ABSOLUTELY $(q, 1)$-SUMMING OPERATORS ACTING IN $C(K)$-SPACES AND THE WEIGHTED ORLICZ PROPERTY FOR BANACH SPACES

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ABSTRACT. We provide a new separation-based proof of the domination theorem for $(q, 1)$-summing operators. This result gives the celebrated factorization theorem of Pisier for $(q, 1)$-summing operators acting in $C(K)$-spaces. As far as we know, none of the known versions of the proof uses the separation argument presented here, which is essentially the same that proves Pietsch Domination Theorem for p-summing operators. Based on this proof, we propose an equivalent formulation of the main summability properties for operators, which allows to consider a broad class of summability properties in Banach spaces. As a consequence, we are able to show new versions of the Dvoretzky-Rogers Theorem involving other notions of summability, and analyze some weighted extensions of the q-Orlicz property.

Dedicated to our esteemed Professor ***Andreas Defant****, who long ago posed the problem that originated this work as a question to the second author.*

Summability and Orlicz property and factorization space 46E30 and 47B38 and 46B42

1. Introduction

The property of infinite dimensional Banach spaces containing weakly p-sumable series that are not norm p-sumable is one of the most relevant facts in functional analysis. The so called Dvoretzky-Rogers Theorem gives the strict limits of this result, establishing that actually this property is an exclusive feature of infinite dimensional spaces. Although this is a classical theorem, modern functional analysis provides a simple and elegant proof using the Pietsch's Factorization Theorem for p-summing operators: the identity map in a Banach space is p-summing if and only if it is finite dimensional. However, this is not true if we consider (q, p)-summing operators instead; we write $\Pi_{q,p}$ for this class. Recall that an operator $T : G \to E$ is (q, p)-summing $(1 \leq p, q < \infty)$, that is, $T \in \Pi_{q,p}(G, E)$, if there is a constant Q such that

$$\Big(\sum_{i=1}^n \|T(x_i)\|^q \Big)^{1/q} \le Q \sup_{x' \in B_{G^*}} \Big(\sum_{i=1}^n |\langle x_i, x' \rangle|^p \Big)^{1/p},$$

for every finite set $x_1, ..., x_n \in G$. As usual, we write $\pi_{q,p}(T)$ for the norm of T in the Banach space $\Pi_{q,p}(G, E)$, which coincides with the infimum of all the constants Q in the inequality above. If $q = p$ we have the ideal of p-summing operators. A direct computation shows that, in the case that the domain is a $C(K)$ space, the right hand side of the inequality above can be substituted by the simpler expression $Q \|(\sum_{i=1}^n |x_i|^p)^{1/p}\|_{C(K)}$.

There are a lot of examples of relevant (q, p)-summing operators. For instance, there are infinite dimensional Banach spaces —e.g. Hilbert spaces and in general cotype q-spaces— satisfying the q-Orlicz property, that is, the identity map is $(q, 1)$-summing (see the papers [18, 19] by Talagrand, see also [3, Chs.8,31,32] and [6, Ch.11]). The case $q = 2$ gives the so called Orlicz property, that is satisfied for all cotype 2 spaces (see [19], [3, 32.11] and the references therein).

A more general version of the Dvoretzky-Rogers Theorem can be found in [6, Th.10.5]. It states that for $1 \le p \le q < \infty$ such that $1/p - 1/q < 1/2$, every infinite dimensional Banach space contains a weakly p-summable sequence that fails to be norm q-summable, and the relation among the indices is optimal. The present paper is an attempt of giving a more general point of view by introducing homogeneous weights in the definition of weak summability. Our ideas are inspired by the characterization of the class of $(q, 1)$-summing operators acting in $C(K)$-spaces that provides Pisier's Factorization Theorem through Lorentz spaces $L^{q,1}(\lambda)$ (see [16]). Pisier's result can be presented as follows: an operator $T : C(K) \to E$ is $(q, 1)$-summing if and only if there are a constant $Q > 0$ and a probability measure $\lambda \in \mathcal{M}(K)$ —the space of all Borel regular measures on K— such that for every $f \in C(K)$, $\|T(f)\| \le Q \, (\int_K |f| \, d\lambda)^{1/q} \, \|f\|^{1/q'}$. An easy calculation and the standard separation argument that proves the Pietsch's Domination Theorem for p-summing operators show that this is also equivalent to the inequality

$$\Big(\sum_{i=1}^n \big\|T(f_i)\big\|^q \Big)^{1/q} \le Q \sup_{\eta \in B_{\mathcal{M}(K)}} \Big(\sum_{i=1}^n \big|\langle f_i, \eta \rangle\big| \, \big\|f_i\big\|^{q-1} \Big)^{1/q}, \qquad (1)$$

to hold for all finite sets $f_1, ..., f_n \in C(K)$; this will be shown in the Preliminaries section of the present paper. However, this equivalence is not true anymore if we consider a general Banach space instead of $C(K)$.

