2n}\sum_{k=1}^{n}p(\Sigma^k(e_{n+1}))\right|\\
&=\frac{1}{2n}\left|\sum_{k=1}^{n}p(\Sigma^k(e_{n+1}))\right|=\frac{1}{2n}\left|\sum_{k=1}^{n}p\left(e_{n+1-k}\right)\right|=\frac{1}{2n}\left|\sum_{k=1}^{n}1\right|=\frac{1}{2}.
\end{aligned}
$$

This, as in Example 2.12, shows that $\big((C_\Sigma)_{[n]}(p)\big)_n$ is not Cauchy and that C_Σ is not mean ergodic.

Example 2.14. For fixed $1<p<\infty$ we take $0<\alpha<\frac{1}{p'}:=1-\frac{1}{p}$ and define the weighted forward shift $\psi_\alpha\colon \ell_p\to\ell_p$ by

$$\psi_\alpha(e_k)=\Big(\frac{k+1}{k}\Big)^\alpha e_{k+1}\text{ for } k\geq 1\,,$$

that is, $\psi_\alpha(x_1,x_2,\dots)=(0,w_2x_1,w_3x_2,\dots)$, where, as before, $w_k=\big(\frac{k}{k-1}\big)^\alpha$ for $k\geq 2$. We consider the composition operator $C_{\psi_\alpha}\colon \mathscr{P}(^1\ell_p)_{\|\cdot\|}\to\mathscr{P}(^1\ell_p)_{\|\cdot\|}$. Since $\mathscr{P}(^1\ell_p)_{\|\cdot\|}=\ell_p'=\ell_{p'}$, for each $u\in\ell_{p'}$ and $x\in\ell_p$ we have

$$C_{\psi_\alpha}(u)(x)=u\big(\psi_\alpha(x)\big)=u(0,w_2x_1,w_3x_2,\dots)=\sum_{k=1}^{\infty}x_kw_{k+1}u_{k+1}=\varphi_\alpha(u)(x)\,.$$

This shows that $C_{\psi_\alpha}=\varphi_\alpha\colon \ell_{p'}\to\ell_{p'}$. By [7, Corollary 2.7] ($\ell_{p'}$ is reflexive), C_{ψ_α} is mean ergodic. On the other hand, we already saw in Example 2.8 that there is some $x_0\in\ell_p'$ such that $\big(\varphi_\alpha^n(x_0)\big)_n$ is dense in ℓ_p'. Then C_{ψ_α} is not power bounded.

Chapter 3

Composition operators on spaces of holomorphic functions

3.1 Introduction

Our aim now is to perform an analogous study to the one in Chapter 2, replacing the space of homogeneous polynomials with other spaces of holomorphic functions. More precisely, we consider the composition operator C_φ defined on

- $H(B_X)$, endowed with the compact-open topology τ_0;
- $H_b(B_X)$, endowed with the topology τ_b of uniform convergence on B_X-bounded sets;
- $H^\infty(B_X)$, endowed with the topology of the norm,

and we want to study some ergodic properties, focusing on the interplay between power boundedness and mean ergodicity. We study also the case when X is a Hilbert space for each of the settings considered above. The reason is that in [50] it was introduced a group of automorphisms on the unit ball of a Hilbert space which allows to use similar properties as in the one-dimensional case to study the dynamics of the composition operator.

The main results of this chapter are collected in [38].

3.2 Automorphisms in the ball of a Hilbert space

As we will see later, when we consider spaces of holomorphic functions defined on the open unit ball of a Hilbert spaces we can say much more about the composition operator. One of the reasons is that in this case we have a group of automorphisms that, in some sense, work as the Möbius transforms in the unit disc. This group of automorphisms was introduced in [50]. Let us recall the definition and prove the main property that we use later.

Let H be a Hilbert space and B_H its open unit ball. From [50, Proposition 1] we know that, given $a \in B_H$, the linear operator $\gamma_a : H \to H$ defined by

$$\gamma_a(x) := \frac{1}{1+\nu(a)} a\langle x, a\rangle + \nu(a)x,$$

where $\nu(a) = \sqrt{1-\|a\|^2} < 1$, satisfies $\|\gamma_a(x)\| \le \|x\|$ for all $x \in H$ and $\gamma_a(a) = a$. Once we have this, for each $a \in B_H$ we can define an automorphism $\alpha_a : B_H \to B_H$ by

$$\alpha_a(x) = \gamma_a\left(\frac{a-x}{1-\langle x, a\rangle}\right). \tag{3.1}$$

This satisfies:

- $\alpha_a(0) = \gamma_a(a) = a$.
- $\alpha_a(a) = \gamma_a(0) = 0$.
- $\alpha_a^{-1} = \alpha_a$ follows by [50, Proposition 1].

The following result is a consequence of [50, (9')]; we include a proof for the sake of completeness.

Lemma 3.1. *For each* $0 < r < 1$ *there is* $0 < \rho < 1$ *such that*

$$\alpha_a(rB_H) \subseteq \rho B_H, \tag{3.2}$$

for every $a \in rB_H$.

Proof. For an arbitrary $x \in B_H$ with $\|x\| < r$, we put $y := \alpha_a(x)$. We want to find $\varepsilon_r > 0$ independent of x such that $1 - \|y\|^2 \ge \varepsilon_r$.

First observe that

$$\begin{aligned}
\|y\|^2 &= \left\langle \frac{\gamma_a(a-x)}{1-\langle x, a\rangle}, \frac{\gamma_a(a-x)}{1-\langle x, a\rangle} \right\rangle = \frac{\langle a-\gamma_a(x), a-\gamma_a(x)\rangle}{|1-\langle x, a\rangle|^2} \\
&= \frac{\|a\|^2 + \|\gamma_a(x)\|^2 - \langle \gamma_a(x), a\rangle - \langle a, \gamma_a(x)\rangle}{|1-\langle x, a\rangle|^2} \\
&= \frac{\|a\|^2 + \|\gamma_a(x)\|^2 - 2\,\mathrm{Re}\langle \gamma_a(x), a\rangle}{|1-\langle x, a\rangle|^2}.
\end{aligned}$$

Now we check that

$$\begin{aligned}
\langle \gamma_a(x), a\rangle &= \frac{1}{1+\nu(a)}\langle a, a\rangle\langle x, a\rangle + \nu(a)\langle x, a\rangle \\
&= \frac{\|a\|^2 + (1+\nu(a))\nu(a)}{1+\nu(a)}\langle x, a\rangle \\
&= \frac{\|a\|^2 + \nu(a) + 1 - \|a\|^2}{1+\nu(a)}\langle x, a\rangle \\
&= \langle x, a\rangle.
\end{aligned} \tag{3.3}$$

Then

$$\begin{aligned}\|\gamma_a(x)\|^2 =&\Big\|\frac{1}{1+\nu(a)}a\langle x,a\rangle\Big\|^2+\|\nu(a)x\|^2\\ &+\Big\langle\frac{1}{1+\nu(a)}a\langle x,a\rangle,\nu(a)x\Big\rangle+\Big\langle\nu(a)x,\frac{1}{1+\nu(a)}a\langle x,a\rangle\Big\rangle\\ =&\frac{|\langle x,a\rangle|^2\|a\|^2}{(1+\nu(a))^2}+(1-\|a\|^2)\|x\|^2+\frac{2\nu(a)|\langle x,a\rangle|^2}{1+\nu(a)}\\ =&(1-\|a\|^2)\|x\|^2+\frac{|\langle x,a\rangle|^2\|a\|^2+(1+\nu(a))\nu(a)2|\langle x,a\rangle|^2}{2+2\nu(a)-\|a\|^2}\\ =&(1-\|a\|^2)\|x\|^2+|\langle x,a\rangle|^2\frac{\|a\|^2+2\nu(a)+2-2\|a\|^2}{2+2\nu(a)-\|a\|^2}\\ =&(1-\|a\|^2)\|x\|^2+|\langle x,a\rangle|^2.\end{aligned}$$

Using (3.3) and the last equality we obtain that

$$\|y\|^2=\frac{\|a\|^2+(1-\|a\|^2)\|x\|^2+|\langle x,a\rangle|^2-2\operatorname{Re}\langle x,a\rangle}{|1-\langle x,a\rangle|^2}.$$

Since $|1-\langle x,a\rangle|^2=1+|\langle x,a\rangle|^2-2\operatorname{Re}\langle x,a\rangle$, we get

$$\begin{aligned}1-\|y\|^2&=\frac{|1-\langle x,a\rangle|^2-\|a\|^2-(1-\|a\|^2)\|x\|^2-|\langle x,a\rangle|^2+2\operatorname{Re}\langle x,a\rangle}{|1-\langle x,a\rangle|^2}\\ &=\frac{1-\|a\|^2-(1-\|a\|^2)\|x\|^2}{|1-\langle x,a\rangle|^2}=\frac{(1-\|a\|^2)(1-\|x\|^2)}{|1-\langle x,a\rangle|^2}.\end{aligned}$$

Since $|1-\langle x,a\rangle|^2\le(1+\|x\|\|a\|)^2\le(1+r)^2$, we deduce

$$1-\|y\|^2\ge(1-r^2)^2(1+r)^{-2}=(1-r)^2=:\varepsilon_r,$$

as we wanted. The conclusion follows taking $\rho:=\sqrt{1-\varepsilon_r}$. □

3.3 Stable and B_X-stable orbits

As we know (recall Theorem 1.19) the symbol having stable orbits plays a major role in the study of composition operators on $H(B_X)$ when X is finite dimensional. We will see later (in Theorem 3.11) that this is also the case for infinite dimensional spaces.

A self map $f: B_X\to B_X$ is said to have *stable orbits* if for every compact subset K of B_X there is a compact subset $L\subset B_X$ such that $f^n(K)\subseteq L$ for every $n\in\mathbb{N}$ or, equivalently, if $\overline{\bigcup_{n=0}^\infty f^n(K)}$ is compact in B_X for every compact set $K\subseteq B_X$ (cf. Section 2.3).

In order to deal with the space $H_b(B_X)$ we need a similar notion, which we introduce here. We say that f has *B_X-stable orbits* if for every B_X-bounded set $A\subset B_X$

there is a B_X-bounded set $L \subset B_X$ such that $f^n(A) \subseteq L$ for every $n \in \mathbb{N}$ (equivalently, $\bigcup_{n=0}^{\infty} f^n(A)$ is B_X-bounded for every B_X-bounded set $A \subseteq B_X$). Observe that when X is finite dimensional the notions of stable orbits and B_X-stable orbits are equivalent. Therefore the corresponding results coincide.

Remark 3.2. The orbit $\{f^n(x) \colon n \in \mathbb{N}\}$ of each point $x \in B_X$ is relatively compact if f has stable orbits and B_X-bounded if f has B_X-stable orbits.

The notion of a function having B_X-stable orbits is, to our best knowledge, new. However, it is not hard to find functions with this property. In fact, the following well known version of the Schwarz Lemma immediately gives examples.

Lemma 3.3. *Let $\varphi \colon B_X \to B_X$ be holomorphic such that $\varphi(0) = 0$. Then $\|\varphi(x)\| \leq \|x\|$ for every $x \in B_X$.*

Proof. Consider the family of functions

$$\Big\{[\lambda \in \mathbb{D} \mapsto u\big(\varphi(\lambda x/\|x\|)\big)] \colon u \in X',\ \|u\| \leq 1,\ 0 < \|x\| < 1\Big\}.$$

Observe that these are holomorphic functions that map $\mathbb{D}$ into itself and fix 0. Fix $x \in B_X$ and $u \in X'$ with $\|u\| \leq 1$. Applying the classical Schwarz Lemma to the corresponding function we have

$$\Big|u\big(\varphi(\lambda x/\|x\|)\big)\Big| \leq |\lambda|,$$

for every $|\lambda| < 1$. Since the inequalities hold for every $u \in X'$ with $\|u\| \leq 1$, this gives $\|\varphi(\lambda x/\|x\|)\| \leq |\lambda|$ for every $|\lambda| < 1$. Taking $\lambda = \|x\|$ completes the proof. □

Proposition 3.4. *Let $\varphi \colon B_X \to B_X$ be a holomorphic mapping such that $\varphi(0) = 0$. Then φ has B_X-stable orbits.*

Proof. Lemma 3.3 clearly implies $\|\varphi^n(x)\| \leq \|x\|$ for all $n \in \mathbb{N}$ and all $x \in B_X$ and, therefore, for each $0 < r < 1$ we have

$$\varphi^n(rB_X) \subseteq rB_X,$$

for all $n \in \mathbb{N}$. This gives the claim. □

As a consequence, every continuous homogeneous polynomial $P \colon X \to X$ (in particular every linear operator) with $\|P\| \leq 1$ has B_X-stable orbits.

Example 3.5. If X is either c_0 or ℓ_p with $1 \leq p \leq \infty$, we recall the forward and backward shift operators

$$F(x_1, x_2, \dots) = (0, x_1, x_2, \dots) \text{ and } \Sigma(x_1, x_2, \dots) = (x_2, x_3, \dots). \tag{3.4}$$

Both are linear and clearly have norm less or equal 1, hence have B_X-stable orbits. It is not difficult to see that Σ has stable orbits. It all relays on the nice description of compact sets in these spaces.

First of all, if X is either c_0 or ℓ_∞, given a compact set $K \subset B_X$, we can find a sequence of positive numbers $a = (a_n)_n \in B_{c_0}$ such that $\sup_{x\in K} |x_n| \leq a_n$, for every $n \in \mathbb{N}$. Now, consider the sequence $b = (b_n)_n$ given by $b_n = \sup_{i\geq n} a_i$. Clearly, $b \in B_{c_0}$ and satisfies

$$\sup_{x\in\bigcup_m \Sigma^m(K)} |x_n| = \sup_{m\in\mathbb{N}} \sup_{x\in\Sigma^m(K)} |x_n| = \sup_{m\geq n} \sup_{x\in K} |x_m| \leq \sup_{m\geq n} a_m = b_n,$$

for every $n \in \mathbb{N}$. Then $\bigcup_m \Sigma^m(K)$ is a relatively compact set in B_X and Σ has stable orbits.

Now, consider $X = \ell_p$ with $1 \leq p < \infty$ and fix $K \subset B_{\ell_p}$ a compact set. By [23, p. 6], K is a relatively compact set in ℓ_p if and only if $\lim_{i\to\infty} \sup_{x\in K} \sum_{n=i}^{\infty} |x_n|^p = 0$. Now, for every $m \in \mathbb{N}$ we have

$$\sup_{y\in\Sigma^m(K)} \sum_{n=i}^{\infty} |y_n|^p = \sup_{x\in K} \sum_{n=i+m}^{\infty} |x_n|^p \leq \sup_{x\in K} \sum_{n=i}^{\infty} |x_n|^p.$$

This gives that $\bigcup_m \Sigma^m(K)$ is a relatively compact set in B_{ℓ_p} and Σ has stable orbits.

Now, for the forward shift, we have that the set

$$\left\{F^n\left(\frac{e_1}{2}\right) : n \in \mathbb{N}\right\} = \left\{\frac{e_n}{2} : n > 1\right\}$$

is not relatively compact in X and, by Remark 3.2, F does not have stable orbits.

Example 3.6. Consider the mapping $\phi : B_{c_0} \to B_{c_0}$ defined as

$$\phi(x_1, x_2, \dots) = \left(\tfrac{x_1}{2} + \tfrac{1}{2}, 0, 0, \dots\right).$$

Note that $\phi^n(0) = \left(\sum_{i=1}^{n} \frac{1}{2^i}, 0, 0, \dots\right)$ and, therefore,

$$\lim_{n\to\infty} \|\phi^n(0)\| = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2^i} = 1.$$

Hence ϕ has neither stable nor B_{c_0}-stable orbits.

3.3.1 The Hilbert-space case

If H is a Hilbert space, for each $a \in B_H$ the automorphism $\alpha_a : B_H \to B_H$ defined in (3.1) satisfies $\alpha_a^{-1} = \alpha_a$. Hence

$$\bigcup_{n=0}^{\infty} \alpha_a^n(A) = A \cup \alpha_a(A),$$

for every $A \subseteq B_H$. If A is compact, $\alpha_a(A)$ is again compact, and if A is B_H-bounded, by Lemma 3.1, so also is $\alpha_a(A)$. This shows that α_a has both stable and B_H-stable orbits.

Using these automorphisms, in the case of Hilbert spaces we can extend Proposition 3.4 showing that every holomorphic function with a fixed point has B_H-stable orbits.

Lemma 3.7. *If $\varphi : B_H \to B_H$ has stable orbits (respectively B_H-stable orbits), then the mapping $\psi = \alpha_a \circ \varphi \circ \alpha_a$ has stable orbits (respectively B_H-stable orbits) for every $a \in B_H$.*

Proof. If $K \subseteq B_H$ is compact, then $\alpha_a(K)$ is compact and, having φ stable orbits, we can find a compact set $L \subseteq B_H$ so that $\varphi^n(\alpha_a(K)) \subseteq L$ for each $n \in \mathbb{N}$. Then $\alpha_a(L) \subseteq B_H$ is compact and $\alpha_a\big(\varphi^n(\alpha_a(K))\big) \subseteq \alpha_a(L)$. Since $\psi^n = \alpha_a \circ \varphi^n \circ \alpha_a$ (because $\alpha_a^2 = \mathrm{id}$), ψ has stable orbits.

The argument if φ has B_H-stable orbits is exactly the same, using that $\alpha_a(A)$ is B_H-bounded for every B_H-bounded set A by Lemma 3.1. □

Proposition 3.8. *Let $\varphi : B_H \to B_H$ be a holomorphic mapping with a fixed point. Then φ has B_H-stable orbits.*

Proof. Take $a \in B_H$ with $\varphi(a) = a$. The holomorphic function $\psi = \alpha_a \circ \varphi \circ \alpha_a : B_H \to B_H$ satisfies $\psi(0) = \alpha_a(\varphi(\alpha_a(0))) = \alpha_a(\varphi(a)) = \alpha_a(a) = 0$. Then, by Proposition 3.4 the function ψ has B_H-stable orbits, and Lemma 3.7 gives the conclusion. □

3.4 The space of holomorphic functions

Proposition 3.9. *Let $\varphi : B_X \to B_X$ be a continuous mapping. Then $C_\varphi : H(B_X) \to H(B_X)$ is well defined if and only if φ is holomorphic.*

Proof. If φ is holomorphic the composition operator is well defined (and continuous). On the other hand, if C_φ is well defined, $u \circ \varphi$ is holomorphic for every $u \in X'$. By Theorem 1.4 φ is holomorphic. □

In view of this result there is no restriction to assume that φ is holomorphic.

Remark 3.10. Suppose that the composition operator C_φ is topologizable. Then for every compact set $K \subseteq B_X$ there is some compact $L \subseteq B_X$ and some sequence $(a_n)_n$ of positive numbers such that

$$\sup_{x \in K} |f(\varphi^n(x))| \leq a_n \sup_{x \in L} |f(x)|,$$

for every $f \in H(B_X)$ and every $n \in \mathbb{N}$ (see (1.5)). Now fix $f \in H(B_X)$. Since $f^m \in H(B_X)$ for every $m \in \mathbb{N}$ the following holds

$$\sup_{x \in K} |f(\varphi^n(x))| = \sqrt[m]{\sup_{x \in K} |f^m(\varphi^n(x))|} \leq \sqrt[m]{a_n \sup_{x \in L} |f^m(x)|} = \sqrt[m]{a_n} \sup_{x \in L} |f(x)|,$$

for every $f \in H(B_X)$ and every $m, n \in \mathbb{N}$. Letting $m \to \infty$ yields

$$\sup_{x\in K} |f(\varphi^n(x))| \leq \sup_{x\in L} |f(x)|,$$

and, in particular, C_φ is power bounded. Our aim now is to show that in fact this implication can be characterised in terms of the symbol.

Following [54] and [15] (cf. Lemma 2.6), given a family $\mathscr{F}$ of $\mathbb{C}$-valued holomorphic functions defined on an open set U, the $\mathscr{F}$-hull of $A \subseteq U$ is denoted by

$$\widehat{A}_{\mathscr{F}} := \{x \in U : |f(x)| \leq \sup_{y\in A} |f(y)|, \text{ for all } f \in \mathscr{F}\}. \tag{3.5}$$

Stable orbits of the symbol is the property that characterises the power boundedness of the composition operator. The result in Proposition 2.7 is similar to the following, however in $H(B_X)$ we can say more.

Theorem 3.11. *Let $\varphi\colon B_X \to B_X$ be holomorphic. The following assertions are equivalent:*

a) φ has stable orbits on B_X.

b) $C_\varphi\colon H(B_X) \to H(B_X)$ is power bounded.

c) $\left(\frac{1}{n}C_\varphi^n\right)_n$ is equicontinuous in $\mathscr{L}(H(B_X))$.

d) $C_\varphi\colon H(B_X) \to H(B_X)$ is topologizable.

Proof. a)⇒b). If φ has stable orbits, given a compact set $K \subseteq B_X$ there is a compact set $L \subseteq B_X$ such that $\varphi^n(K) \subseteq L$ for every $n \in \mathbb{N}$. Hence

$$\sup_{x\in K} |C_\varphi^n(f)(x)| = \sup_{x\in K} |f(\varphi^n(x))| \leq \sup_{x\in L} |f(x)|,$$

for all $f \in H(B_X)$ and $n \in \mathbb{N}$. So the sequence $(C_\varphi^n)_n$ is equicontinuous, i.e. C_φ is power bounded.

b)⇒c). Suppose now that C_φ is power bounded, then for each compact set $K \subseteq B_X$ we can find $c > 0$ and a compact set $L \subseteq B$ such that

$$\sup_{x\in K} |C_\varphi^n(f)(x)| \leq c \sup_{x\in L} |f(x)|$$

for every $f \in H(B_X)$ and $n \in \mathbb{N}$. This obviously implies

$$\sup_{x\in K} \left|\tfrac{1}{n}C_\varphi^n(f)(x)\right| \leq c \sup_{x\in L} |f(x)|$$

for every f and n, and $\left(\frac{1}{n}C_\varphi^n\right)_n$ is equicontinuous.