From a different point of view, the operator ideal defined by the inequality (1) for general Banach spaces was analyzed by Matter (see [11, 12]) as a consequence of an interpolation procedure developed by Jarchow and Matter ([7]). Using this method, if $0 \le \sigma < 1$, the operator ideal Π_1^σ can be constructed. A linear operator $T : G \to E$ belongs to the corresponding component $\Pi_1^\sigma(G,E)$ of the ideal if there is $Q > 0$ such that for all finite sequences $x_1, \cdots, x_n \in G$ we have

$$\Big(\sum_{i=1}^n \big\|T(f_i)\big\|^{\frac{1}{1-\sigma}}\Big)^{1-\sigma} \le Q \sup_{x' \in B_{G^*}} \Big(\sum_{i=1}^n \big|\langle f_i, x'\rangle\big| \, \big\|f_i\big\|^{\frac{\sigma}{1-\sigma}}\Big)^{1-\sigma}.$$

The key result that connects the characterization of $(q,1)$-summing operators acting in $C(K)$-spaces and the operator ideal defined by the interpolation procedure of Jarchow and Matter is the coincidence of $\Pi_{q,1}(C(K),E)$ and $\Pi_1^\sigma(C(K),E)$ for $q = 1/(1-\sigma)$. This fact was already noticed by Matter, and was specifically studied in [10].

There is a third point of view for approaching factorization of $(q,1)$-summing operators which was introduced by Kalton and Montgomery-Smith in [8]. In this paper, a domination of capacities by measures provides a technique for proving in an alternate way Pisier's Factorization Theorem. The same procedure was the starting point of the results in [5], in which a unified method for proving a broad class of known factorization theorems —Pisier's Theorem, Maurey-Rosenthal Theorem,...—, was given.

In this paper we show that in fact Pisier's Theorem can be proved using the same standard separation argument that proves Pietsch's Theorem and Maurey-Rosenthal's Theorem, just by adapting the proof of Pisier's Theorem shown in the excellent presentation given in the book of Diestel, Jarchow and Tongue ([6, Ch.10]). This determines our methodological approach, that will be applied for showing our generalization of the Dvoretzky-Rogers Theorem. Our main result is a characterization of the class of homogeneous weights for which the coincidence of weak weighted summable and norm summable sequences in a Banach space implies that it is finite dimensional (Theorem 3.7). In order to do this, we introduce the notion of weighted (q,ϕ)-Orlicz property for a Banach space as follows. Consider a Banach space G and a (positive) homogeneous function $\phi : G \to \mathbb{R}^+$. *The space G has the weighted (q,ϕ)-Orlicz property $(1 \le q < \infty)$ if there is a constant Q such that*

$$\Big(\sum_{i=1}^n \|x_i\|^q\Big)^{1/q} \le Q \sup_{x' \in B_{G^*}} \Big(\sum_{i=1}^n |\langle x_i, x'\rangle| \, \phi(x_i)^{q-1}\Big)^{1/q},$$

for every finite set $x_1, ..., x_n \in G$.

2. Preliminaries: A separation argument for proving Pisier's theorem on $(q,1)$-summing operators

As usual, for $1 \leq q < \infty$ we write q' for the real number given by $1/q + 1/q' = 1$. If E is a Banach space, we write B_E for its unit ball, S_E for its unit sphere, E^* for its dual space and $\mathcal{M}(B_{E^*})$ for the space of regular Borel measures on the unit ball B_{E^*}. We use standard Banach space definitions and notation.

We will work with (q,p)-summing operators ($1 \leq p, q < \infty$); the corresponding operator ideal is denoted by $\Pi_{q,p}$; we have already given the definition. The interpolated ideals introduced by Matter in [11] and referred to in the Introduction are defined as follows. An operator $T : G \to E$ is (q,σ)-absolutely continuous ($0 \leq \sigma \leq 1$), that is $T \in \Pi_1^\sigma(G,E)$, if and only if there is a constant Q such that

$$\Big(\sum_{i=1}^n \|T(x_i)\|^{\frac{q}{1-\sigma}} \Big)^{\frac{1-\sigma}{q}} \leq Q \sup_{x' \in B_{G^*}} \Big(\sum_{i=1}^n |\langle x_i, x' \rangle|^q \|x_i\|^{\frac{q\sigma}{1-\sigma}} \Big)^{\frac{1-\sigma}{q}},$$

for every finite set $x_1, ..., x_n \in G$.

The following result is of course well-known —it is Pisier's Theorem—. From the point of view of the general techniques of factorization of operators, it was a bit disappointing that the procedures for proving such results —at least, three of them, as was explained in the Introduction— did not use a Hahn-Banach/Ky Fan separation argument, as in the other known factorizations. What is relevant in the following proof is that it is supported by such an argument.

A family Ψ of real functions defined on a non-empty set is called concave if, for every convex combination ψ_c of every finite set $\{\psi_1, ..., \psi_n\} \subseteq \Psi$, there exists $\psi \in \Psi$ such that $\psi_c \leq \psi$.

Theorem 2.1. *Let $T : C(K) \to E$ be a bounded operator and $1 < q < \infty$. The following statements are equivalent.*

(i) *T is $(q,1)$-summing.*

(ii) *There are a regular Borel probability measure μ_0 on K and a constant $Q > 0$ such that for every function $f \in C(K)$,*

$$\|T(f)\| \leq Q \Big(\int_K |f| \, d\mu_0 \Big)^{1/q} \|f\|^{1/q'}.$$

(iii) *There is a constant $Q' > 0$ such that for every finite set of functions $f_1, ... f_n \in C(K)$,*

$$\Big(\sum_{i=1}^n \|T(f_i)\|^q \Big)^{1/q} \leq Q' \Big\| \sum_{i=1}^n \|f_i\|^{q-1} |f_i| \Big\|^{1/q}.$$

Proof. (i) $\Rightarrow$ (ii) Assume w.l.o.g. that $\pi_{q,1}(T) = 1$. Take $0 < \omega < 1$, and consider the concave set of convex functions $\Psi : (1+2\omega)\mathcal{P}(K) \to \mathbb{R}$ defined as

$$\Psi(\nu) := \sum_{i=1}^{n} \|T(f_i)\|^q - q \int_K \|f_i\|^{q-1} |f_i| \, d\nu,$$

where $(1+2\omega)\mathcal{P}(K) \subset (1+2\omega)B_{(C(K))^*}$ is the space of positive measures with variation less than or equal to $(1+2\omega)$ acting on K and considered with the weak* topology. By definition, all these functions are weak*-continuous.

Let us show that for each function Ψ there is a measure $\nu \in (1+2\omega)\,\mathcal{P}(K)$ such that $\Psi(\nu) \leq 0$. The following computations are based on an argument similar to the one presented in [6, Th.10.8]. The inequality $1-|1-x|^q \leq qx$ for $0 \leq x \leq 1$ is used. Take a finite set $f_1, ..., f_n \in C(K)$. Let $w_1, ..., w_n \in K$ such that $f_i(w_i) = \|f_i\|$, $i = 1, ..., n$, and δ_{w_i} the corresponding Dirac's deltas. Fix $\varepsilon > 0$, and assume w.l.o.g. that $\varepsilon < q\omega/n$ and $\varepsilon < \omega$. Recall that $\pi_{q,1}(T) = 1$, and let $h_1, ..., h_m \in C(K)$ such that $\sum_{k=1}^m |h_k| \leq (1+\varepsilon)^{1/q}$ and $\sum_{k=1}^m \|T(h_k)\|^q = 1$. Take $b_1^*, ..., b_m^* \in E^*$ such that $\sum_{k=1}^m \|b_k^*\|^{q'} = 1$ and $\sum_{k=1}^m \langle T(h_k), b_k^* \rangle = 1$. The function $g \mapsto \mu(g) := \sum_{k=1}^m \langle T(h_k\, g), b_k^* \rangle$, $g \in C(K)$, is clearly an element of $C(K)^* = \mathcal{M}(K)$ of variation $|\mu|$ less than or equal to $(1+\varepsilon)^{1/q}$, and such that $\mu(1) = 1$. Note that for a fixed i, taking into account that $\pi_{q,1}(T) = 1$, we have

$$\Big(\|T(f_i)\|^q + \sum_{k=1}^m \|T(h_k(\|f_i\| - |f_i|))\|^q \Big)^{1/q} \leq \Big\| |f_i| + \sum_{k=1}^m |h_k(\|f_i\| - |f_i|)| \Big\|$$

$$\leq \Big\| |f_i| + (\|f_i\| - |f_i|) \sum_{k=1}^m |h_k| \Big\| \leq \Big\| |f_i| + (\|f_i\| - |f_i|)(1+\varepsilon)^{1/q} \Big\| \leq (1+\varepsilon)^{1/q} \, \|f_i\|.$$

Then, computing the power q in the expression above and summing for all i we get

$$\sum_{i=1}^n \|T(f_i)\|^q \leq \Big(\sum_{i=1}^n (1+\varepsilon)\|f_i\|^q \Big) - \sum_{i=1}^n \sum_{k=1}^m \|T(h_k(\|f_i\| - |f_i|))\|^q$$

$$\leq \sum_{i=1}^n \Big((1+\varepsilon)\|f_i\|^q - \Big| \sum_{k=1}^m \langle T(h_k(\|f_i\| - |f_i|)), b_k^* \rangle \Big|^q \Big)$$

$$\leq \varepsilon \Big(\sum_{i=1}^n \|f_i\|^q \Big) + \sum_{i=1}^n \|f_i\|^q \left(1 - \left|\mu(1 - \frac{|f_i|}{\|f_i\|})\right|^q\right)$$

$$\leq \varepsilon \Big(\sum_{i=1}^n \|f_i\|^q \Big) + \sum_{i=1}^n \|f_i\|^q \left(1 - \left|\mu(1) - \mu(\frac{|f_i|}{\|f_i\|})\right|^q\right)$$

$$\leq n\,\varepsilon\Big(\int_K \sum_{i=1}^n |f_i|(w)^q \, d\frac{\sum_{i=1}^n \delta_{w_i}}{n}\,(w)\Big) + \sum_{i=1}^n \|f_i\|^q\,\big(1-\big|1-|\mu|(\frac{|f_i|}{\|f_i\|})\big|^q\big)$$

$$\leq n\,\varepsilon\Big(\int_K \sum_{i=1}^n |f_i|(w)^q \; d\eta\,(w)\Big) + q\,\sum_{i=1}^n \|f_i\|^q|\mu|(\frac{|f_i|}{\|f_i\|})$$