That c)⇒d) follows just taking $a_n = cn$ in (1.5).

d)⇒a). Fix some compact set $K \subseteq B_X$. Since C_φ is topologizable, we can find some compact set $W \subseteq B_X$, and $(a_n)_n$ with $a_n > 0$ such that,

$$\sup_{x \in K} |f(\varphi^n(x))| \leq a_n \sup_{x \in W} |f(x)|, \tag{3.6}$$

for all $f \in H(B_X)$ and $n \in \mathbb{N}$. By [47, Corollary 10.7 and Theorem 11.4], the set $L = \widehat{W}_{H(B_X)}$ (recall (3.5)) is compact and contains W. We see that $\varphi^n(K) \subseteq L$ for every n. Suppose that this is not the case and take $x_0 \in K$ and $n_0 \in \mathbb{N}$ such that $\varphi^{n_0}(x_0) \notin L$. Then there is $f \in H(B_X)$ such that $|f(\varphi^{n_0}(x_0))| > \sup_{y \in W} |f(y)|$, and there exists $m \in \mathbb{N}$ such that

$$\sup_{y \in W} \frac{|f(y)|^m}{|f(\varphi^{n_0}(x_0))|^m} < \frac{1}{a_{n_0}}.$$

But the function $g = f^m$ satisfies (3.6) for n_0, thus we obtain

$$|f(\varphi^{n_0}(x_0))|^m \leq \sup_{x \in K} |f(\varphi^{n_0}(x))|^m \leq a_{n_0} \sup_{y \in W} |f(y)|^m,$$

which gives a contradiction. □

Proposition 3.12. *Let $\varphi\colon B_X \to B_X$ be holomorphic. If $C_\varphi\colon H(B_X) \to H(B_X)$ is power bounded, then it is uniformly mean ergodic.*

Proof. Since $H(B_X)$ is semi-Montel, by Proposition 1.15, we have that every power bounded operator is uniformly mean ergodic in $H(B_X)$. □

3.5 The space of holomorphic functions of bounded type

If $\varphi\colon B_X \to B_X$ is holomorphic of bounded type, then clearly $C_\varphi\colon H_b(B_X) \to H_b(B_X)$ is well defined. On the other hand, we observe that $X' \subseteq H_b(B_X)$ because every functional is trivially holomorphic and bounded in B_X. So, if the composition operator is well defined (as a self map on $H_b(B_X)$), then the argument in Remark 3.9 shows that φ has to be holomorphic. Furthermore, [33, Proposition 3] shows that φ is of bounded type.

Our first goal in this section is to characterise the power boundedness of composition operators on $H_b(B_X)$. In this case having B_X-stable orbits is the key property. As in Remark 3.10, if the composition operator C_φ is topologizable, then for every B_X-bounded set U there is some B_X-bounded set V such that

$$\sup_{x \in U} |f(\varphi^n(x))| \leq \sup_{x \in V} |f(x)|,$$

and C_φ is power bounded. We go further.

Lemma 3.13. *Let A be a B_X-bounded set then, $\widehat{A}_{H_b(B_X)}$ is B_X-bounded.*

Proof. By the definition of $\widehat{A}_{H_b(B_X)}$, for each $u \in X'$, we have

$$\sup_{x \in \widehat{A}_{H_b(B_X)}} |u(x)| \leq \sup_{x \in A} |u(x)|.$$

Now, proceeding as in Lemma 2.6 we obtain $\widehat{A}_{H_b(B_X)} \subseteq \overline{\text{co}}(A)$. Since B_X is absolutely convex, [14, Remark, p. 527] gives that $\overline{\text{co}}(A)$ is B_X-bounded, which completes the proof.

This argument also works replacing B_X by any absolutely convex open set in X. □

With exactly the same proof as in Theorem 3.11, replacing 'compact' by 'B_X-bounded' we have the following.

Theorem 3.14. *Let $\varphi : B_X \to B_X$ be a holomorphic mapping of bounded type. The following assertions are equivalent*

a) φ has B_X-stable orbits.

b) $C_\varphi : H_b(B_X) \to H_b(B_X)$ is power bounded.

c) $\left(\frac{1}{n} C_\varphi^n\right)_n$ is equicontinuous in $\mathscr{L}(H_b(B_X))$.

d) $C_\varphi : H_b(B_X) \to H_b(B_X)$ is topologizable.

We now show that in this case every mean ergodic composition operator is power bounded, and there are power bounded operators that are not mean ergodic.

Proposition 3.15. *Let $\varphi : B_X \to B_X$ a holomorphic mapping of bounded type. If $C_\varphi : H_b(B_X) \to H_b(B_X)$ is mean ergodic, then C_φ is power bounded.*

Proof. First, by (1.10) we have

$$\frac{1}{n} C_\varphi^n = \frac{n+1}{n} (C_\varphi)_{[n+1]} - (C_\varphi)_{[n]},$$

for every $n \in \mathbb{N}$. The mean ergodicity of C_φ gives that the sequence $(\frac{1}{n} C_\varphi^n)$ tends to zero (pointwise), so it is pointwise bounded. Since $H_b(B_X)$ is barrelled (see Section 1.3), it is also equicontinuous on $H_b(B_X)$. This, in view of Theorem 3.14, gives the conclusion. □

We want to find now composition operators that are power bounded but not mean ergodic. The shifts in (3.4) provide us with such examples.

Proposition 3.16. *The composition operators $C_\Sigma : H_b(B_{c_0}) \to H_b(B_{c_0})$ and $C_F : H_b(B_{\ell_1}) \to H_b(B_{\ell_1})$ are power bounded but not mean ergodic.*

Proof. We already noted in Example 3.5 that Σ has B_{c_0}-stable orbits which, in view of Theorem 3.14, shows that C_Σ is power bounded.

We now see that C_Σ is not mean ergodic. The rest of the proof is based in [8, Theorem 1.2]. We begin by observing that $H_b(B_X)$ contains a complemented copy of X' for every Banach space X. As in the proof of Proposition 1.5, now we can see that the mappings $P\colon H_b(B_X) \to X'$ and $J: X' \to H_b(B_X)$, defined by $P(f) = df_0$ and $J(u) = u|_{B_X}$, are continuous and give our claim.

We consider now the restriction of C_Σ to $J(\ell_1)$ (recall that $c_0' = \ell_1$) and we have, for each $u \in \ell_1$ and $x \in c_0$,

$$\begin{aligned}\langle C_\Sigma u, x\rangle &= u(\Sigma(x)) = \langle u, \Sigma(x)\rangle = \langle (u_1,u_2,u_3,\dots),(x_2,x_3,x_4,\dots)\rangle \\ &= u_1x_2 + u_2x_3 + u_3x_4 + \cdots = \langle (0,u_1,u_2,u_3,\dots),(x_1,x_2,x_3,x_4,\dots)\rangle = \langle Fu, x\rangle\,.\end{aligned}$$

Thus $F = P \circ C_\Sigma \circ J$, which is not mean ergodic in ℓ_1 (see, for instance, [8]). This implies that C_Σ is not mean ergodic in $H_b(B_{c_0})$.

For the forward shift, in Example 3.5 we showed that F has B_{ℓ_1}-stable orbits. Taking the restriction of C_F to $\ell_1' = \ell_\infty$, for each $u \in \ell_\infty$ and $x \in \ell_1$, we have

$$\begin{aligned}\langle C_F u, x\rangle &= \langle u, F(x)\rangle = \langle (u_1,u_2,u_3,\dots),(0,x_1,x_2,\dots)\rangle \\ &= u_2x_1 + u_3x_2 + u_4x_3 + \cdots = \langle (u_2,u_3,u_4,\dots),(x_1,x_2,x_3,\dots)\rangle = \langle \Sigma u, x\rangle\,.\end{aligned}$$

Thus $\Sigma = P \circ C_F \circ J$, which is not mean ergodic in ℓ_∞ (see also [8]). □

We look now for sufficient conditions for a given power bounded composition operator to be mean ergodic.

We denote by $\mathscr{P}(X)$ the algebra of all continuous polynomials on X (these are finite sums of homogeneous polynomials), and by $\sigma(X,\mathscr{P}(X))$ the coarsest topology making all $P \in \mathscr{P}(X)$ continuous. This is a Hausdorff topology satisfying $\|\cdot\| \succeq \sigma(X,\mathscr{P}(X)) \succeq \sigma(X,X')$, and the concepts of (relatively) countably compact, (relatively) sequentially compact and (relatively) compact for a subset of X all agree with respect to this topology [29].

Proposition 3.17. *Let $\varphi\colon B_X \to B_X$ be holomorphic, having B_X-stable orbits and such that $\varphi(A)$ is relatively $\sigma(X,\mathscr{P}(X))$-compact for every B_X-bounded set A. Then $C_\varphi\colon H_b(B_X) \to H_b(B_X)$ is mean ergodic.*

Proof. By Theorem 3.14 we have that the composition operator C_φ is power bounded. By Remark 1.9 we have $\big((C_\varphi)_{[n]}\big)_n$ is equicontinuous. This gives that the set $\big((C_\varphi)_{[n]}(f)\big)_n$ is bounded for every $f \in H_b(B_X)$. Now, by [31, Theorem 2.9], C_φ maps bounded sets of $H_b(B_X)$ into relatively $\sigma(H_b(B_X),H_b(B_X)')$-compact sets. So, for every $f \in H_b(B_X)$ the set

$$C_\varphi\Big(\big((C_\varphi)_{[n]}(f)\big)_n\Big) = \left\{\frac{1}{n}\sum_{k=1}^{n} C_\varphi^k(f) : n \in \mathbb{N}\right\}$$

is relatively $\sigma(H_b(B_X), H_b(B_X)')$-compact. Our aim now is to see that $\big((C_\varphi)_{[n]}(f)\big)_n$ is a relatively $\sigma(H_b(B_X), H_b(B_X)')$-compact set for all $f \in H_b(B)$, which in view of Proposition 1.13, gives that C_φ is mean ergodic.

Note that

$$(C_\varphi)_{[n]}(f) = \frac{1}{n}(f - C_\varphi^n(f)) + \frac{1}{n}\sum_{k=1}^{n} C_\varphi^k(f),$$

for every $n \in \mathbb{N}$. The fact that C_φ is power bounded implies that $\big(\frac{1}{n}(\mathrm{id}(f) - C_\varphi^n(f))\big)_n$ tends to 0 as $n \to \infty$. Thus, $\big(\frac{1}{n}(\mathrm{id}(f) - C_\varphi^n(f))\big)_n$ is a relatively compact set. In particular, it is relatively $\sigma(H_b(B_X), H_b(B_X)')$-compact. Since $\big((C_\varphi)_{[n]}(f)\big)_n$ is contained in the sum of two relatively $\sigma(H_b(B_X), H_b(B_X)')$-compact sets we have finished the proof. □

Corollary 3.18. *Let X be a Banach space such that every B_X-bounded set is relatively $\sigma(X, \mathscr{P}(X))$-compact. Then $C_\varphi : H_b(B_X) \to H_b(B_X)$ is power bounded if and only if C_φ is mean ergodic.*

An example of a Banach space which satisfies such a property is the Tsirelson space T^* (see [44, Example 2.e.1]): it is known that T^* is reflexive and the polynomials on T^* are weakly sequentially continuous [24, p. 121]. Hence, any sequence in the unit ball of T^* has a weakly convergent subsequence, which converges in the topology $\sigma(T^*, \mathscr{P}(T^*))$. Since $\sigma(T^*, \mathscr{P}(T^*))$ is angelic [29, p. 150] (for the definition of angelic, see also [26, Definition 3.53]), the unit ball is also relatively $\sigma(T^*, \mathscr{P}(T^*))$-compact.

So far we have analysed the relationship between power boundedness and mean ergodicity. We move one step further, looking at uniform mean ergodicity. To be more precise, we look now for conditions that ensure that a composition operator $C_\varphi : H_b(B_X) \to H_b(B_X)$ is uniformly mean ergodic. Here C_0 denotes the composition operator defined by the constant function 0 (i.e. $C_0(f) = f(0)$ for every f).

Theorem 3.19. *Let $\varphi : B_X \to B_X$ be holomorphic so that for every $0 < t < 1$ there exists $0 < \rho < t$ such that*

$$\varphi(tB_X) \subseteq \rho B_X\,. \tag{3.7}$$

Then $C_{\varphi^n} \to C_0$ in τ_b. In particular, $(C_\varphi)_{[n]} \to C_0$ in τ_b, i.e. $C_\varphi : H_b(B_X) \to H_b(B_X)$ is uniformly mean ergodic.

Proof. Fix some $0 < t < 1$. First of all, (3.7) implies, on the one hand, that $\varphi^n(tB_X) \subset \rho B_X$ for every $n \in \mathbb{N}$ and, on the other hand, that $\varphi(0) = 0$. We can then apply Lemma 3.3 to the function $[x \mapsto \frac{1}{\rho}\varphi(tx)]$ and get

$$\|\varphi^n(x)\| \leq \Big(\frac{\rho}{t}\Big)^n \|x\|, \tag{3.8}$$

for every $x \in tB_X$ and $n \in \mathbb{N}$. Now, given $f \in H_b(B_X)$, we obviously have $\|f \circ \varphi^n\|_{tB_X} \leq \|f\|_{tB_X}$ for every $n \in \mathbb{N}$. We define $g : B_X \to \mathbb{D}$ by $g(x) = \frac{1}{2\|f\|_{tB_X}}\big(f(\varphi(tx)) - f(0)\big)$.

This is clearly holomorphic and satisfies $g(0) = 0$. Then we can apply Lemma 3.3 to g and (3.8) to obtain

$$\begin{aligned} \|C^n_\varphi(f) - C_0(f)\|_{tB_X} &= \sup_{x \in tB_X} |f(\varphi^n(x)) - f(0)| \\ &\leq 2\|f\|_{tB_X} \sup_{x\in tB_X} \|\varphi^{n-1}(x)\| \leq 2\|f\|_{tB_X} \Big(\frac{\rho}{t}\Big)^{n-1}. \end{aligned}$$

This implies, for every $0 < t < 1$ and every bounded set $A \subseteq H_b(B_X)$,

$$\lim_{n\to\infty} \sup_{f \in A} \sup_{x \in tB_X} |C_{\varphi^n}(f)(x) - f(0)| = 0.$$

Hence, $C_{\varphi^n} \to C_0$ in τ_b. Once we have this, a standard argument for numerical sequences gives $(C_\varphi)_{[n]} \to C_0$ in τ_b. □

Remark 3.20. If $\varphi\colon B_X \to B_X$ is holomorphic and satisfies

$$\varphi(B_X) \subseteq rB_X \text{ for some } 0 < r < 1 \text{ and } \varphi(0) = 0, \tag{3.9}$$

then, applying Lemma 3.3 to the function $[x \mapsto \frac{1}{r}\varphi(x)]$, we get $\|\varphi(x)\| \leq r\|x\|$ for every $x \in B_X$, and this implies that φ satisfies (3.7) with $\rho = tr$.

There are, however, functions satisfying (3.7) but not (3.9). To see this just consider the restriction to B_X of any m-homogeneous polynomial (for $m > 1$) $P\colon X \to X$ with $\|P\| \leq 1$. For a fixed $0 < t < 1$ take any $0 < \varepsilon < t - t^m$ and notice that

$$\|P(tx)\| \leq t^m \|x\|^m \leq (t - \varepsilon)\|x\|,$$

for every $x \in B_X$. That is, every homogeneous polynomial with norm ≤ 1 satisfies (3.7). If $\|P\| = 1$ and attains its norm, that is, there is x_0 with $\|x_0\| = 1$ such that $\|P(x_0)\| = \|P\|$, then

$$\Big\|P\big((1-\tfrac{1}{n})x_0\big)\Big\| = \Big(1 - \frac{1}{n}\Big)^m,$$

and there is no $0 < r < 1$ such that $P(B_X) \subseteq rB_X$. For a concrete example of such a polynomial just consider the 2-homogeneous one $P\colon \ell_2 \to \ell_2$ given by $P\big((x_n)_n\big) = (x_n^2)_n$ (in this case one can take $x_0 = e_1$).

In particular, we have that, if $m > 1$ and P is an m-homogeneous polynomial with $\|P\| \leq 1$, then $C_P\colon H_b(B_X) \to H_b(B_X)$ is uniformly mean ergodic. For $m = 1$, that is, for linear operators, this property does not hold, as Proposition 3.24 shows.

One may also ask if in (3.9) we can drop the condition on the fixed point and still get (3.7) just assuming that $\varphi(B_X) \subseteq rB_X$ for some $0 < r < 1$. But this is not the case: fix some $x_0 \in B_X$ and consider the constant function $\varphi(x) = x_0$ for every $x \in B_X$.

Now, we provide an example of composition operator on $H_b(B_{c_0})$ that is mean ergodic but not uniformly mean ergodic. We use two well known results. The first one can be found in [42, §39,4(1), p. 138].

Lemma 3.21. *Let $(T_n)_n$ be an equicontinuous sequence of operators on a locally convex space E. If (T_n) is pointwise convergent to a continuous operator T on some dense subset, then $(T_n)_n$ is pointwise convergent to T in E.*

For the second one we need the notion of monomial on c_0. Each multi-index $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_0^N$ (with $N \in \mathbb{N}$) defines a function $h_\alpha \colon c_0 \to \mathbb{C}$, called *monomial*, by

$$h_\alpha(x) = x^\alpha := x_1^{\alpha_1} \cdot \cdots \cdot x_N^{\alpha_N}, \qquad \text{for every } x \in c_0. \tag{3.10}$$

Theorem 3.22. [20, Theorem 15.60]. *For each $m \in \mathbb{N}$, the set*

$$A_m := \{h_\alpha : \alpha \in \mathbb{N}_0^N, \alpha_1 + \cdots + \alpha_N = m; N \in \mathbb{N}\}$$

of monomials generates a dense subspace of $\mathscr{P}({}^m c_0)_{\|\cdot\|}$.

Remark 3.23. As a consequence of Remark 1.7, we have that the polynomials are dense in $H_b(B_X)$, for every Banach space X. In the space c_0, we additionally have Theorem 3.22. Since $\mathscr{P}({}^m c_0)_{\|\cdot\|} = (\mathscr{P}({}^m X), \tau_b)$, a combination of both results yields that

$$\operatorname{span}\{h_\alpha : \alpha \in \mathbb{N}_0^N; N \in \mathbb{N}\}$$

is dense in $H_b(B_{c_0})$.

Proposition 3.24. *Let $F \colon B_{c_0} \to B_{c_0}$ be the forward shift. The composition operator $C_F \colon H_b(B_{c_0}) \to H_b(B_{c_0})$ is mean ergodic but not uniformly mean ergodic.*

Proof. First, we see that C_F is mean ergodic. We follow a similar scheme to that in [5, Theorem 2.2], using that $\operatorname{span}\{h_\alpha : \alpha \in \mathbb{N}_0^N; N \in \mathbb{N}\}$ is a dense subspace of $H_b(B_{c_0})$ (Remark 3.23) together with Lemma 3.21. Since C_F is power bounded on $H_b(B_{c_0})$ (because it has B_{c_0}-stable orbits), $(C_F^n)_n$ is equicontinuous. Therefore, $\big((C_F)_{[n]}\big)_n$ is also equicontinuous on $H_b(B_{c_0})$ (see Remark 1.9). Since $C_F(1) = 1 = C_0(1)$ for any constant mapping (this is in fact true for any composition operator), it remains to see that $\big((C_F)_{[n]}(h)\big)_n$ τ_b-converges to $C_0(h)$ for every $h \in A_m$ and $m > 0$ (in these cases $C_0(h) = 0$). It is enough to check this for the monomials h_α defined in (3.10). Given any $\alpha \in \mathbb{N}_0^N$, for each $n \geq N$ we clearly have $C_F^n(h_\alpha)(x) = (F^n(x))^\alpha = 0$ for every $x \in B_{c_0}$, and the claim follows.

As in the proof of Proposition 3.16, one can see that $P \circ C_F \circ J = \Sigma$, where $\Sigma \colon \ell_1 \to \ell_1$ is the backward shift (recall (3.4)). If C_F were uniformly mean ergodic on $H_b(B_{c_0})$, then $\Sigma \colon \ell_1 \to \ell_1$ would be uniformly mean ergodic, but this is not the case. Indeed, since $\Sigma^j x$ tends to 0 in ℓ_1 for all $x \in \ell_1$, the only possible value for the limit projection of $\frac{1}{n}\sum_{j=0}^{n-1} \Sigma^j$ is 0. But, for each $n \in \mathbb{N}$, we have

$$\sup_{\|x\| \leq 1} \Big\| \frac{1}{n} \sum_{j=0}^{n-1} \Sigma^j(x) \Big\|_{\ell_1} \geq \frac{1}{n} \Big\| \sum_{j=0}^{n-1} \Sigma^j(e_n) \Big\|_{\ell_1} = \frac{1}{n} \big\| (1, \overset{(n)}{\ldots}, 1, 0, \ldots) \big\|_{\ell_1} = 1.$$

Hence, we have $\Sigma_{[n]} \not\to 0$ in τ_b. □

3.5.1 The Hilbert-space case

Now, we recall the Earle-Hamilton fixed point theorem (see [25]) which allows us to find conditions to obtain that some composition operators are uniformly mean ergodic.