$$\leq n\,\varepsilon\Big(\int_K \sum_{i=1}^n \|f_i\|^{q-1}|f_i|(w)\; d\eta\,(w)\Big) + q\Big(\int_K \sum_{i=1}^n \|f_i\|^{q-1}|f_i|\,d|\mu|\Big)$$

$$\leq q\int_K \sum_{i=1}^n \|f_i\|^{q-1}|f_i| \; d\,(\omega\,\eta+|\mu|),$$

where $\eta=\frac{\sum_{i=1}^n \delta_{w_i}}{n}$ is a probability measure, and so $(\omega\,\eta+|\mu|)(K)<1+\omega+\varepsilon<1+2\omega$. An application of Ky Fan's Lemma gives a probability measure μ_0 such that for every $f\in C(K)$,

$$\|T(f)\|\leq \big((1+2\omega)\,q\big)^{1/q}\Big(\int_K |f|\,d\mu_0\Big)^{1/q}\|f\|^{1/q'},$$

which proves (ii). Note that the constant $\big((1+2\omega)\,q\big)^{1/q}$ appearing in the final inequality tends to $q^{1/q}$ for $\omega\to 0$; a sequence can be defined with the associated measures, and a weak* limit can be found. This provides the probability measure in the statement and gives the estimate $q^{1/q}$ for the corresponding constant in the original proof of Pisier.

A direct computation shows that (ii) implies (iii). Finally, for (iii) implies (i) just consider the following calculations. If $f_1,...,f_n\in C(K)$, we have

$$\Big(\sum_{i=1}^n \big\|T(f_i)\big\|^q\Big)^{1/q}\leq Q'\,\Big\|\sum_{i=1}^n \|f_i\|^{q-1}|f_i|\Big\|^{1/q}$$

$$\leq Q'\,\Big\|\sum_{i=1}^n |f_i|\Big\|^{1/q}\cdot\max_{i=1,\dots,n}\|f_i\|^{1/q'}$$

$$\leq Q'\,\Big\|\sum_{i=1}^n |f_i|\Big\|^{1/q}\,\Big\|\sum_{i=1}^n |f_i|\Big\|^{1/q'} = Q'\,\Big\|\sum_{i=1}^n |f_i|\Big\|.$$

□

Similar arguments prove also the result for the case of the (q,p)-summing operators for $1\leq p<q$, which coincide with the $(q,1)$-summing operators (see Theorem 10.9 in [6]).

For the case of $(q,1)$-summing operators defined on a $C(K)$ space it is well-known that it is enough to consider the inequalities in the definition to hold for disjoint functions. Both statements —with general sets of functions or just with sets of disjoint functions— are equivalent, as a consequence of

Theorem 2.4 and Proposition 2.8 in [16], where the equivalences between (i) and (ii) and (ii) and (iii) in our Theorem 2.1, respectively, can be found.

Corollary 2.2. *For an operator $T : C(K) \to E$ the following statements are equivalent.*

(i) *There is a constant $Q > 0$ such that for every finite set of disjoint functions $f_1, ...f_n \in C(K)$,*

$$\Big(\sum_{i=1}^{n} \big\| T(f_i) \big\|^q \Big)^{1/q} \leq Q \, \Big\| \sum_{i=1}^{n} \|f_i\|^{q-1} |f_i| \Big\|^{1/q}.$$

(ii) *There is a constant $Q' > 0$ such that for every finite set of functions $f_1, ...f_n \in C(K)$,*

$$\Big(\sum_{i=1}^{n} \big\| T(f_i) \big\|^q \Big)^{1/q} \leq Q' \, \Big\| \sum_{i=1}^{n} \|f_i\|^{q-1} |f_i| \Big\|^{1/q}.$$

Proof. The proof follows easily from Proposition 16.12 of [6], where it is shown that the inequalities that give that an operator is $(q,1)$-summing hold if and only if they hold for disjoint functions. Indeed, note that for disjoint functions $f_1, ..., f_n \in C(K)$,

$$\Big\| \sum_{i=1}^{n} |f_i| \Big\| \leq \max_{i=1,\dots,n} \|f_i\| \leq \Big\| \sum_{i=1}^{n} \|f_i\|^{q-1} |f_i| \Big\|^{1/q}$$

$$\leq \Big\| \sum_{i=1}^{n} |f_i| \Big\|^{1/q} \max_{i=1,\dots,n} \|f_i\|^{1/q'} \leq \Big\| \sum_{i=1}^{n} |f_i| \Big\|.$$

□

3. Summability of operators and the weighted (q,ϕ)-Orlicz property for general Banach spaces

The equivalence between (i) and (iii) in Theorem 2.1 allows us to consider two different formulations of the notion of $(q,1)$-summability, and opens the door to a new class of domination closely related to this kind of summability for operators and spaces. Indeed, statement (i) in Theorem 2.1 when applied to the identity operator on a Banach space E gives the definition of the q-Orlicz property for E. However, (iii) gives a different summability property. Some simple calculations that we write below in Remark 3.3 show that the identity map on a Banach space G is weighted $(q, \| \cdot \|)$-summing— that is, $(1,\sigma)$-absolutely continuous for $\sigma = 1/q'$— if and only if G is finite dimensional. It is natural to ask which requirements for a weight function ϕ appearing in the summability inequality lead to the same conclusion.

Therefore, in this section we will analyze which are the properties that are needed for assuring that the identity map in a Banach space is weighted (q,ϕ)-summing. The question that we face is in which sense the Dvoretzky-Rogers Theorem works for other types of weighted summability. Let us fix first the main definitions.

Definition 3.1. Let E, G be Banach spaces. Fix a homogeneous function $\phi : G \to \mathbb{R}^+$. Let $1 \leq q < \infty$. We say that an operator $T : G \to E$ is weighted (q,ϕ)-summing if and only if there is $K > 0$ such that for every finite sequence $x_1, ..., x_n \in G$,

$$\Big(\sum_{i=1}^n \|T(x_i)\|^q \Big)^{1/q} \leq K \sup_{x' \in B_{G^*}} \Big(\sum_{i=1}^n |\langle x_i, x' \rangle| \phi(x_i)^{q-1} \Big)^{1/q}.$$

Note that for $\phi(\cdot) \leq \|\cdot\|$ we have that being weighted (q,ϕ)-summing implies that the operator is $(q,1)$-summing. Indeed, for $x_1, ..., x_n \in G$,

$$\Big(\sum_{i=1}^n \|T(x_i)\|^q \Big)^{1/q} \leq K \sup_{x' \in B_{G^*}} \Big(\sum_{i=1}^n |\langle x_i, x' \rangle| \phi(x_i)^{q-1} \Big)^{1/q}$$

$$\leq K \max_{i=1,\dots,n} \|x_i\|^{1/q'} \sup_{x' \in B_{G^*}} \Big(\sum_{i=1}^n |\langle x_i, x' \rangle| \Big)^{1/q} \leq K \sup_{x' \in B_{G^*}} \Big(\sum_{i=1}^n |\langle x_i, x' \rangle| \Big).$$

Definition 3.2. Let G be a Banach space. Fix a homogeneous function $\phi : G \to \mathbb{R}^+$. Let $1 \leq q < \infty$. We say that G has the (weighted) (q,ϕ)-Orlicz property if and only if the identity map on G is weighted (q,ϕ)-summing.

We will say that the space G satisfies the weighted (q,ϕ)-Orlicz property, and we will consider it for a general homogeneous function ϕ. However, due to the next remark we can restrict our attention to the case of functions ϕ which do not satisfy an inequality $\phi(\cdot) \leq \|\cdot\|$.

Remark 3.3. Let us show that a domination like the one given by the identity map on G being weighted $(q, \|\cdot\|)$-summing implies that G is finite dimensional. Indeed, for $x_1, ..., x_n \in G$,

$$\Big(\sum_{i=1}^n \|x_i\|^q \Big)^{1/q} \leq K \sup_{x' \in B_{G^*}} \Big(\sum_{i=1}^n |\langle x_i, x' \rangle| \|x_i\|^{q-1} \Big)^{1/q}$$

$$\leq K \sup_{x' \in B_{G^*}} \Big(\sum_{i=1}^n |\langle x_i, x' \rangle|^q \Big)^{1/q^2} \cdot \Big(\sum_{i=1}^n \|x_i\|^{q'(q-1)} \Big)^{1/(q'q)}$$

$$= K \sup_{x' \in B_{G^*}} \Big(\sum_{i=1}^n |\langle x_i, x' \rangle|^q \Big)^{1/q^2} \cdot \Big(\sum_{i=1}^n \|x_i\|^q \Big)^{1/(q'q)}.$$

Thus,

$$\Big(\sum_{i=1}^{n} \|x_i\|^q\Big)^{1/q-1/(q'q)} \leq K \sup_{x' \in B_{G^*}} \Big(\sum_{i=1}^{n} |\langle x_i, x'\rangle|^q\Big)^{1/q^2},$$

that is,

$$\Big(\sum_{i=1}^{n} \|x_i\|^q\Big)^{1/q} \leq K^q \sup_{x' \in B_{G^*}} \Big(\sum_{i=1}^{n} |\langle x_i, x'\rangle|^q\Big)^{1/q}.$$

Thus, the identity map is q-summing, and so G is finite dimensional.

Therefore, we have that for weight functions ϕ that are norm bounded (functions for which there is a constant $K > 0$ such that $\phi(\cdot) \leq K\|\cdot\|$), the space G satisfies the (q, ϕ)-Orlicz property if and only if it is finite dimensional. In the rest of this section we will show how far we can extend this result to spaces having the (q, ϕ)-Orlicz property. The key argument is given by the fact that this kind of domination implies that the operator is completely continuous.

We will write i for the canonical isometric map $G \hookrightarrow C(B_{G^*})$, $x \rightsquigarrow \langle x, \cdot\rangle$, $x \in G$. Let η be a probability measure on B_{G^*}. Let us define the function $\Phi_{q,\phi,\eta} : G \to \mathbb{R}^+$ as

$$\Phi_{q,\phi,\eta}(x) := \Big(\int_{B_{G^*}} |\langle x, \cdot\rangle|\, d\eta(\cdot)\Big)^{1/q} \phi(x)^{1/q'}, \quad x \in G.$$

We also define the space $G_{q,\phi}(\eta)$ as the closure of the quotient of the space G defined by the semi-norm

$$\|x\|_{G_{q,\phi}(\eta)} := \inf\Big\{\sum_{i=1}^{n} \Phi_{q,\phi,\eta}(x_i) \,:\, x = \sum_{i=1}^{n} x_i, x_i \in G\Big\}, \quad x \in G.$$

This is what we call a factorization space for an operator T; the formulation and general results on these spaces can be found in [1]. Indeed, the following result is an immediate consequence of Theorem 2.4 in [1].