Theorem 3.25 (Earle-Hamilton)**.** *Let $U \subseteq X$ be a connected open subset and let $f : U \to U$ be a holomorphic mapping such that:*

- *$f(U)$ is bounded in X.*
- *the distance between $f(U)$ and $X \setminus U$ is strictly positive.*

Then there is a unique $x \in U$ such that $f(x) = x$. Additionally, for each $y \in U$ we have that $f^n(y) \to x$ in X.

Let us go back to (3.9) for a moment. If we only assume $\varphi(B_X) \subseteq rB_X$, the Earle-Hamilton Fixed Point Theorem implies that there exists a unique $a \in B_X$ such that $\varphi(a) = a$. It is then natural to ask if this is enough to ensure that the composition operator is uniformly mean ergodic. If we restrict ourselves to Hilbert spaces H we can say something in this respect. We need the following lemma.

Lemma 3.26. *Let $\varphi : B_H \to B_H$ be holomorphic such that $C_{\varphi^n} \to C_0$ in the topology τ_b of $\mathscr{L}(H_b(B_H))$. Then for every $a \in B_H$ the mapping $\psi = \alpha_a \circ \varphi \circ \alpha_a$ satisfies that $C_{\psi^n} \to C_a$ in the topology τ_b.*

Proof. Since both φ and α_a are of bounded type (see Lemma 3.1), the composition $\alpha_a \circ \varphi \circ \alpha_a$ is of bounded type and $C_\psi : H_b(B_H) \to H_b(B_H)$ is well defined. Observe now that $\psi^n = \alpha_a \circ \varphi^n \circ \alpha_a$ for all $n \in \mathbb{N}$ since $\alpha_a^{-1} = \alpha_a$. Then

$$C_{\psi^n} = C_{\alpha_a \circ \varphi^n \circ \alpha_a} = C_{\alpha_a} \circ C_{\varphi^n} \circ C_{\alpha_a} \to C_{\alpha_a} \circ C_0 \circ C_{\alpha_a} = C_{\alpha_a} \circ C_{\alpha_a(0)} = C_{\alpha_a} \circ C_a = C_a.$$

□

Proposition 3.27. *Let $\varphi : B_H \to B_H$ be holomorphic such that*

$$\varphi(B_H) \subseteq rB_H \text{ for some } 0 < r < 1. \tag{3.11}$$

Then, for the unique $a \in B_H$ such that $\varphi(a) = a$ we have $C_{\varphi^n} \to C_a$ in the topology τ_b. In particular, $C_\varphi : H_b(B_H) \to H_b(B_H)$ is uniformly mean ergodic.

Proof. Define $\phi = \alpha_a \circ \varphi \circ \alpha_a : B_H \to B_H$, which clearly satisfies $\phi(0) = 0$. Also,

$$\phi(B_H) = (\alpha_a \circ \varphi \circ \alpha_a)(B_H) = (\alpha_a \circ \varphi)(B_H) \subseteq \alpha_a(rB_H),$$

and using Lemma 3.1 we can find some $0 < \varepsilon < 1$ such that

$$\phi(B_H) \subseteq (1 - \varepsilon)B_H.$$

Then ϕ satisfies (3.9) and, by Theorem 3.19, $C_{\phi^n} \to C_0$. Since $\varphi = \alpha_a \circ \phi \circ \alpha_a$ (because $\alpha_a^{-1} = \alpha_a$), Lemma 3.26 yields the claim. □

Let us consider any holomorphic self-map $\varphi\colon B_X \to B_X$ (being X any Banach space) such that $\varphi\circ\varphi = \mathrm{id}$. Then

$$C_\varphi^n = \begin{cases} C_\varphi \text{ if } n \text{ is odd,} \\ C_{\mathrm{id}_{B_X}} = \mathrm{id}_{H_b(B_X)} \text{ if } n \text{ is even}, \end{cases}$$

and for each $k\in\mathbb{N}$ we have

$$(C_\varphi)_{[2k-1]} = \frac{1}{2k-1}\sum_{n=0}^{2k-1} C_\varphi^n = \frac{k}{2k-1}\big(C_\varphi + \mathrm{id}_{H_b(B_X)}\big),$$

and

$$(C_\varphi)_{[2k]} = \frac{1}{2k}\sum_{n=0}^{2k} C_\varphi^n = \frac{1}{2}\big(C_\varphi + \mathrm{id}_{H_b(B_X)}\big) + \frac{1}{2k}\,\mathrm{id}_{H_b(B_X)}\,.$$

This implies that $\lim_{n\to\infty}(C_\varphi)_{[n]} = \frac{1}{2}\big(C_\varphi + \mathrm{id}_{H_b(B_X)}\big)$ in the topology τ_b, i.e. $C_\varphi\colon H_b(B_X)\to H_b(B_X)$ is uniformly mean ergodic. Note that $\alpha_a\colon B_H\to B_H$ (now H being a Hilbert space) satisfies this condition, so $C_{\alpha_a}\colon H_b(B_H)\to H_b(B_H)$ is uniformly mean ergodic. However, α_a does not satisfy neither (3.7) nor (3.11).

3.6 The space of bounded holomorphic functions

We consider now the space $H^\infty(B_X)$ of all holomorphic functions $f\colon B_X\to\mathbb{C}$ that are bounded. With the norm $\|f\|_\infty = \sup_{x\in B_X}|f(x)|$, it becomes a Banach space (recall Section 1.3). We look at composition operators $C_\varphi\colon H^\infty(B_X)\to H^\infty(B_X)$. If $\varphi\colon B_X\to B_X$, then

$$\|C_\varphi^n(f)\|_\infty = \sup_{x\in B_X}|C_\varphi^n(f)(x)| = \sup_{x\in B_X}|f(\varphi^n(x))| \le \sup_{x\in B_X}|f(x)| = \|f\|_\infty\,,$$

and $\|C_\varphi^n\|\le 1$ for all $n\in\mathbb{N}$. Hence every C_φ that is well defined on $H^\infty(B_X)$ is power bounded. Since $(X',\|\cdot\|) = (X',\tau_b)$, the dual space X' is also complemented in $H^\infty(B_X)$, and the same arguments as in Proposition 3.16 give examples of composition operators $C_\varphi\colon H^\infty(B_X)\to H^\infty(B_X)$ which are not mean ergodic. Let us emphasize that X' is in general not complemented in $H(B_X)$ since $(X',\|\cdot\|)\neq(X',\tau_0)$ and these arguments do not work for $H(B_X)$ (cf. Proposition 1.5).

We give now conditions on the symbol to get a uniformly mean ergodic composition operator on $H^\infty(B_X)$.

Proposition 3.28. *Let $\varphi\colon B_X\to B_X$ be holomorphic and such that $\varphi(B_X)\subseteq rB_X$ for some $0<r<1$ with $\varphi(0)=0$. Then $C_{\varphi^n}\to C_0$ in the norm operator topology of $\mathscr{L}(H^\infty(B_X))$. In particular, $C_\varphi\colon H^\infty(B_X)\to H^\infty(B_X)$ is uniformly mean ergodic.*

Proof. Take some $f \in H^\infty(B_X)$ with $\|f\|_\infty \le 1$. Defining $g\colon B_X \to \mathbb{D}$ by $g(x) = \frac{1}{2}(f(x) - f(0))$ and using Lemma 3.3 we get

$$|f(x) - f(0)| \le 2\|x\|,$$

for every $x \in B_X$. Proceeding as in (3.8), we get that $\|\varphi^n(x)\| \le r^n\|x\|$ for every $x \in B_X$ and $n \in \mathbb{N}$. This yields

$$\left|f\big(\varphi^n(x)\big) - f(0)\right| \le 2\|\varphi^n(x)\| \le 2r^n\|x\|.$$

Therefore

$$\|C_\varphi^n - C_0\|_{\mathscr{L}(H^\infty(B))} = \sup_{\|f\|_\infty \le 1} \sup_{x \in B_X} \left|f\big(\varphi^n(x)\big) - f(0)\right| \le 2 \sup_{x \in B_X} \|\varphi^n(x)\| \le 2r^n,$$

which gives the claim. □

We observe that the hypothesis in Proposition 3.28 is exactly the same one as (3.9) in Remark 3.20. One can ask if the result also holds assuming (3.7) instead . This is not the case. We already saw in Remark 3.20 that the mapping $P\colon B_{\ell_2} \to B_{\ell_2}$ given by $P\big((x_n)_n\big) = (x_n^2)_n$ satisfies (3.7). Then, by Theorem 3.19 the Cesàro means of C_P converge to C_0. And, $C_P\colon H_b(B_{\ell_2}) \to H_b(B_{\ell_2})$ is uniformly mean ergodic.

However, the operator $C_P\colon H^\infty(B_{\ell_2}) \to H^\infty(B_{\ell_2})$ is not even mean ergodic. If C_P were mean ergodic, then $\big((C_P)_{[n]}(f)\big)_n$ should converge (in $H^\infty(B_{\ell_2})$) for every f. But by what we have just seen, this limit should be $C_0(f) = f(0)$. In other words, $(C_P)_{[n]} \to C_0$ in the topology τ_s. Take $f \in H^\infty(B_{\ell_2})$ given by $f\big((x_n)_n\big) = x_1$ and consider $z_m = (1 - \frac{1}{m})e_1 \in B_{\ell_2}$ for each $m \in \mathbb{N}$. Then $P^k(z_m) = (1 - \frac{1}{m})^{2^k} e_1$ for every k and

$$(C_P)_{[n]}(f)(z_m) - C_0(f)(z_m) = \frac{1}{n}\sum_{k=0}^{n-1} f\big((1 - \frac{1}{m})^{2^k} e_1\big) - f(0) = \frac{1}{n}\sum_{k=0}^{n-1}\big(1 - \frac{1}{m}\big)^{2^k}.$$

Thus

$$\sup_{x \in B_{\ell_2}} |(C_P)_{[n]}(f)(x) - C_0(f)(x)| \ge \sup_{m \in \mathbb{N}} \frac{1}{n}\sum_{k=0}^{n-1}\big(1 - \frac{1}{m}\big)^{2^k} = 1,$$

and $\big((C_P)_{[n]}(f)\big)_n$ does not converge in norm to $C_0(f)$. This finally shows that $C_P\colon H^\infty(B_{\ell_2}) \to H^\infty(B_{\ell_2})$ is not mean ergodic.

The same argument as in Lemma 3.26 and Proposition 3.27 shows the following result. It can be seen as a partial extension of Theorem 1.20 to $H^\infty(B_H)$.

Proposition 3.29. *Let $\varphi\colon B_H \to B_H$ be holomorphic such that $\varphi(B_H) \subsetneq rB_H$ for some $0 < r < 1$. Then, for the unique $a \in B$ such that $\varphi(a) = a$, we have $C_{\varphi^n} \to C_a$ in the norm of $\mathscr{L}(H^\infty(B_H))$. In particular, $(C_\varphi)_{[n]} \to C_a$ in this topology, i.e. C_φ is uniformly mean ergodic.*

Open problems. The following questions have arisen in this chapter and remain open.

- We do not know so far whether having stable orbits implies having B_X-stable orbits.
- We do not know whether the symbol φ satisfies (3.11), taking B_X instead of B_H, then the composition operator C_φ is uniformly mean ergodic (that is, Proposition 3.27 extends to arbitrary Banach spaces).
- It would also be interesting to find examples of the following situations:
  - A composition operator on $H(B_X)$ which is mean ergodic but not uniformly mean ergodic.
  - A composition operator on $H(B_X)$ which is mean ergodic but not power bounded.
  - A composition operator on $H^\infty(B_X)$ which is mean ergodic but not uniformly mean ergodic.

Chapter 4

Compact weighted composition operators

4.1 Introduction

Compactness of weighted composition operators defined on spaces of functions of one variable has been extensively studied, and there is a huge related literature (see, for example, [10, 17, 18, 19, 30, 52] and the references therein). However, for spaces of holomorphic functions on infinite dimensional spaces the literature is much more scarce. Compactness of the composition operator C_φ defined on $H^\infty(B)$ was studied in [3] and on $H_b(B)$ in [31].

In this chapter, we are interested in the study of continuity and (weak) compactness of $C_{\psi,\varphi}\colon H^\infty(B) \to H^\infty(B)$, in terms of the properties of the *weight* ψ and the *symbol* φ. On the other hand, we also study when $C_{\psi,\varphi}\colon H_b(B) \to H_b(B)$ is continuous, bounded, reflexive, Montel and (weakly) compact, also in terms of different properties of ψ and φ.

4.2 Continuity

Our first aim is to know when the weighted composition operator $C_{\psi,\varphi}$ is well defined and continuous in different spaces of holomorphic functions. We begin by paying attention to the case of the space $H(B)$. In the rest of the chapter, we always assume that ψ is non-zero to avoid the trivial case.

Let $\varphi\colon B \to B$ be a holomorphic mapping and $\psi\colon B \to \mathbb{C}$ any function. If the operator $C_{\psi,\varphi}\colon H(B) \to H(B)$ is well defined, then $C_{\psi,\varphi}(f) \in H(B)$ for all $f \in H(B)$. In particular, for the constant function $1 \in H(B)$ we have

$$C_{\psi,\varphi}(1) = \psi \cdot (1 \circ \varphi) = \psi \in H(B). \tag{4.1}$$

Conversely, if we assume $\psi \in H(B)$ and take an arbitrary $f \in H(B)$ we obtain that

$\psi \cdot (f \circ \varphi) \in H(B)$. Then, the operator $C_{\psi,\varphi} : H(B) \to H(B)$ is well defined. Moreover, the operator is also continuous. Indeed, for any compact subset $K \subset B$ there is a constant $c > 0$ such that $\sup_{x\in K} |\psi(x)| \le c$. Since φ is continuous the set $L = \varphi(K)$ is compact in B. We obtain

$$\sup_{x\in K} |C_{\psi,\varphi}(f)(x)| = \sup_{x\in K} |\psi(x) f(\varphi(x))| \le \sup_{x\in K} |\psi(x)| \sup_{x\in K} |f(\varphi(x))| \le c \sup_{x\in L} |f(x)|,$$

for every $f \in H(B)$. $C_{\psi,\varphi} : H(B) \to H(B)$ is continuous. Thus, without loss of generality we can always assume that ψ and φ are holomorphic.

Proposition 4.1. *Assume ψ is non-zero. Then $C_{\psi,\varphi} : H_b(B) \to H_b(B)$ is continuous if and only if ψ and φ are of bounded type.*

Proof. Assume first that both ψ and φ are of bounded type. Given an arbitrary $0 < r < 1$ we can find $0 < s < 1$ as in (BTa) and $M > 0$ such that $\sup_{\|x\|<r} |\psi(x)| \le M$ (see (BTb)). Then, for an arbitrary $f \in H_b(B)$ we have

$$\sup_{\|x\|<r} |C_{\psi,\varphi}(f)(x)| = \sup_{\|x\|<r} |\psi(x)| \cdot |f(\varphi(x))| \le M \sup_{x\in\varphi(rB)} |f(x)| \le M \sup_{\|x\|<s} |f(x)|,$$

so $C_{\psi,\varphi}$ is continuous.

Conversely, assume $C_{\psi,\varphi}$ is continuous. Then, for each $0 < r < 1$, we can find $0 < s < 1$ and $c > 0$ such that

$$\sup_{\|x\|<r} |C_{\psi,\varphi}(f)(x)| = \sup_{\|x\|<r} |\psi(x)| \cdot |f(\varphi(x))| \le c \sup_{\|x\|<s} |f(x)|. \tag{4.2}$$

for every $f \in H_b(B)$. In particular, for $f = 1 \in H_b(B)$, we have $\sup_{\|x\|<r} |\psi(x)| \le c$. Hence, ψ is of bounded type. In order to see that φ is of bounded type, pick $u \in X'$ with $\|u\|_{X'} = 1$ and, for each $n \in \mathbb{N}$ consider $u^n \in H_b(B)$. For $\|x\| < r$ with $\psi(x) \neq 0$ we have, using (4.2),

$$|\psi(x)| \cdot |u(\varphi(x))|^n \le c \sup_{\|y\|<s} |u(y)|^n \le c s^n,$$

for every $n \in \mathbb{N}$. This clearly implies that $|u(\varphi(x))| \le s$, for $\|x\| < r$ with $\psi(x) \neq 0$. Since u is arbitrary, this implies $\|\varphi(x)\| \le s$ for every $\|x\| < r$ with $\varphi(x) \neq 0$. By the Identity Principle (see [47, Proposition 5.7]), the set $\{x \in rB : \psi(x) \neq 0\}$ is dense in rB, so $\|\varphi(x)\| \le s$ for every $\|x\| < r$. □

The case of $H^\infty(B)$ is much simpler to handle. Since it is an algebra which contains the constant function 1 and the composition of bounded functions is a bounded function, the following result is immediate.

Proposition 4.2. *Given $\varphi : B \to B$ a holomorphic mapping and $\psi \in H(B)$, the operator $C_{\psi,\varphi} : H^\infty(B) \to H^\infty(B)$ is continuous if and only if $\psi \in H^\infty(B)$.*

From what we have just seen we can easily deduce when a multiplication operator defined on any of these spaces is continuous. Note that the identity $\mathrm{id}_B\colon B \to B$ is a holomorphic mapping of bounded type. As we already pointed out, we can write $M_\psi = C_{\psi,\mathrm{id}_B}$. This allows us to obtain the following result.

Proposition 4.3. *Let $\mathscr{H}$ be either $H^\infty(B)$, $H_b(B)$ or $H(B)$. Then $M_\psi\colon \mathscr{H} \to \mathscr{H}$ is well defined and continuous if and only if $\psi \in \mathscr{H}$.*

4.3 Compactness on $H^\infty(B)$

Since $C_{\psi,\varphi} = M_\psi \circ C_\varphi$, if either M_ψ or C_φ is compact, then $C_{\psi,\varphi}$ is compact. We recall that $C_\varphi\colon H^\infty(B) \to H^\infty(B)$ is compact if and only if there is $0 < s < 1$ such that $\varphi(B) \subseteq sB$ and the set $\varphi(B)$ is relatively compact in X (Theorem 1.21).

We consider the point evaluation functional $\delta_x\colon H^\infty(B) \to \mathbb{C}$ on $H^\infty(B)$, defined as $\delta_x(f) = f(x)$ for $x \in B$. It belongs to the dual space $H^\infty(B)'$ and, moreover, $\|\delta_x\|_{H^\infty(B)'} = 1$ for every $x \in B$ since $|\delta_x(f)| \le \|f\|_\infty$ for every $f \in H^\infty(B)$ and the constant mapping 1 is in $H^\infty(B)$.

Given a Banach space Y and $M \subseteq Y$ a bounded and absolutely convex subset, we write $\|\cdot\|_M$ for the *Minkowski gauge* defined by M as

$$\|x\|_M := \inf\{\lambda > 0 : x \in \lambda M\}.$$

We denote Y_M for the normed space $(Y, \|\cdot\|_M)$. More details can be found in, e.g. [49, Chapter 3.2]. As an immediate consequence of Grothendieck's Precompactness Lemma [41, §21, 7(1)] we have:

Lemma 4.4. *Let Y be a Banach space and $M \subset Y$, $N \subset Y'$ be bounded closed absolutely convex sets. Then M° is precompact in $(Y')_N$ if and only if N° is precompact in Y_M.*

Lemma 4.5. *The operator $C_{\psi,\varphi}\colon H^\infty(B) \to H^\infty(B)$ is compact if and only if the set*

$$\{\psi(x)\delta_{\varphi(x)} : x \in B\}$$

is relatively compact in $H^\infty(B)'$.

Proof. Suppose first that $C_{\psi,\varphi}$ is compact. Then its adjoint $C^t_{\psi,\varphi}\colon H^\infty(B)' \to H^\infty(B)'$ is compact, and therefore the set $C^t_{\psi,\varphi}(\{\delta_x : x \in B\})$ is relatively compact in $H^\infty(B)'$. For $f \in H^\infty(B)$ we have

$$\langle C^t_{\psi,\varphi}(\delta_x), f\rangle = \langle \delta_x, C_{\psi,\varphi}(f)\rangle = \psi(x) f(\varphi(x)) = \psi(x)\langle \delta_{\varphi(x)}, f\rangle = \langle \psi(x)\delta_{\varphi(x)}, f\rangle.$$

Hence $C^t_{\psi,\varphi}(\{\delta_x : x \in B\}) = \{\psi(x)\delta_{\varphi(x)} : x \in B\}$ is relatively compact in $H^\infty(B)'$.