Proposition 3.4. *An operator $T : G \to E$ is weighted (q, ϕ)-summing if and only if there is a probability measure η and a factorization for it as*

$$\begin{array}{ccc} G & \xrightarrow{\quad T \quad} & E \\ {\scriptstyle i}\downarrow & & \uparrow{\scriptstyle T_0} \\ i(G) & \xrightarrow{[i]_{G_{q,\phi}(\eta)}} & G_{q,\phi}(\eta). \end{array}$$

Proof. The standard separation argument that is used in this kind of results works also in this case (for example, the one based on Ky Fan's Lemma explained in Theorem 2.1). We do not write an explicit proof since it can be obtained as a direct application of Theorem 2.4 in [1] (see also Propositions

2.2 and 2.3 in this paper), as can be easily seen just by defining Φ in this theorem as

$$\Phi(\langle x,\cdot\rangle) = \big(|\langle x,\cdot\rangle|\,\phi(x)^{q-1}\big)^{1/q}, \quad x \in G.$$

□

Note that in Proposition 3.4, the operator $[i]_{G_{q,\phi}(\eta)}$ does not need to be continuous, although it is clearly well-defined. The following property, together with Lemma 3.6 provides the requirement for the operator to satisfy this property.

Definition 3.5. We will say that the function ϕ is **weakly null bounded** if for every weakly null sequence (x_i), $\sup_i \phi(x_i) < \infty$.

Lemma 3.6. *Let $1 < q < \infty$. Let $\phi : G \to \mathbb{R}^+$ be a positive homogeneous function. The following statements are equivalent for a sequence (x_i) of G.*

(i) *The sequence $(x_i\phi(x_i)^{q/q'})$ is weakly null.*
(ii) *$\Phi_{q,\phi,\eta}(x_i) \to_i 0$ for every probability measure $\eta \in \mathcal{M}(B_{G^*})$.*

Consequently, if η is a probability measure and the function ϕ is weakly null bounded then the map $[i]_{G_{q,\phi}(\eta)} : E \hookrightarrow G_{q,\phi}(\eta)$ is (well-defined, continuous and) completely continuous.

Proof. (i) ⇒ (ii) is given by a standard integration argument. Consider a probability measure η. Note that by the Uniform Boundedness Principle, the weakly null sequence $(x_i\phi(x_i)^{q/q'})$ is norm bounded by a real number r, that is, $\sup_i \|x_i\|\phi(x_i)^{q/q'} \le r$. We have that $\lim_i |\langle x_i, x'\rangle|\phi(x_i)^{q/q'} = 0$ for each $x' \in G^*$, and the function $r\cdot\chi_{B_{G^*}}(x')$ is a pointwise bound for all the functions $B_{G^*} \ni x' \mapsto |\langle x_i, x'\rangle|\phi(x_i)^{q/q'}$. Thus, an application of the Dominated Convergence Theorem gives

$$\Phi_{q,\phi,\eta}(x_i) = \Big(\int_{B_{G^*}} |\langle x_i,\cdot\rangle|\phi(x_i)^{q/q'}\,d\eta(\cdot)\Big)^{1/q} \to_i 0.$$

(ii) ⇒ (i). Take a sequence (x_i) and an element $x_0' \in B_{G^*}$. Suppose that $\Phi_{q,\phi,\eta}(x_i) \to_i 0$ for all probability measures η. In particular, for the Dirac's delta $\delta_{x_0'}$ we have that $\Phi_{q,\phi,\delta_{x_0'}}(x_i)$ equals

$$\Big(\int_{B_{G^*}} |\langle x_i, x'\rangle|\phi(x_i)^{q/q'}\,d\delta_{x_0'}(x')\Big)^{1/q} = |\langle x_i, x_0'\rangle|^{1/q}\phi(x_i)^{1/q'} \to_i 0,$$

which gives (i).

We prove now the final remark. Consider a probability measure η and a norm convergent sequence (x_i) in G. We can assume that it is a null sequence. Of course, it is also weakly null and norm bounded by a real number k. Then, since $r := \sup_i \phi(x_i)$ is also bounded by hypothesis, we have that

$$\lim_i |\langle x_i, x'\rangle| \phi(x_i) = 0 \quad \text{for each} \quad x' \in G^*.$$

Then

$$\|x_i\|_{G_{q,\phi}(\eta)} \leq \Big(\int_{B_{G^*}} |\langle x_i, \cdot\rangle| d\eta(\cdot) \Big)^{1/q} \phi(x_i)^{1/q'} \leq \Big(\int_{B_{G^*}} |\langle x_i, \cdot\rangle| d\eta(\cdot) \Big)^{1/q} \cdot r^{1/q'}.$$

Since for each $x' \in B_{G^*}$ we have that $|\langle x_i, x'\rangle| \to_i 0$ and $|\langle x_i, x'\rangle| \leq \sup_i \|x_i\| \leq k$, an application of the Dominated Convergence Theorem gives that

$$\|x_i\|_{G_{q,\phi}(\eta)} \leq \Big(\int_{B_{G^*}} |\langle x_i, \cdot\rangle| d\eta(\cdot) \Big)^{1/q} \cdot r^{1/q'} \to_i 0.$$

Consequently, (x_i) converges to 0 in the norm, and so the operator $[i]_{G_{q,\phi}(\eta)}$ is continuous. Note that the same argument —starting with a weakly null sequence instead of a norm null sequence— gives that the operator is completely continuous.

□

Theorem 3.7. *Let G be a Banach space. The following statements are equivalent.*

(i) *G is a reflexive space with the weighted (q,ϕ)-Orlicz property for a weakly null bounded homogeneous function $\phi : G \to \mathbb{R}^+$.*

(ii) *G is a reflexive space with the weighted (q,ϕ)-Orlicz property for every weakly null bounded homogeneous function $\phi : G \to \mathbb{R}^+$ satisfying that*
$$0 < \inf_{x \in S_G} \phi(x) \leq \sup_{x \in S_G} \phi(x) < \infty.$$

(iii) *G is finite dimensional.*

Proof. (iii) ⇒ (ii). Consider a weakly null bounded homogeneous function ϕ. Take a basis of norm one elements $\{b_1^*, ..., b_n^*\}$ for the finite dimensional space G^*, and the measure $\delta := \sum_{i=1}^n \delta_{b_i^*}/n$. By assumption there are positive constants K_0, K_1 such that for every $x \in S_G$,

$$K_0 \leq \phi(x) \leq K_1.$$

Thus, we have that there are positive constants A and B such that for all $x \in G$,

$$A \|x\|_G \leq \Big(\int_{B_{G^*}} |\langle x, x'\rangle| \, \phi(x)^{q/q'} \, d\delta(x') \Big)^{1/q} \leq B\|x\|_G.$$

Thus, for each finite set $x_1, ..., x_m \in G$ we have

$$A^q \sum_{i=1}^m \|x_i\|^q \leq \int_{B_{G^*}} \sum_{i=1}^m |\langle x_i, x'\rangle| \phi(x_i)^{q/q'} \, d\delta(x') \leq \sup_{x' \in B_{G^*}} \sum_{i=1}^m |\langle x_i, x'\rangle| \phi(x_i)^{q/q'},$$

and so id is weighted (q,ϕ)-summing.

(ii) $\Rightarrow$ (i). It is enough to consider $\phi(\cdot) = \|\cdot\|_G$, which provides a weakly null bounded homogeneous function that satisfies the inequality in (ii).

(i) $\Rightarrow$ (iii) is given by Lemma 3.6. It implies that id is (continuous and) completely continuous, and so compact by the factorization given in Proposition 3.4 and the reflexivity of G. Thus we have that G is finite dimensional.

□

To finish this section, let us show that this result is optimal, in the sense that all the requirements are needed for the result to hold under the assumption of the corresponding weighted Orlicz property. Let us see what happens in case we remove the property for ϕ to be weakly null bounded or the requirement for G to be reflexive.

After the results that we have shown, it seems natural to consider homogeneous functions $\phi : G \to \mathbb{R}^+$ satisfying the following requirements to know whether or not a Dvoretzky-Rogers Theorem still holds. For a certain measure η such that $i(G)$ can be "injectively" included in $L^1(\eta)$, we consider functions ϕ satisfying

$$A\,\|x\| \;\leq\; \phi(x) \;\leq\; B\,\frac{\|x\|^2}{\int_{B_{G^*}} |\langle x, x'\rangle|\, d\eta(x')}, \qquad x \in G, \tag{2}$$

for positive constants A and B. Next examples show what happens when a function ϕ is defined as the right hand bound of these inequalities.

Remark 3.8. *A reflexive infinite dimensional space with a weighted Orlicz property for a function ϕ that is not weakly null bounded.* Suppose that E is an infinite dimensional Banach space with separable dual and fix just for the aim of simplicity $q = 2$. Then we can easily define a measure η_0 in $\mathcal{M}(B_{E^*})$ as

$$\eta_0 : B_{E^*} \to \mathbb{R}^+, \quad \eta_0 = \sum_{i=1}^{\infty} \frac{1}{2^i}\,\delta_{x_i'},$$

where $\{x_i' : i \in \mathbb{N}\}$ is a countable set that is dense in B_{E^*}. Clearly, in this case we have that the inclusion map $i : E \to L^1(\eta_0)$ is injective. We can consider now the (positively) homogeneous function defined by

$$\phi_0(x) := \frac{\|x\|^2}{\int_{B_{E^*}} |\langle x, x'\rangle|\, d\eta_0}.$$

Then we have that for every $x \in E$,

$$\|x\| = \Big(\int_{B_{E^*}} |\langle x, x'\rangle|\, d\eta_0\Big)^{1/2} \phi_0(x)^{1/2}.$$

Note that for every finite set $x_1, ..., x_n \in E$,

$$\Big(\sum_{i=1}^{n} \|x_i\|^2\Big)^{1/2} = \Big(\sum_{i=1}^{n} \Big(\big(\int_{B_{E^*}} |\langle x, x'\rangle|\, d\eta_0\big)^{1/2} \big(\frac{\|x\|^2}{\int_{B_{E^*}} |\langle x, x'\rangle|\, d\eta_0}\big)^{1/2}\Big)^2\Big)^{1/2}$$

$$\leq \sup_{x' \in B_{E^*}} \Big(\sum_{i=1}^{n} |\langle x_i, x'\rangle| \phi_0(x_i)\Big)^{1/2}.$$