Let us assume now that $A = \{\psi(x)\delta_{\varphi(x)} : x \in B\}$ is relatively compact in $H^\infty(B)'$. Then $N = \overline{\mathrm{co}}(A) = A^{\circ\circ}$ is compact in $H^\infty(B)'$ by the Bipolar Theorem. Note that taking $M = \overline{B_{H^\infty(B)'}}$ (the closed unit ball of $H^\infty(B)'$) we obviously have $H^\infty(B)' = (H^\infty(B)')_M$. Observe that

$$N^\circ = A^\circ = \{f \in H^\infty(B) : |\psi(x) f(\varphi(x))| \leq 1 \text{ for every } x \in B\}.$$

In other words, we have that $(N^\circ)^\circ$ is compact in $(H^\infty(B)')_M$. By Lemma 4.4 the set M° is precompact in $(H^\infty(B))_{N^\circ} = (H^\infty(B))_{A^\circ}$. Note that $M^\circ = \overline{B_{H^\infty(B)}}$. Then, for each sequence $(f_n)_n \subset \overline{B_{H^\infty(B)}}$, there is a subsequence $(f_{n_k})_k$ and $f_0 \in \overline{B_{H^\infty(B)}}$ such that $f_{n_k} \to f_0$ in $(H^\infty(B))_{A^\circ}$. That is, for each $\varepsilon > 0$ there is k_0 such that for every $k \geq k_0$ we have

$$f_{n_k} - f_0 \in \varepsilon N^\circ.$$

Then for each $\varepsilon > 0$ there is k_0 such that for every $k \geq k_0$ we have

$$\sup_{x \in B} |\langle f_{n_k} - f_0, \psi(x)\delta_{\varphi(x)}\rangle| < \varepsilon,$$

or equivalently,

$$\|C_{\psi,\varphi}(f_{n_k}) - C_{\psi,\varphi}(f_0)\|_\infty < \varepsilon.$$

Therefore $C_{\psi,\varphi} : H^\infty(B) \to H^\infty(B)$ is compact. □

From this formal result we can deduce an important necessary condition. This will be used later to characterise the compactness of $C_{\psi,\varphi}$ in terms of relatively compact sets in X (see Theorem 4.8).

Proposition 4.6. *Let $C_{\psi,\varphi} : H^\infty(B) \to H^\infty(B)$ be compact. Then $(\psi \cdot \varphi)(B)$ is relatively compact in X.*

Proof. Assume $(\psi \cdot \varphi)(B)$ is not relatively compact in X. Then there is $\varepsilon > 0$ and $(x_n)_n \subseteq B$ such that

$$\|\psi(x_n)\varphi(x_n) - \psi(x_m)\varphi(x_m)\| > \varepsilon,$$

for every $n \neq m$. By the Hahn-Banach Theorem there is $u_{n,m} \in X'$ with $\|u_{n,m}\|_{X'} \leq 1$ such that

$$\left|u_{n,m}\left(\psi(x_n)\varphi(x_n) - \psi(x_m)\varphi(x_m)\right)\right| > \varepsilon, \tag{4.3}$$

for every $n \neq m$. On the other hand, by Lemma 4.5 $\{\psi(x)\delta_{\varphi(x)} : x \in B\}$ is relatively compact in $H^\infty(B)'$. Hence we can find a subsequence $(n_k)_k$ such that

$$\begin{aligned}
\frac{\varepsilon}{2} &> \sup_{\|f\|_\infty \leq 1} \left|\langle f, \psi(x_{n_k})\delta_{\varphi(x_{n_k})} - \psi(x_{n_j})\delta_{\varphi(x_{n_j})}\rangle\right| \\
&\geq \sup_{u \in (u_{n,m})_{n\neq m}} \left|\psi(x_{n_k}) u(\varphi(x_{n_k})) - \psi(x_{n_j}) u(\varphi(x_{n_j}))\right| \\
&= \sup_{u \in (u_{n,m})_{n\neq m}} \left|u\left(\psi(x_{n_k})\varphi(x_{n_k}) - \psi(x_{n_j})\varphi(x_{n_j})\right)\right|,
\end{aligned}$$

for every $k, j \in \mathbb{N}$. Since we can choose k, j such that $n_k \neq n_j$, we obtain a contradiction with (4.3) and this completes the proof. □

Our last step in our way to a characterisation of compactness is the following lemma.

Lemma 4.7. *Let $\psi \in H^\infty(B)$, $\psi \neq 0$, and $\varphi\colon B \to B$ be holomorphic. Assume $(\psi \cdot \varphi)(B)$ is relatively compact in X and that one of the following conditions holds:*

a) There is $0 < s < 1$ such that $\varphi(B) \subseteq sB$,

b) $\lim_{r\to 1^-} \sup_{\|\varphi(x)\|>r} |\psi(x)| = 0$.

Then, for each sequence $(x_n)_n$ such that $(\varphi(x_n))_n$ is not relatively compact in B, there is a subsequence $(x_{n_k})_k$ such that

$$\lim_{k\to\infty} |\psi(x_{n_k})| = 0.$$

Proof. We begin by fixing some $(x_n)_n$ such that $(\varphi(x_n))_n$ is not relatively compact in B. We find two different situations. Assume there is $(x_{n_k})_k$ such that $\lim_{k\to\infty} \|\varphi(x_{n_k})\| = 1$. Then, by condition b), we obtain $\lim_k |\psi(x_{n_k})| = 0$. Otherwise, we can find $0 < r_0 < 1$ such that $(\varphi(x_n))_n \subseteq r_0 B$. Thus the set $(\varphi(x_n))_n$ is neither relatively compact in X. We claim that the set

$$A_m := \varphi\Big(\big\{x \in B \,:\, |\psi(x)| > \frac{1}{m}\big\}\Big)$$

is relatively compact in X for each $m \in \mathbb{N}$. Indeed, for a fixed $m \in \mathbb{N}$, if $\varphi(x) \in A_m$ we have that $\frac{1}{\|\psi\|_\infty} \leq \frac{1}{|\psi(x)|} < m$. For $(\varphi(y_n))_n \subseteq A_m$, we can find a subsequence, $z \in \mathbb{C}$ and $y \in X$ such that $\lim_k \frac{1}{\psi(y_{n_k})} = z$ and $\lim_k \psi(y_{n_k})\varphi(y_{n_k}) = y$. Therefore, we obtain

$$\varphi(y_{n_k}) = \frac{1}{\psi(y_{n_k})} \cdot \psi(y_{n_k})\varphi(y_{n_k}) \to zy \in X,$$

and the claim holds.

Now, since the set $(\varphi(x_n))_n$ is not relatively compact in X we can find $n_1 \in \mathbb{N}$ such that $\varphi(x_{n_1}) \notin A_1$. Therefore $|\psi(x_{n_1})| \leq 1$. Now, the set $(\varphi(x_n))_{n>n_1}$ is not relatively compact in X and we can find $n_2 > n_1$ such that $\varphi(x_{n_2}) \notin A_2$. This element satisfies that $|\psi(x_{n_2})| \leq \frac{1}{2}$. Repeating this procedure we obtain a subsequence such that $|\psi(x_{n_k})| \leq 1/k$, for every $k \in \mathbb{N}$, which finishes the proof. □

We can finally give the characterisation of the compactness of the weighted composition operator on $H^\infty(B)$ in terms of properties of the symbol and the weight we were aiming at. This extends, at the same time, Theorem 1.21 to weighted composition operators and Theorem 1.23 to $H^\infty(B)$.

Theorem 4.8. *Let $\psi \in H^\infty(B)$ and $\varphi\colon B \to B$ be holomorphic. Then the following conditions are equivalent:*

a) $C_{\psi,\varphi}: H^\infty(B) \to H^\infty(B)$ *is compact,*

b) $C_{\psi,\varphi}: H^\infty(B) \to H^\infty(B)$ *is weakly compact and* $(\psi \cdot \varphi)(B)$ *is relatively compact in* X*,*

c) $(\psi \cdot \varphi)(B)$ *is relatively compact in* X *and one of the following properties hold:*

(i) There is $0 < s < 1$ *such that* $\varphi(B) \subseteq sB$*,*

(ii) $\lim_{r\to 1^-} \sup_{\|\varphi(x)\|>r} |\psi(x)| = 0.$

Proof. a)⇒b) follows immediately from Proposition 4.6 and the fact that every compact operator is weakly compact.

To see that b)⇒c), assume that neither (i) nor (ii) hold. Then there is a sequence $(x_j)_j \subset B$ such that $\|\varphi(x_j)\| \geq 1 - 1/j$ for all j and $(\psi(x_j))_j$ converges to some $z \in \mathbb{C} \setminus \{0\}$. Without loss of generality we may assume $z = 1$ and, since $(\psi \cdot \varphi)(B)$ is relatively compact in X, that $(\psi(x_j)\varphi(x_j))_j$ converges to some $y_0 \in X$ with $\|y_0\| = 1$. Note that, in fact, $(\varphi(x_j))_j$ converges to y_0. Choose $u \in X'$ such that $u(y_0) = 1 = \|u\|$. The set $\{u^n|_B : n \in \mathbb{N}\}$ is bounded in $H^\infty(B)$ and $C_{\psi,\varphi}$ is weakly compact, so we may assume, passing to a subsequence if necessary, that $\psi(u^n \circ \varphi)$ converges weakly to some $f \in H^\infty(B)$. Since for every j, the point evaluation map $\delta_{x_j} \in H^\infty(B)'$ has norm equal to 1, by Alaoglu's theorem, the sequence $(\delta_{x_j})_j$ has a $\sigma(H^\infty(B)', H^\infty(B))$-cluster point $\gamma \in H^\infty(B)'$. Passing to a subsequence if necessary, for every n, we have $1 = u(y_0)^n = \lim_j \psi(x_j)u(\varphi(x_j))^n = \lim_j \delta_{x_j}(\psi(u^n \circ \varphi)) = \gamma(\psi(u^n \circ \varphi))$. Now, since $\|\varphi(x_j)\| < 1$ for every j, $0 = \lim_n \psi(x_j)u^n(\varphi(x_j)) = f(x_j) = \delta_{x_j}(f)$. Hence

$$0 = \lim_j \delta_{x_j}(f) = \gamma(f) = \lim_n \gamma(\psi(u^n \circ \varphi)) = 1,$$

and we have a contradiction. This proves our claim.

To show that c)⇒a), assume that $C_{\psi,\varphi}$ is not a compact operator. Then there is a sequence $(f_n)_n \subset H^\infty(B)$ with $\|f_n\|_\infty \leq 1$ and there is $\varepsilon > 0$ so that

$$\|\psi \cdot (f_n \circ \varphi) - \psi \cdot (f_m \circ \varphi)\|_\infty > 2\varepsilon,$$

for every $n < m$. We can select a set $\{x_{n,m} : n < m\} \subseteq B$ satisfying

$$\left|\psi(x_{n,m})f_n(\varphi(x_{n,m})) - \psi(x_{n,m})f_m(\varphi(x_{n,m}))\right| > \varepsilon, \quad \text{for every} \quad n < m. \tag{4.4}$$

We claim that the set $(\varphi(x_{n,m}))_{n<m}$ is not relatively compact in B. Assume this is not the case. Since $(f_n)_n$ is bounded in $H^\infty(B)$ it is also τ_0-bounded and, therefore τ_0-relatively compact. In particular, there is a subnet $(f_\alpha)_\alpha$ that is τ_0-Cauchy. Then we can find α_0 such that

$$\sup_{y\in(\varphi(x_{n,m}))_{n<m}} |f_\alpha(y) - f_\beta(y)| < \frac{\varepsilon}{2\|\psi\|_\infty} \quad \text{for every} \quad \alpha, \beta > \alpha_0.$$

We can take $\alpha, \beta > \alpha_0$ such that $f_\alpha = f_{n_0}$ and $f_\beta = f_{m_0}$ with $n_0 < m_0$. This implies

$$\frac{\varepsilon}{2} > \|\psi\|_\infty \sup_{y\in(\varphi(x_{n,m}))_{n<m}} |f_{n_0}(y) - f_{m_0}(y)| \geq |\psi(x_{n_0,m_0})| \cdot |f_{n_0}(\varphi(x_{n_0,m_0})) - f_{m_0}(\varphi(x_{n_0,m_0}))|.$$

This contradicts (4.4), so the set $(\varphi(x_{n,m}))_{n<m}$ is not relatively compact in B.

Now, we take any bijection $\Phi\colon \{(n,m) \in \mathbb{N}\times\mathbb{N} : n < m\} \to \mathbb{N}$ and we denote $x_i = x_{n,m}$, where $n < m$ and $i = \Phi(n,m)$. Then the sequence $(\varphi(x_i))_i$ is not relatively compact in B. By Lemma 4.7 there is a subsequence $(x_{i_k})_k$ such that

$$\lim_{k\to\infty} |\psi(x_{i_k})| = 0.$$

In particular, there is $i_{k_0} \in \mathbb{N}$ so that $|\psi(x_{i_{k_0}})| < \frac{\varepsilon}{4}$. For the unique pair $n_1 < m_1$ such that $\Phi(n_1, m_1) = i_{k_0}$, we obtain, from (4.4),

$$\begin{aligned}\varepsilon <& \left|\psi(x_{n_1,m_1}) f_{n_1}(\varphi(x_{n_1,m_1})) - \psi(x_{n_1,m_1}) f_{m_1}(\varphi(x_{n_1,m_1}))\right| \\ \leq& |\psi(x_{i_{k_0}})| \cdot (\|f_{n_1}\|_\infty + \|f_{m_1}\|_\infty) \leq |\psi(x_{i_{k_0}})| \cdot 2 < \frac{\varepsilon}{2},\end{aligned}$$

which is a contradiction. □

4.4 Compactness on $H_b(B)$

We now look for conditions for a weighted composition operator on $H_b(B)$ to be compact. To this aim, we begin by characterising the boundedness of the operator (see Proposition 4.11 bellow).

First, we observe that the sets

$$U_r := \{f \in H_b(B) : \sup_{\|x\|<r} |f(x)| \leq 1\},$$

for $0 < r < 1$, build a basis of 0-neighbourhoods of $H_b(B)$ and then $C_{\psi,\varphi}\colon H_b(B) \to H_b(B)$ is bounded if and only if there is $0 < s < 1$ such that for every $0 < r < 1$ there is $M_r > 0$ with

$$\sup_{\|x\|<r} |\psi(x) f(\varphi(x))| \leq M_r, \quad \text{for every } f \in U_s. \tag{4.5}$$

Theorem 4.9. *Let $\varphi\colon B \to B$ be holomorphic of bounded type. The following are equivalent:*

a) There is $0 < s < 1$ such that $\varphi(B) \subseteq sB$.

b) $C_\varphi\colon H_b(B) \to H_b(B)$ is bounded.

c) $C_{\psi,\varphi}\colon H_b(B) \to H_b(B)$ is bounded for some $\psi \in H_b(B)$, $\psi \neq 0$.

d) $C_{\psi,\varphi} \colon H_b(B) \to H_b(B)$ *is bounded for every* $\psi \in H_b(B)$*.*

Proof. Clearly d)⇒b)⇒c).

Let us see that a)⇒d). Take an arbitrary $\psi \in H_b(B)$, $\psi \neq 0$. By Proposition 4.1, the operator $C_{\psi,\varphi}$ is continuous. For $0 < r < 1$ we can take $M_r = \sup_{\|x\|<r} |\psi(x)|$ to have, for $f \in U_s$,

$$\sup_{\|x\|<r} |\psi(x) f(\varphi(x))| \leq \sup_{\|x\|<r} |\psi(x)| \sup_{x \in \varphi(rB)} |f(x)| \leq M_r \sup_{\|x\|<s} |f(x)| \leq M_r.$$

So $C_{\psi,\varphi}$ is bounded.

To see c)⇒a) we follow some of the ideas in [31, Theorem 2.1]. Suppose that there is some $\psi \in H_b(B)$ such that $C_{\psi,\varphi}$ is bounded. Choose $0 < s < 1$ satisfying (4.5) and let us see that $|f(\varphi(x))| \leq 1$ for every $f \in U_s$ and $\|x\| < 1$. First of all observe that for $n \in \mathbb{N}$ and $f \in U_s$ the function f^n is again in U_s. Then, if $\|x_0\| < 1$ is such that $\psi(x_0) \neq 0$ we can take $\|x_0\| < r < 1$ and, using (4.5), find $M_r > 0$ such that $|\psi(x_0) f(\varphi(x_0))^n| \leq M_r$, for every $n \in \mathbb{N}$. Therefore, since x_0 is arbitrary, $|f(\varphi(x))| \leq 1$ for $\|x\| < 1$ with $\psi(x) \neq 0$. Suppose now that $|f(\varphi(x_0))| > 1$ for some x_0. By continuity $|f \circ \varphi| > 1$ on some open set, and this (by what we just have seen) forces $\psi = 0$ on this open set. By the Identity Principle for holomorphic functions on Banach spaces (see e.g. [47, Proposition 5.7]) this forces ψ to be identically zero, but this is not the case. Hence $\sup_{\|x\|<1} |f(\varphi(x))| \leq 1$, for every $f \in U_s$.
Let us see that this implies $\varphi(B) \subseteq \frac{s+1}{2} B$, which will complete the proof. Suppose that this is not the case and pick $\|x_0\| < 1$ with $\|\varphi(x_0)\| > \frac{s+1}{2}$. In particular $\varphi(x_0) \notin \overline{sB}$ and, by the Hahn-Banach Theorem, we can find $u \in X'$ with $|u(\varphi(x_0))| > 1$ and $|u(x)| \leq 1$ for every $x \in \overline{sB}$. Observe that $u \in U_s$. This is a contradiction and completes the proof. □

Suppose now that B_N is the open unit ball of $\mathbb{C}^N$ with any norm. Note that in this case trivially $H(B_N) = H_b(B_N)$ and, by Montel's theorem for several variables $C_{\psi,\varphi} \colon H(B_N) \to H(B_N)$ is compact if and only if it is bounded. This altogether yields the following known consequence.

Corollary 4.10. *Let* B_N *be the open unit ball of* $\mathbb{C}^N$ *with some norm. Let* $\varphi \colon B_N \to B_N$ *be holomorphic. The following are equivalent:*

a) There is $0 < s < 1$ *such that* $\varphi(B_N) \subseteq sB_N$*.*

b) $C_\varphi \colon H(B_N) \to H(B_N)$ *is compact.*

c) $C_{\psi,\varphi} \colon H(B_N) \to H(B_N)$ *is compact for some* $\psi \in H(B_N)$, $\psi \neq 0$.

d) $C_{\psi,\varphi} \colon H(B_N) \to H(B_N)$ *is compact for every* $\psi \in H(B_N)$*.*

We mention the following property (see [12, Remark (2)] and [31, Lemma 2.6]):

Proposition 4.11. *Let E be a quasinormable Fréchet space. If $T: E \to E$ is a bounded linear operator which is also Montel (reflexive), then T is compact (weakly compact).*

As an immediate consequence we obtain:

Corollary 4.12. *Let $0 \neq \psi \in H_b(B)$ and $\varphi: B \to B$ be holomorphic of bounded type. Then $C_{\psi,\varphi}: H_b(B) \to H_b(B)$ is compact (weakly compact) if and only if the following two conditions hold:*

a) $C_{\psi,\varphi}$ *is Montel (reflexive),*

b) there is $0 < s < 1$ *such that* $\varphi(B) \subseteq sB$.

We find now conditions that ensure that $C_{\psi,\varphi}$ is Montel. Let (E, τ) be a Hausdorff topological space and let $(x_i)_i \subset E$ be a sequence. We recall that that $x \in E$ is a *cluster point* of $(x_i)_i$ if for every $V \subset E$ τ-neighbourhood of x and $i \in \mathbb{N}$ there is $j > i$ such that $x_j \in V$. We denote the set of cluster points of $(x_i)_i$ in the topology τ by $\text{cluster}_\tau(x_i)_i$.

Remark 4.13. Let (E, τ) and (F, σ) be two Hausdorff topological spaces and let $T: (E, \tau) \to (F, \sigma)$ be a continuous linear operator. Take $(x_i)_i \subset E$ a sequence such that $x \in \text{cluster}_\tau(x_i)_i$. Now, fix $i \in \mathbb{N}$ and $U \subset F$ σ-neighbourhood of Tx. Since T is continuous in x we can find a τ-neighbourhood $V \subset E$ of x such that $T(V) \subset U$. We can find $j > i$ such that $x_j \in V$ and we obtain that $Tx_j \in T(V) \subset U$. Since i and U were arbitrary, this gives $Tx \in \text{cluster}_\sigma(Tx_i)_i$.

Lemma 4.14. *Let $T: H_b(B) \to H_b(B)$ be a continuous linear operator such that it is also (τ_0, τ_0)-continuous. Consider the following statements:*

a) $T: H_b(B) \to H_b(B)$ *is Montel,*

b) Every bounded sequence $(f_i)_i \subset H_b(B)$ *with* $0 \in \text{cluster}_{\tau_0}(f_i)_i$ *has a subsequence such that* $Tf_{i_k} \to 0$ *in* $H_b(B)$,

c) If $(f_i)_i \subset H_b(B)$ *is bounded and* $f_i \overset{\tau_0}{\to} 0$, *then* $Tf_i \to 0$ *in* $H_b(B)$.

Then a)⇔b)⇒c). If moreover every compact set in $H(B)$ is sequentially compact, then c) is equivalent to both a) and b).

Proof. Let us see first that a)⇒b). Choose some bounded sequence $(f_i)_i \subset H_b(B)$ such that $0 \in \text{cluster}_{\tau_0}(f_i)_i$. Then $0 = T0 \in \text{cluster}_{\tau_0}(Tf_i)_i$ because T is (τ_0, τ_0)-continuous (Remark 4.13). Now, being T Montel, the set $A := \overline{(Tf_i)_i}$ is compact in $H_b(B)$ and then, since τ_0 is a coarser Hausdorff topology, [41, 1. §3 (6), p. 8] gives that $\tau_b|_A = \tau_0|_A$. Therefore, A is also τ_0-compact and metrizable. Then, since $0 \in A$ (because A is τ_0-closed), we can find a subsequence so that $Tf_{i_k} \to 0$ in τ_0. This gives the claim.

Suppose now that b) holds and let us see that T is Montel. Take $C \subseteq H_b(B)$ bounded and let us see that $T(C)$ is relatively compact. To see this, we pick $(f_i)_i \subseteq C$

and we have to find a subsequence so that $(Tf_{i_k})_k$ converges in $H_b(B)$. Let us note first that C is also τ_0-bounded and, being $H(B)$ a semi-Montel space, it is also relatively τ_0-compact, and $\mathrm{cluster}_{\tau_0}(f_i)_i$ is non-empty. Take $f \in \mathrm{cluster}_{\tau_0}(f_i)_i$ and let us see that $f \in H_b(B)$. Since C is bounded, for each fixed $0 < r < 1$ we can find $M_r > 0$ such that $\sup_{\|x\|<r} |f_i(x)| \leq M_r$ for every $i \in \mathbb{N}$. For any $f \in \mathrm{cluster}_{\tau_0}(f_i)_i$ we have $\sup_{\|x\|<r} |f(x)| \leq M_r$ (since τ_0-convergence implies pointwise convergence), and f is of bounded type. Finally, the sequence $(f_i - f)_i$ is bounded and $0 \in \mathrm{cluster}_{\tau_0}(f_i - f)_i$. Then, by hypothesis there is a subsequence such that $T(f_{i_k} - f) \to 0$ in $H_b(B)$. Then $(Tf_{i_k})_k$ converges to $Tf \in H_b(B)$, which completes the proof.

This shows the equivalence between a) and b). We see now that a)⇒c). Let us assume first that T is Montel and take a bounded sequence $(f_i)_i \subset H_b(B)$ such that $f_i \stackrel{\tau_0}{\to} 0$. Since T is (τ_0, τ_0)-continuous we have

$$Tf_i \to 0 \text{ in } \tau_0. \tag{4.6}$$

Let us see that it also converges to 0 in τ_b. Suppose that this is not the case. Then, (since $H_b(B)$ is metrizable) we can find a subsequence and a 0-neighbourhood $U \subseteq H_b(B)$ such that

$$Tf_{i_k} \notin U \text{ for every } k \in \mathbb{N}. \tag{4.7}$$

Since $(f_{i_k})_k$ is bounded, $(Tf_{i_k})_k$ is relatively compact in $H_b(B)$, and there is a further subsequence $(Tf_{i_{k_l}})_l$ that converges in $H_b(B)$. By (4.6) it necessarily converges to 0. But then we can find $l_0 \in \mathbb{N}$ such that $Tf_{i_{k_l}} \in U$ for every $l > l_0$. This contradicts (4.7) and yields the claim.

Now we assume that c) holds and that every compact set in $H(B)$ is sequentially compact. Since $H_b(B)$ is metrizable, in order to see that T is Montel it is enough to show that for every bounded sequence $(f_i)_i \subset H_b(B)$ there is a subsequence satisfying that $(Tf_{i_k})_k$ converges in $H_b(B)$. But $(f_i)_i$ is τ_b-bounded, so also τ_0-bounded and, by Montel's theorem, it is also relatively τ_0-compact. By our assumption on the space, $(f_i)_i$ is relatively sequentially τ_0-compact, and there is a subsequence $(f_{i_k})_k$ and $f \in H(B)$ with $f_{i_k} \to f$ in τ_0. For each $0 < r < 1$ we can find $M_r > 0$ such that $\sup_{\|x\|<r} |f_{i_k}(x)| \leq M_r$ for every $k \in \mathbb{N}$, therefore f is of bounded type. Now, it is enough to apply c) to the sequence $(f_{i_k} - f)_k$. □

The question of whether the assumption that every compact set in $H(B)$ is sequentially compact is needed remains open. It is easy to see that this condition is satisfied whenever the Banach space X is separable. Indeed, take $(x_i)_i$ dense in B and consider the seminorms on $H(B)$ defined by

$$p_n(f) := \sup_{1 \leq i \leq n} |f(x_i)|.$$

If τ denotes the topology induced by these seminorms, it is clear that τ is coarser than τ_0. If $f \in H(B)$ is such that $f(x_i) = 0$ for all $i \in \mathbb{N}$, the continuity of f implies that

$f \equiv 0$. This shows that $(H(B),\tau)$ is Hausdorff. Now given a τ_0-compact set $C \subset H(B)$ we have that $\tau|_C = \tau_0|_C$ (see [41, 1. §3 (6), p. 8]). Thus τ_0 is metrizable on C and therefore $C \subset H(B)$ is sequentially compact.

We mention that in [16, Corollary 1.1] the authors use more general conditions to ensure that every compact set in $H(B)$ is sequentially compact. This property is not satisfied by every Banach space, as the following example shows.

Example 4.15. From [27, Exercise 3.110] we know that $\overline{B_{\ell'_\infty}}$ (the closed unit ball of ℓ'_∞) is not sequentially $\sigma(\ell'_\infty,\ell_\infty)$-compact. From this we can deduce that there are compact sets in $H(B_{\ell_\infty})$ that are not sequentially compact.

To see this from a general point of view we take any Banach space X. By Banach-Alaoglu $\overline{B_{X'}}$ is a $\sigma(X',X)$-compact set. Denote by τ_k the topology in X' of uniform convergence on compact sets of X. Since $\overline{B_{X'}}$ is equicontinuous it is also τ_k-compact (see [41, Section 21,6.(3)]). Consider now the mapping $\Phi\colon (X',\tau_k) \to (H(B),\tau_0)$ given by $\Phi(u) = u|_B$ which is linear and continuous. Then $\Phi(\overline{B_{X'}}) \subset H(B)$ is τ_0-compact and $\Phi\colon \overline{B_{X'}} \to \Phi(\overline{B_{X'}})$ is an homeomorphism. As a consequence, if every τ_0-compact set in $H(B)$ is sequentially τ_0-compact we immediately obtain that $\overline{B_{X'}}$ is sequentially $\sigma(X',X)$-compact.

Since, as we have just pointed out, this is not the case for $\overline{B_{\ell'_\infty}}$, we deduce that in $H(B_{\ell_\infty})$ there are τ_0-compact sets that are not sequentially τ_0-compact.

The following result is a formal characterisation for $C_{\psi,\varphi}$ to be Montel in $H_b(B)$, and also gives a necessary condition for such an operator to be reflexive. It can be seen as an analogue of Lemma 4.5.

Proposition 4.16. *Assume that $C_{\psi,\varphi}\colon H_b(B) \to H_b(B)$ is continuous. If $C_{\psi,\varphi}$ is (reflexive) Montel, then for each $0 < r < 1$ the set $A_r := \{\psi(x)\delta_{\varphi(x)} : \|x\| \le r\}$ is (weakly) relatively compact in $H_b(B)'$. Conversely, if A_r is relatively compact in $H_b(B)'$ for each $0 < r < 1$, then $C_{\psi,\varphi}$ is Montel.*

Proof. By [22, Corollary 2.4] and [21, Theorem 1.2], $C_{\psi,\varphi}$ is (reflexive) Montel if and only if its transpose $C^t_{\psi,\varphi}\colon H_b(B)'_\beta \to H_b(B)'_\beta$ is (reflexive) Montel. Then $C^t_{\psi,\varphi}$ maps bounded sets into (weakly) relatively compact sets. For a fixed $0 < r < 1$ observe that $\{\delta_x : \|x\| \le r\} \subset H_b(B)'$ is an equicontinuous set. Indeed, since $\langle \delta_z, f\rangle = f(z)$ the following is satisfied

$$\{\delta_x : \|x\| \le r\} \subseteq \Big\{f \in H_b(B) : \sup_{\|x\|\le r} |f(x)| \le 1\Big\}^\circ.$$

Hence it is bounded. Therefore the set

$$C^t_{\psi,\varphi}(\{\delta_x : \|x\| \le r\}) = \{\psi(x)\delta_{\varphi(x)} : \|x\| \le r\}$$

is (weakly) relatively compact.

Conversely, assume that for each $0 < r < 1$ the set A_r is relatively $\beta(H_b(B)', H_b(B))$-compact. We start fixing a bounded sequence $(f_i)_i \subset H_b(B)$ such that $0 \in \text{cluster}_{\tau_0}(f_i)_i$. Since $(f_i)_i$ is bounded, if we consider each f_i as an element in the dual of $H_b(B)'$, we have that $(f_i)_i$ is equicontinuous on $H_b(B)'$. In particular, $(f_i)_i$ is equicontinuous on the compact set $K_r := \overline{A_r}$ for each $0 < r < 1$. We also have that $(f_i)_i \subset \mathscr{C}(K_r)$, the space of continuous functions on K_r, is a pointwise bounded set. Since the compact sets in the dual of a Fréchet space are metrizable (see [16, Theorem 2 and Examples 1.2 C)]) we can use the Ascoli-Arzelà Theorem (see [53, Corollary 3.146]) to obtain that $(f_i)_i$ is relatively compact in $\mathscr{C}(K_r)$ with the natural norm topology. Additionally, since $0 \in \text{cluster}_{\tau_0}(f_i)_i$ we have that 0 is a cluster point of $(f_i)_i$ for the topology of pointwise convergence on $\mathscr{C}(K_r)$ and therefore, 0 is a cluster point for $(f_i)_i$ in the topology of uniform convergence on $\mathscr{C}(K_r)$. Let $(r(k)) \subset (0,1)$ be a strictly increasing sequence such that $\lim_k r(k) = 1$. For $k = 1$ we obtain a subsequence converging to 0 uniformly in $K_{r(1)}$. Then we can find i_1 such that $\sup_{u \in K_{r(1)}} |f_{i_1}(u)| < 1$. Now for $k = 2$ we obtain a subsequence of the subsequence above, that we denote the same, converging to 0 uniformly in $K_{r(2)}$. Then we can find $i_2 > i_1$ such that $\sup_{u \in K_{r(2)}} |f_{i_2}(u)| < 1/2$. Hence iteratively, for each $k \in \mathbb{N}$, we can find $i_k \in \mathbb{N}$ such that $\sup_{u \in K_{r(k)}} |f_{i_k}(u)| < 1/k$ and satisfies $i_k > i_{k-1}$. Now, for a fixed $0 < r < 1$ there is $k_0 \in \mathbb{N}$ such that $r(k_0) > r$ and

$$\sup_{\|x\| \leq r} |\psi(x) f_{i_k}(\varphi(x))| \leq \sup_{u \in K_r} |f_{i_k}(u)| \leq \sup_{u \in K_{r(k)}} |f_{i_k}(u)| < \frac{1}{k},$$

for every $k > k_0$. Hence $C_{\psi,\varphi}(f_{i_k}) \to 0$ in $H_b(B)$. An application of Lemma 4.14 completes the proof. □

In what remains we are going to use the Eberlein-Šmulian theorem several times. Let us recall the statement (see [23, Chapter 3 p. 18] for a proof).

Theorem 4.17 (Eberlein-Šmulian). *A subset of a Banach space is weakly relatively compact if and only if it is weakly relatively sequentially compact.*

In particular, a subset of a Banach space is weakly compact if and only if it is weakly sequentially compact.

Lemma 4.18. *Let $\psi \in H_b(B)$ and let $\varphi : B \to B$ be holomorphic of bounded type. If there is a B-bounded set D such that $A_D := \{\psi(x)\delta_{\varphi(x)} : x \in D\}$ is (weakly) relatively compact in $H_b(B)'$, then the set $(\psi \cdot \varphi)(D)$ is (weakly) relatively compact in X.*

Proof. a) First assume that A_D is relatively compact in $H_b(B)'$ but $(\psi \cdot \varphi)(D)$ is not relatively compact in X. Then there are $\varepsilon > 0$ and $(x_n) \subset D$ such that

$$\|\psi(x_n)\varphi(x_n) - \psi(x_m)\varphi(x_m)\| > \varepsilon,$$

for every $n \neq m$ (as in the proof of Proposition 4.16 the compact subsets of the strong dual of $H_b(B)$ are metrizable; in fact, even all bounded set, because the Fréchet space

$H_b(B)$ is quasinormable (see Section 1.3). Hence, we can use sequences instead of nets in A_D). By the Hahn-Banach Theorem there is $u_{n,m} \in X'$ with $\|u_{n,m}\|_{X'} \leq 1$ such that

$$\left|u_{n,m}\left(\psi(x_n)\varphi(x_n) - \psi(x_m)\varphi(x_m)\right)\right| > \varepsilon, \tag{4.8}$$

for every $n \neq m$. On the other hand, $(\psi(x_n)\delta_{\varphi(x_n)})_n$ is a relatively compact set in $H_b(B)'$. So, given the bounded set $(u_{n,m})_{n\neq m} \subset H_b(B)$, we can find a subsequence $(n_k)_k$ such that

$$\begin{aligned}
\frac{\varepsilon}{2} &> \sup_{u\in(u_{n,m})_{n\neq m}} \left|\langle u, \psi(x_{n_k})\delta_{\varphi(x_{n_k})} - \psi(x_{n_j})\delta_{\varphi(x_{n_j})}\rangle\right| \\
&= \sup_{u\in(u_{n,m})_{n\neq m}} \left|\psi(x_{n_k})u(\varphi(x_{n_k})) - \psi(x_{n_j})u(\varphi(x_{n_j}))\right| \\
&= \sup_{u\in(u_{n,m})_{n\neq m}} \left|u\left(\psi(x_{n_k})\varphi(x_{n_k}) - \psi(x_{n_j})\varphi(x_{n_j})\right)\right|,
\end{aligned}$$

for every $k, j \in \mathbb{N}$. Since we can choose k, j such that $n_k \neq n_j$ we obtain a contradiction with (4.8).

b) Now we assume that A_D is weakly relatively compact in $H_b(B)'$ but $(\psi\cdot\varphi)(D)$ is not weakly relatively compact in X. By the Eberlein-Šmulian theorem there are $\varepsilon > 0$, $(x_n) \subset D$ and $u \in X'$ such that

$$|u\left(\psi(x_n)\varphi(x_n) - \psi(x_m)\varphi(x_m)\right)| > \varepsilon, \tag{4.9}$$

for every $n \neq m$. Since the sequence $(\psi(x_n)\delta_{\varphi(x_n)})_n$ is weakly relatively compact in $H_b(B)'$, there is a weakly convergent subnet $(\psi(x_\alpha)\delta_{\varphi(x_\alpha)})_\alpha$ and we can choose α and β such that $\psi(x_\alpha)\delta_{\varphi(x_\alpha)} \neq \psi(x_\beta)\delta_{\varphi(x_\beta)}$ and

$$\left|\langle u, \psi(x_\alpha)\delta_{\varphi(x_\alpha)} - \psi(x_\beta)\delta_{\varphi(x_\beta)}\rangle\right| = \left|u\big(\psi(x_\alpha)\varphi(x_\alpha) - \psi(x_\beta)\varphi(x_\beta)\big)\right| < \frac{\varepsilon}{2},$$

which contradicts (4.9). □

If the set A_r in Proposition 4.16 is (weakly) relatively compact in $H_b(B)'$, we then have that the set $(\psi\cdot\varphi)(rB)$ is (weakly) relatively compact in X. This allows us to obtain a necessary condition for $C_{\psi,\varphi}$ to be (reflexive) Montel (cf. Proposition 4.6).

Corollary 4.19. *If $C_{\psi,\varphi} : H_b(B) \to H_b(B)$ is (reflexive) Montel, then the set $(\psi\cdot\varphi)(rB)$ is (weakly) relatively compact in X for every $0 < r < 1$.*

The following result has an analogous proof to that of Lemma 4.7, now also for the case of weak compactness.

Lemma 4.20. *Let $\psi \in H_b(B)$ and $\varphi : B \to B$ be holomorphic of bounded type. If $(\psi\cdot\varphi)(rB)$ is (weakly) relatively compact in X for some $0 < r < 1$. Then, for each sequence $(x_n)_n \subset rB$ such that $(\varphi(x_n))_n$ is not (weakly) relatively compact in B, there is a subsequence $(x_{n_k})_k$ such that*

$$\lim_{k\to\infty} |\psi(x_{n_k})| = 0.$$

Proof. We begin by fixing some $(x_n)_n \subset rB$ such that $(\varphi(x_n))_n$ is not (weakly) relatively compact in B. Since φ is of bounded type, we can find $0 < s < 1$ such that $(\varphi(x_n))_n \subseteq sB$. Thus the set $(\varphi(x_n))_n$ is also not (weakly) relatively compact in X. We claim that the set

$$A_m := \varphi\Big(\big\{x \in rB \,:\, |\psi(x)| > \frac{1}{m}\big\}\Big)$$

is (weakly) relatively compact in X for each $m \in \mathbb{N}$. Indeed, for a fixed $m \in \mathbb{N}$ if $\varphi(x) \in A_m$ we have that $\frac{1}{\|\psi\|_r} \leq \frac{1}{|\psi(x)|} < m$. Now given $(\varphi(y_n))_n \subseteq A_m$, we can find a subsequence $(n_k)_k$, $z \in \mathbb{C}$ and $y \in X$ such that $\lim_k \frac{1}{\psi(y_{n_k})} = z$ and $\lim_k \psi(y_{n_k})\varphi(y_{n_k}) = y$ (in the case of weak compactness, by Eberlein-Šmulian theorem, there is $z_0 \in X$ such that for all $u \in X'$, $\lim_k u\big(\psi(y_{n_k})\varphi(y_{n_k})\big) = u(z_0)$). Therefore we obtain

$$\varphi(y_{n_k}) = \frac{1}{\psi(y_{n_k})} \cdot \psi(y_{n_k})\varphi(y_{n_k}) \to zy \in X,$$

and in the case of weak compactness

$$u(\varphi(y_{n_k})) = \frac{1}{\psi(y_{n_k})} \cdot u(\psi(y_{n_k})\varphi(y_{n_k})) \to z\, u(z_0) = u(z\, z_0),$$

and the claim holds in both cases (again by Eberlein-Šmulian theorem in the case of weak compactness). Since $(\varphi(x_n))_n$ is not (weakly) relatively compact in X we can find $n_1 \in \mathbb{N}$ such that $\varphi(x_{n_1}) \notin A_1$. Therefore $|\psi(x_{n_1})| \leq 1$. Now, the set $(\varphi(x_n))_{n>n_1}$ is not (weakly) relatively compact in X and we can find $n_2 > n_1$ such that $\varphi(x_{n_2}) \notin A_2$. This element satisfies that $|\psi(x_{n_2})| \leq \frac{1}{2}$. Repeating this procedure we obtain a subsequence such that $|\psi(x_{n_k})| \leq 1/k$, for every $k \in \mathbb{N}$ and the proof is complete. □

This result allows to generalize Theorem 1.22 for reflexive and Montel weighted composition operators using a similar procedure to that of Theorem 4.8. In the case of weak compactness, we need an extra assumption on the Banach space X, namely that it has the *Schur property*, that is, that every weakly convergent sequence is norm convergent in X. The space ℓ_1 is a classical space having this property.

Theorem 4.21. *Let ψ and φ be holomorphic of bounded type. We have:*

a) *$C_{\psi,\varphi} : H_b(B) \to H_b(B)$ is Montel if and only if $(\psi \cdot \varphi)(rB)$ is relatively compact in X for every $0 < r < 1$.*

b) *If $C_{\psi,\varphi} : H_b(B) \to H_b(B)$ is reflexive, then $(\psi \cdot \varphi)(rB)$ is weakly relatively compact in X for every $0 < r < 1$.*

c) *If X has the Schur property, then $C_{\psi,\varphi} : H_b(B) \to H_b(B)$ is reflexive if and only if $(\psi \cdot \varphi)(rB)$ is weakly relatively compact in X for every $0 < r < 1$.*

Proof. a) If $C_{\psi,\varphi}$ is Montel, by Corollary 4.19, $(\psi \cdot \varphi)(rB)$ is relatively compact in X for every $0 < r < 1$.

To show the converse we proceed by contradiction. Assume $C_{\psi,\varphi}$ is not a Montel operator. Then there is a bounded sequence $(f_n)_n \subset H_b(B)$, some $0 < r_0 < 1$ and $\varepsilon > 0$ such that

$$\|\psi \cdot (f_n \circ \varphi) - \psi \cdot (f_m \circ \varphi)\|_{r_0} > 2\varepsilon,$$

for every $n < m$. Now, we can proceed as in the proof of Theorem 4.8 to obtain a contradiction (here we use Lemma 4.20 instead of Lemma 4.7).

b) If $C_{\psi,\varphi}$ is reflexive, by Corollary 4.19, $(\psi \cdot \varphi)(rB)$ is weakly relatively compact in X for every $0 < r < 1$.

On the other hand, if $(\psi \cdot \varphi)(rB)$ is weakly relatively compact in X, by the Eberlein-Šmulian theorem, every sequence in this set has a weakly convergent subsequence, which is norm convergent because X has the Schur property. So $(\psi \cdot \varphi)(rB)$ is in fact relatively compact and hence $C_{\psi,\varphi}$ is even a Montel operator. □

Now, as an immediate consequence of Proposition 4.11, Corollary 4.12 and Theorem 4.21 we obtain

Theorem 4.22. *Let ψ and φ be holomorphic of bounded type. Consider the following statements:*

a) $C_{\psi,\varphi} : H_b(B) \to H_b(B)$ *is compact.*

a′) $C_{\psi,\varphi} : H_b(B) \to H_b(B)$ *is weakly compact.*

b) The following two conditions hold:

- *(i)* $(\psi \cdot \varphi)(rB)$ *is relatively compact in X for every* $0 < r < 1$.
- *(ii) There is* $0 < s < 1$ *such that* $\varphi(B) \subseteq sB$.

b′) The following two conditions hold:

- *(i)* $(\psi \cdot \varphi)(rB)$ *is weakly relatively compact in X for every* $0 < r < 1$.
- *(ii) There is* $0 < s < 1$ *such that* $\varphi(B) \subseteq sB$.

Then a) is equivalent to b), a′) implies b′) and, if X has the Schur property, b′) implies a).

Corollary 4.23. *Let φ be holomorphic of bounded type. Then $C_\varphi : H_b(B) \to H_b(B)$ is compact if and only if there is $0 < s < 1$ such that $\varphi(B) \subseteq sB$ and that for each $0 < r < 1$ the set $\varphi(rB)$ is relatively compact in X.*

Proposition 4.24. *Let $\psi \in H_b(B)$ and $\varphi : B \to B$ be holomorphic of bounded type and open. If $C_{\psi,\varphi} : H_b(B) \to H_b(B)$ is Montel then X is finite dimensional. Consequently, $C_\varphi : H_b(B) \to H_b(B)$ is also Montel.*

Proof. The assumptions imply that $C_{\psi,\varphi}$ is continuous. We can find $x_0 \in B$ such that $\psi(x_0) \neq 0$. Then there are $0 < r < 1$ and $\varepsilon, M, m > 0$ such that $m \leq |\psi(x)| \leq M$, for all $x \in D := B(x_0, \varepsilon) \subset rB$. By Proposition 4.16 the set $A := \{\psi(x)\delta_{\varphi(x)} : x \in D\}$ is relatively compact in $H_b(B)'$. We claim that $C := \{\delta_{\varphi(x)} : x \in D\}$ is relatively $\beta(H_b(B)', H_b(B))$-compact in $H_b(B)'$. Indeed, since the compact subsets of $H_b(B)'_\beta$ are metrizable, it is enough to see that this set is relatively sequentially compact. Given a sequence $(\delta_{\varphi(x_n)})_n$ in C, there is a subsequence denoted the same way such that $(\psi(x_n)\delta_{\varphi(x_n)})_n$ converges to $u \in H_b(B)'$ in the topology $\beta(H_b(B)', H_b(B))$. The sequence $\left(\frac{1}{\psi(x_n)}\right)_n$ is bounded in $\mathbb{C}$ (since each $x_n \in D$). Then there is a subsequence such that $\frac{1}{\psi(x_{n_k})} \to z \in \mathbb{C}$ as $k \to \infty$. This implies that

$$\delta_{\varphi(x_{n_k})} = \frac{1}{\psi(x_{n_k})}\psi(x_{n_k})\delta_{\varphi(x_{n_k})} \to zu \in H_b(B)',$$

as $k \to \infty$, and we obtain the claim. Taking the function 1 as the weight in Lemma 4.18 we obtain that $\varphi(D)$ is relatively compact in X. Since φ is open we have found an open relatively compact set in X. The topology of every locally convex space is translation invariant, therefore X has an open 0-neighbourhood which is relatively compact. Hence is finite dimensional.

Assuming X is finite dimensional, we have that the set $\varphi(rB)$ is B-bounded and therefore relatively compact on X, for every $0 < r < 1$. By Theorem 4.21 we obtain that C_φ is Montel. □

4.5 Examples

We finish this chapter giving some examples of weighted composition operators that have or do not have some of the properties that we have been studying. Next example follows directly from Theorem 1.23.

Example 4.25. Let $\varphi : \mathbb{D} \to \mathbb{D}$ and $\psi \in H^\infty(\mathbb{D})$ defined by

$$\varphi(z) = \frac{1+z}{2} \quad \text{and} \quad \psi(z) = 1 - z.$$

In view of Theorem 4.8, the set $(\psi \cdot \varphi)(\mathbb{D})$ is relatively compact in $\mathbb{C}$. However, $\varphi(\mathbb{D})$ is not contained in a ball of radius strictly smaller than 1. Observe that the weight ψ satisfies

$$\lim_{r \to 1^-} \sup_{|\varphi(z)|>r} |\psi(z)| = 0,$$

but the constant weight 1 does not.

Then $C_{\psi,\varphi} : H^\infty(\mathbb{D}) \to H^\infty(\mathbb{D})$ is compact, but $C_\varphi = C_{1,\varphi} : H^\infty(\mathbb{D}) \to H^\infty(\mathbb{D})$ is not compact.

Examples 4.26. Assume that X is an infinite dimensional Banach space. Consider $\varphi\colon B \to B$ defined by $\varphi(x) = \frac{1}{2}x$.

Observe that $\varphi(B) = \frac{1}{2}B$ is not a relatively compact set since X is infinite dimensional. Thus, by Theorem 1.21, the operator $C_\varphi\colon H^\infty(B) \to H^\infty(B)$ is not compact (Montel). By Theorem 4.9 the operator $C_\varphi\colon H_b(B) \to H_b(B)$ is bounded. However, Theorem 4.21 implies that C_φ is not Montel on $H_b(B)$. Then, we have the following

a) The operator $C_\varphi\colon H^\infty(B) \to H^\infty(B)$ is continuous but it is not compact, therefore it is bounded but it is not Montel.

b) The operator $C_\varphi\colon H_b(B) \to H_b(B)$ is bounded but it is not Montel.

On [31, Example 2.13] the authors give an example of a composition operator on $H_b(B)$, when X is the Tsirelson space, which is weakly compact but not compact. The following example is based on [31, Example 2.16].

Example 4.27. Let $\varphi\colon B_{c_0} \to B_{c_0}$ be defined by

$$\varphi(x) = \left(\frac{x_1}{2}, \frac{x_2^2}{2}, \frac{x_3^3}{2}, \dots\right).$$

Then the composition operator C_φ is compact on $H_b(B_{c_0})$, but it is not compact on $H^\infty(B_{c_0})$. First, observe that $\varphi(rB_{c_0})$ is relatively compact in c_0 for each $0 < r < 1$. Indeed, if we take an arbitrary $x \in c_0$ with $\|x\|_\infty < r < 1$ we have that the n-th coordinate of $\varphi(x)$ satisfies

$$\frac{1}{2}|x_n|^n < \frac{1}{2}r^n \xrightarrow{n} 0,$$

and this implies that $\varphi(rB_{c_0}) \subseteq \frac{1}{2}B_{c_0}$ is a relatively compact set of B_{c_0}. Corollary 4.23 gives that C_φ is compact on $H_b(B_{c_0})$. On the other hand, for the sequence

$$y_n = \left(1 - \frac{1}{n}\right)^{1/n} e_n \in B_{c_0}, \quad n \in \mathbb{N},$$

we have $(\varphi(y_n))_n = \left(\frac{1}{2}\left(1 - \frac{1}{n}\right)e_n\right)_n$ is not relatively compact in B_{c_0}. By Theorem 1.21 (or Theorem 4.8) the operator C_φ is not compact in $H^\infty(B_{c_0})$.

Based on the previous example we can construct a composition operator on $H_b(B_{c_0})$ which is Montel but not compact.

Example 4.28. Let $\varphi\colon B_{c_0} \to B_{c_0}$ defined by $\varphi(x) = \left(x_1, x_2^2, x_3^3, \dots\right)$. The composition operator $C_\varphi\colon H_b(B_{c_0}) \to H_b(B_{c_0})$ is Montel, but not bounded and hence, not compact either.

Open problem. We do not know if there is $C_{\psi,\varphi}\colon H_b(B) \to H_b(B)$ Montel (compact) such that $C_\varphi\colon H_b(B) \to H_b(B)$ is not Montel (compact). However, in Proposition 4.24 we already seen that this cannot be the case when φ is holomorphic of bounded type and open.

Chapter 5

Ergodic properties of weighted composition operators

5.1 Introduction

In this chapter we obtain different results about the ergodicity of weighted composition operators when acting on the spaces $H(B)$, $H_b(B)$ and $H^\infty(B)$, as well as about the compactness and the ergodicity of the multiplication operator in terms of the weight. Mean ergodicity and related properties of weighted composition operators acting on spaces of holomorphic functions on domains of finite dimension (such as the unit disc $\mathbb{D}$) have been studied in [6, 9, 36].

Recall that the operator $C_{\psi,\varphi}\colon H(B)\to H(B)$ is continuous if and only if $\psi\in H(B)$ and $\varphi\colon B\to B$ is holomorphic.

We begin with some simple observations. First, setting $\varphi^0 := id_B$, note that the iterates of the operator $C_{\psi,\varphi}\colon H(B)\to H(B)$ can be written as follows

$$(C_{\psi,\varphi})^n f(x)=\psi(x)\cdots\psi(\varphi^{n-1}(x))f(\varphi^n(x))=\left(\prod_{m=0}^{n-1}(\psi\circ\varphi^m)(x)\right)f(\varphi^n(x)),$$

for $f\in H(B)$, $n\in\mathbb{N}$ and $x\in B$. We denote $\psi^{[n]} := \prod_{m=0}^{n-1}(\psi\circ\varphi^m)$. Then we have the following identity

$$(C_{\psi,\varphi})^n(f)=\psi^{[n]}\cdot(f\circ\varphi^n) \tag{5.1}$$

for all $n\in\mathbb{N}$ and $f\in H(B)$. Then, it seems natural to assume that the elements $\psi^{[n]}$ are not zero for infinitely many $n\in\mathbb{N}$. If this is not the case, the iterates $(C_{\psi,\varphi})^n$ will be the zero operator for n big enough.

Remark 5.1. Consider a weight $\psi\in H(B)$ and a holomorphic symbol $\varphi\colon B\to B$ such that there is $n_0\in\mathbb{N}$ with $\psi\circ\varphi^{n_0}\equiv 0$, then $\psi^{[n]}\equiv 0$ for all $n>n_0$. This implies that the operator $C_{\psi,\varphi}\colon H(B)\to H(B)$ satisfies $(C_{\psi,\varphi})^n=0$ for all $n>n_0$. Since the constant function 1 is in $H(B)$ the converse also holds.

Remark 5.2. Take ψ and φ such that $\psi \circ \varphi^{n_0} \equiv 0$ for some $n_0 \in \mathbb{N}$. If we additionally assume that φ is an open map then $\psi \equiv 0$. Indeed, since $\psi \circ \varphi^{n_0} \equiv 0$, we have that $\psi \circ \varphi^{n_0-1}(\varphi(B)) = \{0\}$. But $\varphi(B)$ is an open nonempty set, then by the Identity Principle $\psi \circ \varphi^{n_0-1} \equiv 0$. Recursively we apply this argument to obtain $\psi \equiv 0$.

Of course, we can have

$$\psi \circ \varphi^n \not\equiv 0 \tag{5.2}$$

for all $n \in \mathbb{N}_0$ but φ is not open. On the open unit ball of $\mathbb{C}^2$, consider the mappings $\psi((x_1, x_2)) = x_1(x_2 + 1)$ and $\varphi((x_1, x_2)) = (x_1, 0)$. Clearly (5.2) is satisfied for all $n \in \mathbb{N}_0$.

The assumption of φ being not open is not interesting on $H(\mathbb{D})$ since the constant symbols are the only holomorphic mappings that are not open. However, on Banach spaces of higher dimension the situation is very different (see Examples 5.26).

Now, we briefly focus on weighted composition operators acting on spaces of homogeneous polynomials, which are natural subspaces of $H(B)$. In some cases we can rewrite the weighted composition operator as a composition operator whose symbol is a simple modification of the original one.

Proposition 5.3. *Let $\psi : X \to \mathbb{C}$ and $\varphi : X \to X$ be two continuous mappings. Assume there is $m \in \mathbb{N}$ and a continuous function $\widehat{\psi} : X \to \mathbb{C}$ such that*

$$\psi(x) = \left(\widehat{\psi}(x)\right)^m \quad \textit{for every } x \in X. \tag{5.3}$$

Then $C_{\psi,\varphi} : \mathscr{P}(^mX) \to \mathscr{P}(^mX)$ is well defined if and only if the map $\widehat{\psi} \cdot \varphi : X \to X$ is a continuous linear operator. In addition, the operator $C_{\psi,\varphi}$ coincides with the composition operator $C_{\widehat{\psi}\cdot\varphi}$.

Proof. Fix $p \in \mathscr{P}(^mX)$ and $x \in X$, by (5.3) we have

$$p\left(\widehat{\psi}(x) \cdot \varphi(x)\right) = \left(\widehat{\psi}(x)\right)^m p(\varphi(x)) = \psi(x) p(\varphi(x)).$$

Since p and x were arbitrary we obtain that the operator $C_{\psi,\varphi}$ coincides with the composition operator $C_{\widehat{\psi}\cdot\varphi}$. Observe that the symbol $\widehat{\psi} \cdot \varphi : X \to X$ is continuous and applying Proposition 2.3 we obtain the result. □

The constant functions are examples of weights satisfying (5.3). In this case if there is $c \in \mathbb{C} \setminus \{0\}$ such that $\psi(x) = c$ for every $x \in X$, we can find $z \in \mathbb{C} \setminus \{0\}$ with $c = z^m$. Thus, the weighted composition operator is well defined if and only if $z \cdot \varphi : X \to X$ is a linear operator, and this forces φ to be a linear operator. We can construct more elaborated examples. For each $u \in X' \setminus \{0\}$ observe that the function $\psi = u^m$ satisfies (5.3) taking, for example, $\widehat{\psi} = u$. Now if $\varphi(x) = x_0 \in X$ for every $x \in X$, we have that $C_{u^m,x_0} : \mathscr{P}(^mX) \to \mathscr{P}(^mX)$ coincides with the composition operator $C_{u\cdot x_0}$.

Properties of composition operators acting on spaces of m-homogeneous polynomials have already been discussed in Chapter 2. Then, for weighted composition operators whose symbol and weight satisfy Proposition 5.3, the results of Chapter 2 clearly apply.

However, there are combinations of weights and symbols that do not satisfy the assumption on Proposition 5.3, but the associated weighted composition operator is well defined on $\mathscr{P}(^m X)$. Indeed, consider $\psi \in \mathscr{P}(^2\mathbb{C}^2)$ given by $\psi(z_1, z_2) = z_1 z_2$. Clearly, ψ does not satisfy (5.3) since $\widehat{\psi}(1,z)$ would be a determination of the square root continuous on $\mathbb{C}$. Now, taking the constant symbol $\varphi(z_1,z_2) = (1,1)$ for every $z_1, z_2 \in \mathbb{C}$, we have for any $p \in \mathscr{P}(^2\mathbb{C}^2)$ that

$$\psi(z_1,z_2)\cdot p(\varphi(1,1)) = \psi(z_1,z_2)\cdot p(1,1),$$

for every $(z_1,z_2) \in \mathbb{C}^2$. This is again a 2-homogeneous polynomial and then $C_{\psi,\varphi}$ is well defined.

5.2 Ergodic properties on $H(B)$

We begin by looking at weighted composition operators acting on $H(B)$. Some results obtained in this section will be given in the following sections for the spaces $H_b(B)$ and $H^\infty(B)$.

The following is an extension of Theorem 1.24 to $H(B)$.

Proposition 5.4. *Let $\psi \in H(B)$ and $\varphi : B \to B$ be holomorphic.*

a) If $C_{\psi,\varphi} : H(B) \to H(B)$ is power bounded then $C_{\psi,\varphi}$ is uniformly mean ergodic and $(\psi^{[n]})_n$ is bounded in $H(B)$.

b) If $C_{\psi,\varphi} : H(B) \to H(B)$ is mean ergodic then $\lim_{n\to\infty} \frac{1}{n}\psi^{[n]} = 0$ in $H(B)$.

Proof. a) If $C_{\psi,\varphi}$ is power bounded then the set $\{C^n_{\psi,\varphi}(f) : n \in \mathbb{N}\}$ is bounded in $H(B)$ for every $f \in H(B)$. In particular, for $f \equiv 1$ we have that $\{\psi^{[n]} : n \in \mathbb{N}\}$ is bounded in $H(B)$. Since $H(B)$ is a semi-Montel space, Proposition 1.15 gives that the operator $C_{\psi,\varphi}$ is uniformly mean ergodic.

b) If $C_{\psi,\varphi}$ is mean ergodic by identity (1.10) we have

$$\lim_{n\to\infty} \frac{1}{n} C^n_{\psi,\varphi}(f) = \lim_{n\to\infty} \left[\frac{n+1}{n} \left(C_{\psi,\varphi}\right)_{[n+1]}(f) - \left(C_{\psi,\varphi}\right)_{[n]}(f) \right] = 0,$$

for every $f \in H(B)$. Therefore $\lim_{n\to\infty} \frac{1}{n} C^n_{\psi,\varphi}(1) = 0$, which yields the assertion. □

As in the case of composition operators (recall Theorem 3.11), φ having stable orbit (recall the definition in Section 3.3) plays a fundamental role. Several authors have used this property to characterize topologizable, power bounded and/or mean ergodic weighted composition operators in different function spaces (see [6, 9, 39]).

Proposition 5.5. *Let $\psi \in H(B)$ and $\varphi\colon B \to B$ be a holomorphic mapping with stable orbits. Then $C_{\psi,\varphi}\colon H(B) \to H(B)$ is topologizable.*

Proof. Fix a compact subset $K \subset B$ and take a compact subset $L \subset B$ such that $\varphi^n(K) \subseteq L$ for all $n \in \mathbb{N}$. Let $a_n = \sup_{x\in K} |\psi^{[n]}(x)| < \infty$. Then, using (5.1) we have

$$\sup_{x\in K} |C^n_{\psi,\varphi}(f)(x)| \le \sup_{x\in K} |\psi^{[n]}(x)| \sup_{x\in K} |f(\varphi^n(x))| \le a_n \sup_{x\in L} |f(x)|$$

and $C_{\psi,\varphi}$ is topologizable. □

In Proposition 5.9, in some sense, we show that the converse implication does hold for $C_{\psi,\varphi}$ acting on $H_b(B)$. However, the argument that we will use there do not work on $H(B)$, since compact subsets of B may have empty interior, as it happens on every infinite dimensional Banach space.

For $C_{\psi,\varphi}$ being power bounded we obtain a similar result.

Proposition 5.6. *Let $\psi \in H(B)$ and $\varphi\colon B \to B$ holomorphic. Consider the following conditions.*

a) φ has stable orbits and $(\prod_{m=0}^{n}(\psi \circ \varphi^m))_n$ is a bounded sequence in $H(B)$.

b) $C_{\psi,\varphi}\colon H(B) \to H(B)$ is power bounded.

c) $(\prod_{m=0}^{n}(\psi \circ \varphi^m))_n$ is a bounded sequence in $H(B)$.

Then a) implies b) and b) implies c).

Proof. First assume a) holds. Fix a compact set $K \subset B$ and consider

$$A = \sup_{n\in\mathbb{N}} \sup_{x\in K} \left| \prod_{m=0}^{n} (\psi \circ \varphi^m)(x) \right| < \infty.$$

Since φ has stable orbits there exists a compact set $L \subset B$ such that $\varphi^n(K) \subseteq L$ for every $n \in \mathbb{N}$. Then

$$\sup_{x\in K} |C^n_{\psi,\varphi}(f)(x)| \le A \sup_{x\in K} |f(\varphi^n(x))| \le A \sup_{x\in L} |f(x)|,$$

for each $n \in \mathbb{N}$ and all $f \in H(B)$. Thus $C_{\psi,\varphi}$ is power bounded.

Now, if b) holds, it follows from Proposition 5.4 that $(\prod_{m=0}^{n}(\psi \circ \varphi^m))_n$ is a bounded sequence in $H(B)$. □

We discuss the mean ergodicity of $C_{\psi,\varphi}$ assuming the operator is Cesàro bounded.

Proposition 5.7. *Let $\psi \in H(B)$ and $\varphi\colon B \to B$ holomorphic. Assume that $C_{\psi,\varphi}\colon H(B) \to H(B)$ is Cesàro bounded. Consider the following conditions.*

a) $\left((C_{\psi,\varphi})_{[n]}f\right)_n$ *is* $\sigma(H(B),H(B)')$*-relatively compact for each* $f\in H(B)$.

b) $\lim_{n\to\infty}\frac{\psi^{[n]}}{n}=0$ *in* $H(B)$.

c) φ *has stable orbits.*

Then if a), b) and c) hold $C_{\psi,\varphi}: H(B)\to H(B)$ *is mean ergodic. Conversely, if* $C_{\psi,\varphi}$ *is mean ergodic then conditions a) and b) are satisfied.*

Proof. Assume a), b) and c) hold. By Proposition 1.11 it is enough to show that b) and c) give that

$$\lim_{n\to\infty}\frac{C^n_{\psi,\varphi}(f)}{n}=0 \tag{5.4}$$

for every $f\in H(B)$.

Fix a compact set $K\subset B$ and take $L\subset B$ compact such that $\varphi^n(K)\subseteq L$ for every $n\in\mathbb{N}$. For an arbitrary $f\in H(B)$ we obtain

$$\begin{aligned}\lim_{n\to\infty}\sup_{x\in K}\left|\frac{C^n_{\psi,\varphi}(f)(x)}{n}\right| &= \lim_{n\to\infty}\sup_{x\in K}\left|\frac{\psi^{[n]}(x)\cdot f(\varphi^n(x))}{n}\right|\\ &\le \lim_{n\to\infty}\sup_{x\in K}\left|\frac{\psi^{[n]}(x)}{n}\right|\sup_{y\in K}|f(\varphi^n(y))|\\ &\le \sup_{y\in L}|f(y)|\lim_{n\to\infty}\sup_{x\in K}\left|\frac{\psi^{[n]}(x)}{n}\right|.\end{aligned}$$

The last term tends to 0 and the result follows.

Now, assume that $C_{\psi,\varphi}$ is mean ergodic. Proposition 1.11 gives that a) and (5.4) are satisfied for every $f\in H(B)$. Taking the constant function 1 and an arbitrary compact set $K\subset B$ we have

$$\lim_{n\to\infty}\sup_{x\in K}\left|\frac{\psi^{[n]}(x)}{n}\right|=\lim_{n\to\infty}\sup_{x\in K}\left|\frac{C^n_{\psi,\varphi}(1)}{n}\right|=0,$$

and b) holds. □

5.3 Ergodic properties on $H_b(B)$

Our goal now is to give analogous results to Propositions 5.5, 5.6 and 5.7 for the weighted composition operator acting on $H_b(B)$. In this case we have more tools at hand, since we have been able to give complete characterizations for topologizable, power bounded and mean ergodic weighted composition operators (Proposition 5.9, Theorem 5.10 and Theorem 5.11).

As it happened for composition operators, the crucial property now for φ is to have B-stable orbits (see Section 3.3).

Proposition 5.8. *Let $\psi \in H_b(B)$ and $\varphi : B \to B$ be holomorphic of bounded type.*

a) If $C_{\psi,\varphi} : H_b(B) \to H_b(B)$ is power bounded, then $(\psi^{[n]})_n$ is bounded in $H_b(B)$.

b) If $C_{\psi,\varphi} : H_b(B) \to H_b(B)$ is mean ergodic, then $\lim_{n\to\infty} \frac{\psi^{[n]}}{n} = 0$ in $H_b(B)$.

Proof. a) If $C_{\psi,\varphi}$ is power bounded then the set $\{C^n_{\psi,\varphi}(f) : n \in \mathbb{N}\}$ is bounded in $H_b(B)$ for every $f \in H_b(B)$. We are done, if we just take the constant function 1.

b) By Proposition 1.12 (see also [1, Theorem 2.4]) if $C_{\psi,\varphi}$ is mean ergodic then $\lim_{n\to\infty} \frac{1}{n} C^n_{\psi,\varphi}(f) = 0$ uniformly on the B-bounded sets, for every $f \in H_b(B)$. Again, taking the constant function 1 yields the assertion. □

The following extends [6, Proposition 3.2].

Proposition 5.9. *Let $\psi \in H_b(B)$ and $\varphi : B \to B$ be holomorphic of bounded type such that $\psi \circ \varphi^n \not\equiv 0$ holds for all $n \in \mathbb{N}_0$. The following assertions are equivalent:*

a) $C_{\psi,\varphi} : H_b(B) \to H_b(B)$ is topologizable.

b) φ has B-stable orbits.

Proof. Assume $C_{\psi,\varphi}$ is topologizable. Then, for every $0 < r < 1$ there exists $0 < s < 1$ such that for every $n \in \mathbb{N}$ there is $a_n > 0$ with

$$\sup_{\|x\|<r} |C^n_{\psi,\varphi}(f)(x)| \le a_n \sup_{\|x\|<s} |f(x)| \tag{5.5}$$

for all $f \in H_b(B)$. On the other hand, by Lemma 3.13 the $H_b(B)$-hull of sB given by

$$M = \widehat{(sB)}_{H_b(B)} := \{x \in B : |f(x)| \le \sup_{y\in sB} |f(y)|, \text{ for every } f \in H_b(B)\}$$

is again a B-bounded set. Since M is the intersection of closed sets, it is closed. Now, we see that $\varphi^n(rB) \subseteq M$ for every $n \in \mathbb{N}$. Suppose that this is not the case and take $n_0 \in \mathbb{N}$ and $z_0 \in rB$ such that $\varphi^{n_0}(z_0) \notin M$. In other words, the set $rB \bigcap \varphi^{-n_0}(B \setminus M)$ is nonempty. By the assumption, the function $\psi^{[n_0]}$ is not 0 and, since $rB \bigcap \varphi^{-n_0}(B \setminus M)$ is an open set, by the Identity Principle we can take x_0 in this set such that $|\psi^{[n_0]}(x_0)| > 0$. The definition of M allows us to find $f_0 \in H_b(B)$ such that $\sup_{y\in sB} |f_0(y)| < |f_0(\varphi^{n_0}(x_0))|$. Then there is $m \in \mathbb{N}$ with

$$\sup_{y\in sB} \frac{|f_0(y)|^m}{|f_0(\varphi^{n_0}(x_0))|^m} < \frac{|\psi^{[n_0]}(x_0)|}{a_{n_0}}.$$

Observe that the function $f = f_0^m \in H_b(B)$ does not satisfy (5.5) and this yields a contradiction.

The converse implication follows with exactly the same proof as in Proposition 5.5, replacing ‘compact’ by ‘B-bounded’ and ‘stable orbits’ by ‘B-stable orbits’. □

As a consequence of Proposition 5.9, we are able to characterize power bounded weighted composition operators in $H_b(B)$ and extend Theorem 1.25.

Theorem 5.10. *Let $\psi \in H_b(B)$ and $\varphi : B \to B$ be holomorphic of bounded type such that $\psi \circ \varphi^n \not\equiv 0$ holds for all $n \in \mathbb{N}_0$. The following assertions are equivalent:*

a) $C_{\psi,\varphi} : H_b(B) \to H_b(B)$ *is power bounded,*

b) φ *has B-stable orbits and* $(\prod_{m=0}^{n}(\psi \circ \varphi^m))_n$ *is a bounded sequence in* $H_b(B)$.

Proof. If $C_{\psi,\varphi}$ is power bounded it is in particular topologizable. Applying Propositions 5.8 and 5.9 we obtain b).

Now, b) implies a) follows as in Theorem 5.6 replacing 'compact' by 'B-bounded' and 'stable orbits' by 'B-stable orbits'. □

In contrast to Proposition 5.7 in the following result we do not need to assume that $C_{\psi,\varphi}$ is Cesàro bounded.

Theorem 5.11. *Let $\psi \in H_b(B)$ and $\varphi : B \to B$ be holomorphic of bounded type such that $\psi \circ \varphi^n \not\equiv 0$ holds for all $n \in \mathbb{N}_0$. Then $C_{\psi,\varphi} : H_b(B) \to H_b(B)$ is mean ergodic if and only if we have*

a) $\left((C_{\psi,\varphi})_{[n]} f\right)_n$ *is* $\sigma(H_b(B), H_b(B)')$*-relatively compact for each* $f \in H_b(B)$,

b) φ *has B-stable orbits, and*

c) $\lim_{n\to\infty} \frac{\psi^{[n]}}{n} = 0$ *on* $H_b(B)$.

Proof. Since $H_b(B)$ is barrelled, Proposition 1.12 is available and it suffices to show that b) and c) are satisfied if and only if we have

$$\lim_{n\to\infty} \frac{C^n_{\psi,\varphi}(f)}{n} = 0 \quad \text{for every } f \in H_b(B). \tag{5.6}$$

Observe that (5.6) gives that the sequence $\left(\frac{C^n_{\psi,\varphi}}{n}\right)_n$ is pointwise bounded in $\mathscr{L}(H_b(B))$ and so it is equicontinuous. Then, for every $0 < r < 1$ there are $c > 0$ and $0 < s < 1$ such that

$$\sup_{\|x\|<r} \left| \frac{C^n_{\psi,\varphi}(f)(x)}{n} \right| \le c \sup_{\|x\|<s} |f(x)|,$$

for every $f \in H_b(B)$ and $n \in \mathbb{N}$. We obtain that $C_{\psi,\varphi}$ is topologizable, since it satisfies (1.5) for $a_n = cn$. Now, by Proposition 5.9, we obtain b) and taking the constant function 1 in (5.6), we obtain c).

Assuming b) and c) hold, we obtain (5.6) as in the first part of Proposition 5.7 replacing 'compact' by 'B-bounded' and 'stable orbits' by 'B-stable orbits'. □

5.4 Ergodic properties on $H^\infty(B)$

We ask again the same questions about the ergodic properties of the weighted composition operator, this time acting $H^\infty(B)$. By Remark 1.18, the operator $C_{\psi,\varphi}$ is always topologizable.

Proposition 5.12. *Let $\psi \in H^\infty(B)$ and $\varphi\colon B \to B$ be holomorphic. Then $C_{\psi,\varphi}\colon H^\infty(B) \to H^\infty(B)$ is always topologizable.*

Again, we can characterise power boundedness for weighted composition operators.

Proposition 5.13. *Let $\psi \in H^\infty(B)$ and $\varphi\colon B \to B$ be holomorphic. Then $C_{\psi,\varphi}\colon H^\infty(B) \to H^\infty(B)$ is power bounded if and only if $(\prod_{m=0}^{n}(\psi \circ \varphi^m))_n$ is a bounded sequence in $H^\infty(B)$.*

Proof. If we assume that $C_{\psi,\varphi}$ is power bounded, then the sequence $\left(C^n_{\psi,\varphi}(f)\right)_n$ is bounded in $H^\infty(B)$ for every $f \in H^\infty(B)$. Taking the constant function 1 we obtain the claim.

Conversely, let

$$c = \sup_{n\in\mathbb{N}} \sup_{x\in B} \left| \prod_{m=0}^{n} (\psi \circ \varphi^m)(x) \right| < \infty.$$

Then, we have

$$\sup_{x\in B} |C^n_{\psi,\varphi}(f)(x)| \le \sup_{x\in B} |\psi^{[n]}| \sup_{x\in B} |f(\varphi^n(x))| \le c \sup_{x\in B} |f(x)|,$$

for all $n \in \mathbb{N}$ and $f \in H^\infty(B)$, which gives that $C_{\psi,\varphi}$ is power bounded since $H^\infty(B)$ is a Banach space. □

The following result is a consequence of Proposition 1.12.

Proposition 5.14. *Let $\psi \in H^\infty(B)$ and $\varphi : B \to B$ be holomorphic such that $\psi \circ \varphi^n \not\equiv 0$ holds for all $n \in \mathbb{N}_0$. Then, $C_{\psi,\varphi} : H^\infty(B) \to H^\infty(B)$ is mean ergodic if and only if the following two conditions are satisfied*

a) $\left((C_{\psi,\varphi})_{[n]} f\right)_n$ *is $\sigma(H^\infty(B), H^\infty(B)')$-relatively compact for each $f \in H^\infty(B)$, and*

b) $\lim_{n\to\infty} \frac{\psi^{[n]}}{n} = 0$ *in $H^\infty(B)$.*

5.5 The multiplication operator

As we already noted, the multiplication operator can be seen as a particular case of weighted composition operator. So, we may use the knowledge gained in the last two chapters to obtain some information on compactness and mean ergodic properties of multiplication operators, when acting on $H(B)$, $H_b(B)$ or $H^\infty(B)$.

5.5.1 Compact multiplication operators

We begin with an observation on the invertibility of the multiplication operator.

Lemma 5.15. *Let $\mathscr{H}$ be either $H^\infty(B)$, $H_b(B)$ or $H(B)$. Then, $M_\psi : \mathscr{H} \to \mathscr{H}$ is invertible if and only if $\frac{1}{\psi} \in \mathscr{H}$.*

Proof. Let us suppose that M_ψ is invertible. Then

$$M_\psi (M_\psi)^{-1}(f)(x) = \psi(x) \cdot (M_\psi)^{-1}(f)(x) = f(x),$$

for every $f \in \mathscr{H}$ and $\|x\| < 1$. Applying this equality to the constant function 1 gives $\psi(x) \neq 0$ for every $\|x\| < 1$. Then

$$(M_\psi)^{-1}(f)(x) = \frac{1}{\psi(x)} f(x),$$

for every $f \in \mathscr{H}$ and $x \in B$. This shows that $M_{\frac{1}{\psi}} = (M_\psi)^{-1}$ which, by Proposition 4.3, implies that $\frac{1}{\psi} \in \mathscr{H}$.

The converse implication also follows from Proposition 4.3. □

The *spectrum* of an operator $T : E \to E$, where E is a lcHs space, is denoted by

$$\sigma(T; E) := \{\lambda \in \mathbb{C} : \lambda \cdot \mathrm{id} - T \text{ is not invertible}\}.$$

Proposition 5.16. *Let $\mathscr{H}$ be either $H^\infty(B)$, $H_b(B)$ or $H(B)$. If $\psi \in \mathscr{H}$ then*

$$\psi(B) \subseteq \sigma(M_\psi; \mathscr{H}) \subseteq \overline{\psi(B)}. \tag{5.7}$$

Moreover, the following hold:

a) $\sigma(M_\psi; H^\infty(B)) = \overline{\psi(B)}$,

b) $\sigma(M_\psi; H(B)) = \psi(B)$,

c) $\sigma(M_\psi; H_b(B)) = \{\lambda \in \mathbb{C} : \inf_{\|x\|<r} |\lambda - \psi(x)| = 0,$ *for some* $0 < r < 1\}$.

Proof. Fix $x \in B$. Then, for each $g \in \mathrm{Im}\big(\psi(x)\,\mathrm{id}_{\mathscr{H}} - M_\psi\big)$, we have $g(x) = 0$. Since we can find $f \in \mathscr{H}$ such that $f(x) \neq 0$ we obtain that $\psi(x)\,\mathrm{id}_{\mathscr{H}} - M_\psi$ is not surjective and, therefore $\psi(x) \in \sigma(M_\psi; \mathscr{H})$. This shows that $\psi(B) \subseteq \sigma(M_\psi; \mathscr{H})$. For the other inclusion, take $\lambda \notin \overline{\psi(B)}$ and let us show that $\lambda \cdot \mathrm{id}_{\mathscr{H}} - M_\psi$ is invertible. Take $\varepsilon > 0$ so that $|\lambda - \psi(x)| \geq \varepsilon$ for all $x \in B$. Then we have that $\frac{1}{\lambda - \psi}$ is a continuous bounded function and it is clearly G-holomorphic, thus it is holomorphic (see Theorem 1.4). Therefore it is in $H^\infty(B) \subseteq \mathscr{H}$. Observe that

$$\lambda \cdot \mathrm{id}_{\mathscr{H}} - M_\psi = M_{\lambda - \psi}. \tag{5.8}$$

By Lemma 5.15 the operator $M_{\lambda-\psi}$ is invertible, which completes the proof of (5.7).

Now we see a). Since $H^\infty(B)$ is a Banach space the spectrum of any operator acting on $H^\infty(B)$ is closed and the conclusion follows from (5.7).

For $M_\psi\colon H(B)\to H(B)$, we consider two cases. If ψ is constant, then $\psi(B)=\overline{\psi(B)}$ and by (5.7), we have that b) trivially holds. Assume that ψ is not constant. In view of Lemma 5.15 and (5.8), it is enough to show that for $\lambda\notin\psi(B)$ the function $\frac{1}{\lambda-\psi}$ is in $H(B)$. Indeed, let K be a compact subset of B. We have that $\lambda\in\mathbb{C}\setminus\psi(K)$ which is an open set. Then there exists $\varepsilon_K>0$ such that $|\lambda-\psi(x)|\ge\varepsilon_K$ for every $x\in K$ and we obtain

$$\sup_{x\in K}\left|\frac{1}{\lambda-\psi(x)}\right|<\infty.$$

Then we have that $\frac{1}{\lambda-\psi}$ is a continuous function which is bounded on every compact subset of B. We conclude that $\frac{1}{\lambda-\psi}$ is holomorphic (using Theorem 1.4) and this completes the proof of b).

The proof of c) is similar to the proof of b). Note that, in any case, $\lambda\notin\sigma(M_\psi;H_b(B))$ if and only if for each $0<r<1$ we have $\inf_{\|x\|<r}|\lambda-\psi(x)|>0$. □

If the space X is finite dimensional, we have $H_b(B)=H(B)$ and then the spectrum $\sigma(M_\psi;H_b(B))=\psi(B)$ is fully characterised. However, we do not know what happens in the infinite dimensional case.

Lemma 5.17. *Let $\mathscr{H}$ be either $H^\infty(B)$, $H_b(B)$ or $H(B)$. If $M_\psi\colon\mathscr{H}\to\mathscr{H}$ is compact, then $\psi\in\mathscr{H}$ is constant.*

Proof. Since M_ψ is compact, by [34, Chapter 5 Part 2 Theorem 4], the spectrum of M_ψ is either finite or the closure of a null sequence.

Now assume that ψ is not constant in B. We can find $y_0\in B$ such that $\psi(0)\neq\psi(y_0)$. Taking $\|y_0\|<r<1$, by [47, Proposition 5.8] the set $\psi(B(0,r))$ is open in $\mathbb{C}$. Using (5.7) we have that $\psi(B(0,r))$ is contained in a discrete set, which is a contradiction. □

As a consequence of the previous results, we can see that there are no non-trivial compact multiplication operators on $H_b(B)$ and $H^\infty(B)$.

Proposition 5.18. *Let $\mathscr{H}H$ be either $H_b(B)$ or $H^\infty(B)$. Then $M_\psi\colon\mathscr{H}\to\mathscr{H}$ is compact if and only if ψ is 0.*

Proof. Assume $M_\psi\colon H^\infty(B)\to H^\infty(B)$ is compact. Lemma 5.17 gives $\psi(x)=c\in\mathbb{C}$ for every $\|x\|<1$. Since we can write the multiplication operator as C_{c,id_B}, by Theorem 4.8, we have

$$0=\lim_{r\to1^-}\sup_{\|x\|>r}|\psi(x)|=\lim_{r\to1^-}\sup_{\|x\|>r}|c|=|c|,$$

which gives the claim.

If $M_\psi : H_b(B) \to H_b(B)$ is compact then, by Lemma 5.17, we have that $\psi(x) = c \in \mathbb{C}$ for every $\|x\| < 1$. Assume $c \neq 0$. Then, the operator C_{c,id_B} is not compact in $H_b(B)$ since $\mathrm{id}_B(B) = B$ is not contained in rB for some $0 < r < 1$ (see Theorem 4.22). This is a contradiction. □

5.5.2 Mean ergodic multiplication operators

We move now to the study of ergodic properties. We follow [13]. First, since the multiplication operator is a weighted composition operator whose symbol is the identity, looking at (5.1) we observe that the iterates of M_ψ are given by

$$M_\psi^n(f) = \psi^n \cdot f = M_{\psi^n}(f),$$

where ψ^n now denotes the n-th power of the function instead of the n-th composition.

Remark 5.19. Let $\psi \in \mathscr{H}$ be different from the zero function, where $\mathscr{H}$ is either $H(B)$, $H_b(B)$ or $H^\infty(B)$. The map $\mathrm{id}_B : B \to B$ clearly has stable orbits and B-stable orbits. Then, by Propositions 5.5, 5.9 and 5.12 the operator $M_\psi : \mathscr{H} \to \mathscr{H}$ is always topologizable.

In order to characterise power boundedness of M_ψ it is enough to characterise when the set $(\psi^n)_n$ is bounded in $H(B)$, $H_b(B)$ and $H^\infty(B)$ (see Proposition 5.6, Theorem 5.10 and Proposition 5.13). We obtain a similar result to [13, Proposition 2.3].

Proposition 5.20. *Let $\psi \in H$ where $\mathscr{H}$ is either $H(B)$, $H_b(B)$ or $H^\infty(B)$. Then $M_\psi : \mathscr{H} \to \mathscr{H}$ is power bounded if and only if $\|\psi\|_\infty \leq 1$.*

Lemma 5.21. *Let $\psi \in \mathscr{H}$ where $\mathscr{H}$ is either $H(B)$, $H_b(B)$ or $H^\infty(B)$. If $M_\psi : \mathscr{H} \to \mathscr{H}$ is mean ergodic, then $\|\psi\|_\infty \leq 1$.*

Proof. Identity (1.10) holds for any operator, in particular we have

$$\frac{M_\psi^n}{n} = \frac{n+1}{n}(M_\psi)_{[n+1]} - (M_\psi)_{[n]}. \tag{5.9}$$

Thus, if M_ψ is mean ergodic we obtain $\lim_{n\to\infty} \frac{M_\psi^n}{n} = 0$ pointwise in $\mathscr{H}$. Take the function $1 \in \mathscr{H}$ and fix $x \in B$. Then $\lim\limits_{n\to\infty} \frac{\psi(x)^n}{n} = 0$. Clearly we have $|\psi(x)| \leq 1$ and, since $x \in B$ was arbitrary, the proof is complete. □

As a consequence we obtain the following:

Proposition 5.22. *Let $\psi \in \mathscr{H}$ where $\mathscr{H}$ is either $H(B)$, $H_b(B)$ or $H^\infty(B)$. If $M_\psi : \mathscr{H} \to \mathscr{H}$ is mean ergodic, then it is also power bounded.*

As a straightforward consequence of these results, together with Proposition 1.15 we have the following characterisation:

Corollary 5.23. *Let $\psi \in H(B)$. Consider $M_\psi \colon H(B) \to H(B)$. Then the following are equivalent:*

a) M_ψ *is power bounded.*

b) M_ψ *is uniformly mean ergodic.*

c) M_ψ *is mean ergodic.*

d) $\|\psi\|_\infty \leq 1$.

For $H_b(B)$ and $H^\infty(B)$ we make first a simple observation. Let $x_0 \in B$ be such that $|\psi(x_0)| \leq 1$. For $\psi(x_0) \neq 1$ we have

$$(M_\psi)_{[n]}(f)(x_0) = \frac{f(x_0)}{n}\sum_{m=0}^{n-1}\psi(x_0)^m = \frac{f(x_0)}{n}\frac{1-\psi(x_0)^n}{1-\psi(x_0)}. \tag{5.10}$$

Hence

$$h_f(x_0) = \lim_{n\to\infty}(M_\psi)_{[n]}(f)(x_0) = \begin{cases} f(x_0), & \text{if } \psi(x_0) = 1, \\ 0, & \text{if } \psi(x_0) \neq 1. \end{cases}$$

Now, for the case $\psi(x_0) = 1$, since we have $|\psi(x_0)| \leq 1$, the Maximum Principle (see [47, Proposition 5.9]) gives that $\psi \equiv 1$. We obtain the operator $M_1 = \mathrm{id}$, which is clearly uniformly mean ergodic. Avoiding this case we obtain a similar result to [13, Proposition 2.8].

Theorem 5.24. *Let $\mathscr{H}$ be either $H_b(B)$ or $H^\infty(B)$. Let $\psi \in \mathscr{H}$ be different from the constant function* 1. *Then the following are equivalent:*

a) M_ψ *is uniformly mean ergodic in* $\mathscr{H}$.

b) M_ψ *is mean ergodic in* $\mathscr{H}$.

c) $\|\psi\|_\infty \leq 1$ *and* $\frac{1}{1-\psi} \in \mathscr{H}$.

Proof. That a) implies b) is trivial.

Assume now that b) holds. Since $\mathscr{H}$ is barrelled we can apply [1, Theorem 2.4] to obtain

$$\mathscr{H} = \ker(\mathrm{id}_{\mathscr{H}} - M_\psi) \oplus \overline{\mathrm{Im}(\mathrm{id}_{\mathscr{H}} - M_\psi)} = \ker(M_{1-\psi}) \oplus \overline{\mathrm{Im}(M_{1-\psi})}.$$

On the other hand, by Lemma 5.21 we have $\|\psi\|_\infty \leq 1$. An application of the Maximum Principle (see [47, Proposition 5.9]) yields $\psi(x) \neq 1$ for all $x \in B$. Necessarily $\ker(M_{1-\psi}) = \{0\}$, then $\mathscr{H} = \overline{\mathrm{Im}(M_{1-\psi})}$. By Proposition 4.3 the operator $M_{1-\psi}$ is continuous. The Closed Graph Theorem implies

$$\mathscr{H} = \mathrm{Im}(M_{1-\psi}).$$

In other words, $M_{1-\psi} = \mathrm{id}_{\mathscr{H}} - M_\psi$ is an isomorphism of $\mathscr{H}$ onto itself. Hence, by Lemma 5.15, we have $\frac{1}{1-\psi} \in \mathscr{H}$ and c) holds.

Assume c) is satisfied for $\mathscr{H} = H_b(B)$. Clearly, $\psi(x) \neq 1$ for all $x \in B$. Let $V \subset H_b(B)$ be a bounded set and fix $0 < r < 1$. Then, by (5.10), we have

$$\lim_{n\to\infty} \sup_{f\in V} \sup_{x\in rB} \left|(M_\psi)_{[n]}(f)(x)\right| = \lim_{n\to\infty} \sup_{f\in V} \sup_{x\in rB} \left|\frac{f(x)}{n} \frac{1-\psi(x)^n}{1-\psi(x)}\right|$$
$$\leq \sup_{f\in V} \sup_{x\in rB} |f(x)| \lim_{n\to\infty} \frac{2}{n} \sup_{x\in rB} \left|\frac{1}{1-\psi(x)}\right| = 0.$$

Since $0 < r < 1$ and V were arbitrary we obtain that M_ψ is uniformly mean ergodic on $H_b(B)$. For $\mathscr{H} = H^\infty(B)$, the proof of c) implies a) follows the same steps, replacing the sets rB with the whole open unit ball B. □

Observe that, by Lemma 5.15 and Proposition 5.16, condition c) in Theorem 5.24 is equivalent to $\|\psi\|_\infty \leq 1$ and $1 \notin \overline{\psi(B)}$ when $\mathscr{H} = H^\infty(B)$. If we consider c) for $\mathscr{H} = H(B)$ we can see that it is equivalent to $\|\psi\|_\infty \leq 1$ and $1 \notin \psi(B)$ or simply $\|\psi\|_\infty \leq 1$ if we take ψ different from the constant 1. This explains why the notions of (uniform) mean ergodicity and power boundedness coincide in Corollary 5.23. However, since the spectrum of M_ψ in $H_b(B)$ is not completely described, the question of whether every power bounded multiplication operator $M_\psi\colon H_b(B) \to H_b(B)$ is mean ergodic remains open.

5.6 Examples

Examples on ergodic properties of $C_{\psi,\varphi}$ can be found on Chapter 3 for the special case of $\psi \equiv 1$. Here we present a composition operator that is power bounded in $H_b(B_{c_0})$, $H^\infty(B_{c_0})$ and on $H(B_{c_0})$, but when taking a particular weight, the weighted composition operator is not power bounded in any of these spaces.

Example 5.25. Let $w \in \mathscr{H}$ defined by $w(x) = 1 + x_1$ (where $\mathscr{H}$ is either $H^\infty(B_{c_0})$, $H_b(B_{c_0})$ or $H(B_{c_0})$). Consider $\Sigma\colon B_{c_0} \to B_{c_0}$ the backward shift (see (3.4)). Then $C_\Sigma\colon \mathscr{H} \to \mathscr{H}$ is power bounded but the operator $C_{w,\Sigma}\colon \mathscr{H} \to \mathscr{H}$ is not power bounded.

Since Σ has stable orbits and B_{c_0}-stable orbits, $C_\Sigma\colon \mathscr{H} \to \mathscr{H}$ is power bounded (see Theorem 3.11 and Theorem 3.14). To see that $C_{w,\Sigma}\colon \mathscr{H} \to \mathscr{H}$ is not power bounded it is enough to show that the set $(w^{[n]})_n$ is not bounded in $\mathscr{H}$ (see Proposition 5.6, Theorem 5.10 and Proposition 5.13). Consider $y = (\frac{1}{i+1})_i \in B_{c_0}$, then for any $n \in \mathbb{N}$

we have

$$w_{[n]}(y) = \prod_{m=0}^{n-1} w(\Sigma^m(y)) = \prod_{m=0}^{n-1} w\left(\left(\frac{1}{i+1+m}\right)_i\right) = \prod_{m=0}^{n-1}\left(1+\frac{1}{2+m}\right)$$
$$= \prod_{m=0}^{n-1}\left(\frac{3+m}{2+m}\right) = \frac{n+2}{2}.$$

And the set $(w^{[n]})_n$ is unbounded in $\mathscr{H}$.

Examples 5.26. The following composition operators are not power bounded but when we take a particular ψ, the operators $C_{\psi,\varphi}$ become power bounded.

a) Consider $\varphi : B_{c_0} \to B_{c_0}$ defined by

$$\phi(x) = \left(\tfrac{x_1}{2} + \tfrac{1}{2}, 0, \dots\right)$$

and the weight $\psi(x) = x_2$. Then $C_{\psi,\phi} : H_b(B_{c_0}) \to H_b(B_{c_0})$ is power bounded, but $C_\phi : H_b(B_{c_0}) \to H_b(B_{c_0})$ is not. In fact, $\psi \circ \phi \equiv 0$ and Remark 5.1 gives that the iterates of $C_{\psi,\phi}$ are eventually 0. But, by Example 3.6, ϕ does not have B_{c_0}-stable orbits and Theorem 3.14 gives that C_ϕ is power bounded.

b) Consider $F : B_{c_0} \to B_{c_0}$ the forward shift (see (3.4)) and the weight $\psi(x) = x_2$. Then $C_{\psi,F} : H(B_{c_0}) \to H(B_{c_0})$ is power bounded, but $C_F : H(B_{c_0}) \to H(B_{c_0})$ is not. In fact $\psi \circ F^2 \equiv 0$ but F does not have stable orbits on B_{c_0} (recall Remark 5.1, Example 3.5 and Theorem 3.14).

Note that in both examples the symbol is not an open map (recall Remark 5.2).

It is not difficult to construct multiplication operators that are neither mean ergodic nor power bounded, but when taking a particular symbol the associated weighted composition operator satisfies both properties.

Example 5.27. Let $\varphi : \mathbb{D} \to \mathbb{D}$ and $\psi \in H^\infty(\mathbb{D})$ defined by

$$\varphi(z) = \frac{z}{2} \quad \text{and} \quad \psi(z) = 2z.$$

Then M_ψ is not power bounded nor mean ergodic in $H^\infty(\mathbb{D})$ or $H(\mathbb{D})$ but $C_{\psi,\varphi}$ is power bounded and mean ergodic in $H^\infty(\mathbb{D})$ and $H(\mathbb{D})$. We check this facts first.

Observe that $\|\psi\|_\infty > 1$. Then M_ψ cannot be power bounded nor mean ergodic (see Proposition 5.20 and Lemma 5.21).

Clearly φ has $\mathbb{D}$-stable orbits. Now we check that the set $(\psi^{[n]})_n$ is bounded in $H^\infty(\mathbb{D})$ and therefore bounded in $H(\mathbb{D})$. Indeed, we have

$$\begin{aligned} |\psi^{[n]}(z)| &= \left|\prod_{m=0}^{n-1}(\psi \circ \varphi^m)(z)\right| = \left|\prod_{m=0}^{n-1} 2\left(z \cdot 2^{-m}\right)\right| \\ &= \left|z^n 2^{\sum_{m=0}^{n-1} 1-m}\right| \le |z^n| 2^{3-n} \le 2^{3-n}, \end{aligned} \tag{5.11}$$

for all $z \in \mathbb{D}$. By Theorem 5.10 and Proposition 5.13 we obtain the power boundedness of $C_{\psi,\varphi}$. Additionally, (5.11) implies that $\lim_{n\to\infty} \|\psi^{[n]}\|_\infty = 0$. For any $0 < r < 1$ since $\varphi^n(r\mathbb{D}) \subseteq r\mathbb{D}$ for every $n \in \mathbb{N}$, for an arbitrary $f \in H(\mathbb{D})$ we have

$$\lim_{n\to\infty} \sup_{|z|<r} \left|\psi^{[n]}(z) \cdot f(\varphi^n(z))\right| \le \sup_{|z|<r} |f(z)| \lim_{n\to\infty} \|\psi^{[n]}\|_\infty = 0.$$

Then $C^n_{\psi,\varphi}$ converges pointwise to 0 in $H(\mathbb{D})$ and $C_{\psi,\varphi}$ is mean ergodic in $H(\mathbb{D})$. The mean ergodicity of $C_{\psi,\varphi}$ in $H^\infty(\mathbb{D})$ can be checked with an analogous procedure.

There are weighted composition operators that are power bounded and not mean ergodic in $H^\infty(B)$. In fact, we give an example of multiplication operator on $H^\infty(\mathbb{D})$ with this property.

Example 5.28. Let $M_\psi : H^\infty(\mathbb{D}) \to H^\infty(\mathbb{D})$ be given by $\psi(z) = \frac{1+z}{2}$.
Then M_ψ is power bounded but not (uniformly) mean ergodic because the function $\frac{1}{1-\psi}$ is not bounded in $\mathbb{D}$ (see Theorem 5.24). In fact, $1 \in \overline{\psi(\mathbb{D})}$.

Open problems. The following questions have arisen in this chapter and remain open.

- If $C_{\psi,\varphi} : H(B) \to H(B)$ is topologizable we do not know whether this implies that φ has stable orbits (cf. Proposition 5.5).
- In Proposition 3.16 we find composition operators in $H_b(B)$ which are power bounded and not mean ergodic, thus the same example is valid for weighted composition operators in $H_b(B)$. However, while every mean ergodic composition operators is power bounded in this space (see Proposition 3.15), we do not know if the same holds for weighted composition operators in $H_b(B)$.
- In contrast to Proposition 5.18, we do not know if the zero operator is the unique multiplication operator that is compact in $H(B)$.
- We do not know if every power bounded multiplication operators on $H_b(B)$ is also mean ergodic. This was the case in $H(B)$ (see Corollary 5.23) but not in $H^\infty(B)$ (see Example 5.28).

Bibliography

[1] A. A. Albanese, J. Bonet, and W. J. Ricker. Mean ergodic operators in Fréchet spaces. *Ann. Acad. Sci. Fenn. Math.*, 34(2):401–436, 2009.

[2] J. M. Ansemil and S. Ponte. An example of a quasinormable Fréchet function space which is not a Schwartz space. In *Functional analysis, holomorphy, and approximation theory (Proc. Sem., Univ. Fed. Rio de Janeiro, Rio de Janeiro, 1978)*, volume 843 of *Lecture Notes in Math.*, pages 1–8. Springer, Berlin, 1981.

[3] R. Aron, P. Galindo, and M. Lindström. Compact homomorphisms between algebras of analytic functions. *Studia Math.*, 123(3):235–247, 1997.

[4] F. Bayart and E. Matheron. *Dynamics of linear operators*, volume 179 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2009.

[5] M. J. Beltrán-Meneu, M. C. Gómez-Collado, E. Jordá, and D. Jornet. Mean ergodic composition operators on Banach spaces of holomorphic functions. *J. Funct. Anal.*, 270(12):4369–4385, 2016.

[6] M. J. Beltrán-Meneu, M. C. Gómez-Collado, E. Jordá, and D. Jornet. Mean ergodicity of weighted composition operators on spaces of holomorphic functions. *J. Math. Anal. Appl.*, 444(2):1640–1651, 2016.

[7] T. Bermúdez, A. Bonilla, V. Müller, and A. Peris. Cesàro bounded operators in Banach spaces. *J. Anal. Math.*, 140(1):187–206, 2020.

[8] J. Bonet, B. de Pagter, and W. J. Ricker. Mean ergodic operators and reflexive Fréchet lattices. *Proc. Roy. Soc. Edinburgh Sect. A*, 141(5):897–920, 2011.

[9] J. Bonet and P. Domański. A note on mean ergodic composition operators on spaces of holomorphic functions. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, 105(2):389–396, 2011.

[10] J. Bonet, P. Domański, M. Lindström, and J. Taskinen. Composition operators between weighted Banach spaces of analytic functions. *J. Austral. Math. Soc. Ser. A*, 64(1):101–118, 1998.

[11] J. Bonet, D. Jornet, D. Santacreu, and P. Sevilla-Peris. Compact weighted composition operators on spaces of holomorphic functions on Banach spaces. *J. Oper. Theory*, accepted for publication.

[12] J. Bonet and M. Lindström. Spaces of operators between Fréchet spaces. *Math. Proc. Cambridge Philos. Soc.*, 115(1):133–144, 1994.

[13] J. Bonet and W. J. Ricker. Mean ergodicity of multiplication operators in weighted spaces of holomorphic functions. *Arch. Math. (Basel)*, 92(5):428–437, 2009.

[14] P. A. Burlandy and L. A. Moraes. The spectrum of an algebra of weakly continuous holomorphic mappings. *Indag. Math. (N.S.)*, 11(4):525–532, 2000.

[15] D. Carando and S. Muro. Envelopes of holomorphy and extension of functions of bounded type. *Adv. Math.*, 229(3):2098–2121, 2012.

[16] B. Cascales and J. Orihuela. On compactness in locally convex spaces. *Math. Z.*, 195(3):365–381, 1987.

[17] M. D. Contreras and S. Díaz-Madrigal. Compact-type operators defined on H^∞. In *Function spaces (Edwardsville, IL, 1998)*, volume 232 of *Contemp. Math.*, pages 111–118. Amer. Math. Soc., Providence, RI, 1999.

[18] M. D. Contreras and A. G. Hernandez-Diaz. Weighted composition operators in weighted Banach spaces of analytic functions. *J. Austral. Math. Soc. Ser. A*, 69(1):41–60, 2000.

[19] C. C. Cowen and B. D. MacCluer. *Composition operators on spaces of analytic functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[20] A. Defant, D. García, M. Maestre, and P. Sevilla-Peris. *Dirichlet Series and Holomorphic Functions in High Dimensions*. New Mathematical Monographs. Cambridge University Press, 2019.

[21] J. C. Díaz and P. Domański. Reflexive operators with domain in Köthe spaces. *Manuscripta Math.*, 97(2):189–204, 1998.

[22] S. Dierolf and P. Domański. Factorization of Montel operators. *Studia Math.*, 107(1):15–32, 1993.

[23] J. Diestel. *Sequences and series in Banach spaces*, volume 92 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1984.

[24] S. Dineen. *Complex analysis on infinite-dimensional spaces*. Springer Monographs in Mathematics. Springer-Verlag London Ltd., London, 1999.

[25] C. J. Earle and R. S. Hamilton. A fixed point theorem for holomorphic mappings. In *Global Analysis (Proc. Sympos. Pure Math., Vol. XVI, Berkeley, Calif., (1968)*, pages 61–65. Amer. Math. Soc., Providence, R.I., 1970.

[26] M. Fabian, P. Habala, P. Hájek, V. Montesinos, and V. Zizler. *Banach space theory*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, New York, 2011. The basis for linear and nonlinear analysis.

[27] M. Fabian, P. Habala, P. Hájek, V. Montesinos Santalucía, J. Pelant, and V. Zizler. *Functional analysis and infinite-dimensional geometry*, volume 8 of *CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC*. Springer-Verlag, New York, 2001.

[28] P. Galindo, T. W. Gamelin, and M. Lindström. Fredholm composition operators on algebras of analytic functions on Banach spaces. *J. Funct. Anal.*, 258(5):1504–1512, 2010.

[29] P. Galindo, M. Lindström, and R. Ryan. Weakly compact composition operators between algebras of bounded analytic functions. *Proc. Amer. Math. Soc.*, 128(1):149–155, 2000.

[30] P. Galindo, M. Lindström, and N. Wikman. Spectra of weighted composition operators on analytic function spaces. *Mediterr. J. Math.*, 17(1):Paper No. 34, 22, 2020.

[31] P. Galindo, L. Lourenço, and L. A. Moraes. Compact and weakly compact homomorphisms on Fréchet algebras of holomorphic functions. *Math. Nachr.*, 236:109–118, 2002.

[32] D. García, M. Maestre, and P. Sevilla-Peris. Composition operators between weighted spaces of holomorphic functions on Banach spaces. *Ann. Acad. Sci. Fenn. Math.*, 29(1):81–98, 2004.

[33] M. González and J. M. Gutiérrez. Schauder type theorems for differentiable and holomorphic mappings. *Monatsh. Math.*, 122(4):325–343, 1996.

[34] A. Grothendieck. *Topological vector spaces*. Notes on Mathematics and its Applications. Gordon and Breach Science Publishers, New York-London-Paris, 1973. Translated from the French by Orlando Chaljub.

[35] J. Horváth. *Topological vector spaces and distributions. Vol. I*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1966.

[36] E. Jordá and A. Rodríguez-Arenas. Ergodic properties of composition operators on Banach spaces of analytic functions. *J. Math. Anal. Appl.*, 486(1):123891, 14, 2020.

[37] D. Jornet, D. Santacreu, and P. Sevilla-Peris. Mean ergodic composition operators in spaces of homogeneous polynomials. *J. Math. Anal. Appl.*, 483(1):123582, 11, 2020.

[38] D. Jornet, D. Santacreu, and P. Sevilla-Peris. Mean ergodic composition operators on spaces of holomorphic functions on a Banach space. *J. Math. Anal. Appl.*, 500(2):Paper No. 125139, 16, 2021.

[39] T. Kalmes. Power bounded weighted composition operators on function spaces defined by local properties. *J. Math. Anal. Appl.*, 471(1-2):211–238, 2019.

[40] H. Keshavarzi. Mean ergodic composition operators on $H^\infty(\mathbb{B}_n)$. *arXiv:2010.06938*, preprint, 2020.

[41] G. Köthe. *Topological vector spaces. I*. Translated from the German by D. J. H. Garling. Die Grundlehren der mathematischen Wissenschaften, Band 159. Springer-Verlag New York Inc., New York, 1969.

[42] G. Köthe. *Topological vector spaces. II*, volume 237 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science]*. Springer-Verlag, New York-Berlin, 1979.

[43] U. Krengel. *Ergodic theorems*, volume 6 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1985. With a supplement by Antoine Brunel.

[44] J. Lindenstrauss and L. Tzafriri. *Classical Banach spaces. I*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 92. Springer-Verlag, Berlin-New York, 1977. Sequence spaces.

[45] R. Meise and D. Vogt. *Introduction to functional analysis*, volume 2 of *Oxford Graduate Texts in Mathematics*. The Clarendon Press, Oxford University Press, New York, 1997. Translated from the German by M. S. Ramanujan and revised by the authors.

[46] M. Miyagi. A linear expression of polynomials on locally convex spaces, and holomorphic functions on (DF)-spaces. *Mem. Fac. Sci. Kyushu Univ. Ser. A*, 40(1):1–18, 1986.

[47] J. Mujica. *Complex analysis in Banach spaces*, volume 120 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1986.

[48] L. Narici and E. Beckenstein. *Topological vector spaces*, volume 296 of *Pure and Applied Mathematics (Boca Raton)*. CRC Press, Boca Raton, FL, second edition, 2011.

[49] P. Pérez Carreras and J. Bonet. *Barrelled locally convex spaces*, volume 131 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1987. Notas de Matemática [Mathematical Notes], 113.

[50] A. Renaud. Quelques propriétés des applications analytiques d'une boule de dimension infinie dans une autre. *Bull. Sci. Math. (2)*, 97:129–159 (1974), 1973.

[51] D. Sarason. *Complex function theory*. American Mathematical Society, Providence, RI, second edition, 2007.

[52] J. H. Shapiro. *Composition operators and classical function theory*. Universitext: Tracts in Mathematics. Springer-Verlag, New York, 1993.

[53] K. R. Stromberg. *Introduction to classical real analysis*. Wadsworth International Mathematics Series. Wadsworth International, Belmont, Calif., 1981.

[54] D. M. Vieira. Spectra of algebras of holomorphic functions of bounded type. *Indag. Math. (N.S.)*, 18(2):269–279, 2007.