The identity map is then weighted $(2, \phi_0)$-summing. If we fix for example E to be $E := L^2[0,1]$, we have that the identity map in this space is weighted $(2, \phi_0)$-summing, the space is reflexive but the function ϕ_0 does not satisfy the requirement on the weak null boundedness property. Indeed, taking a weakly null sequence (x_i) in $L^2[0,1]$ defined by norm one elements, the Dominated Convergence Theorem gives that $\int_{B_{E^*}} |\langle x, x'\rangle|\, d\eta_0$ converges to 0, and so we have that $\phi_0(x_i) \to_i \infty$.

We have shown that the requirement on ϕ to be weakly null bounded is the key property that allows to prove that the factorization map —and so the identity map for spaces with the (q, ϕ)-Orlicz property—, is completely continuous. Next example shows that, even if the identity map is completely continuous and the space satisfies a weighted (q, ϕ)-Orlicz property, the space may be infinite dimensional if it is not reflexive.

Example 3.9. *A non-reflexive Banach space with a weighted* $(2, \phi)$*-Orlicz property and a completely continuous identity map.* Fix again $q = 2$. Due to the Schur's property of ℓ^1, the identity map $id : \ell^1 \to \ell^1$ is completely continuous. A construction similar to the one given in Remark 3.8 provides an example of completely continuous map associated to a space that is not finite dimensional. Indeed, define the measure $\nu : 2^{\mathbb{N}} \to \mathbb{R}^+$ by

$$\nu = \sum_{i=1}^{\infty} \frac{1}{2^i}\, \delta_{e_i},$$

where $\{e_i : i \in \mathbb{N}\}$ are the canonical functionals in $(\ell^1)^* = \ell^\infty$. The inclusion map $i : \ell^1 \to L^1(\nu)$ is obviously continuous and injective. A function ϕ_1 defined as ϕ_0 in Remark 3.8 gives again the equality

$$\|\cdot\|_{\ell^1} = \big(\int_{B_{\ell^\infty}} |\langle \cdot, x'\rangle|\, d\nu\big)^{1/2} \phi_1(\cdot)_{\ell^1}^{1/2}.$$

Therefore, the identity map is weighted $(2, \phi_1)$-summing and it is completely continuous.

4. Applications: weakly weighted absolutely summable Banach spaces

Let E and F be a pair of Banach spaces such that E is included continuously in F. In this section we will consider the weight homogeneous function

$$\phi(x) = \frac{\|x\|_E^2}{\|x\|_F}, \quad x \in E.$$

It allows to define a new version of summability of sequences in Banach spaces, that we call weakly weighted absolute summability. Our aim is to show an example of how our arguments can extend the techniques for proving summability results in Banach spaces, allowing to analyze more general versions of summation of series.

Definition 4.1. Let E and F be a couple of Banach spaces such that the inclusion $E \hookrightarrow F$ is continuous. Let us say that E is weakly weighted absolutely summable with respect to F if there is a constant $K > 0$ such that for every $x_1, ..., x_n \in E$,

$$\sum_{i=1}^n \|x_i\|_E \leq K \sup_{x' \in B_{E^*}} \sum_{i=1}^n |\langle \frac{x_i}{\|x_i\|_F}, x' \rangle| \, \|x_i\|_E.$$

Corollary 4.2. *Let E be a reflexive Banach space that is weakly weighted absolutely summable with respect to F. Suppose that $\phi(\cdot) := \| \cdot \|_E^2 / \| \cdot \|_F$ is weakly null bounded. Then E is finite dimensional.*

Proof. Note that just changing the set of vectors x_i by $x_i \|x_i\|_E$, we have that the inequality is equivalent to

$$\sum_{i=1}^n \|x_i\|_E^2 \leq K \sup_{x' \in B_{E^*}} \sum_{i=1}^n |\langle \frac{x_i}{\|x_i\|_F}, x' \rangle| \, \|x_i\|_E^2 = K \sup_{x' \in B_{E^*}} \sum_{i=1}^n |\langle x_i, x' \rangle| \, \frac{\|x_i\|_E^2}{\|x_i\|_F}.$$

Now, we simply apply (i) $\Rightarrow$ (iii) of Theorem 3.7 for $q = 2$ and ϕ defined as in the statement above. $\square$

Example 4.3. Let us show a concrete example of how Corollary 4.2 can be used. We are going to construct explicitly a function ϕ as the one defined above. Let $1 < q < \infty$. Consider E to be $L^q(\mu)$ with respect to a probability measure space (Ω, Σ, μ). If $f \in L^q(\mu)$, of course the function $|f|^q$ belongs to $L^1(\mu)$, and so the following definition makes sense.

Let η_0 be a regular Borel probability measure in $\mathcal{M}(B_{(L^1(\mu)^*})$ with support only in subsets of the positive cone of $L^1(\mu)^* = L^\infty(\mu)$. We can define a function-(semi)norm as

$$\|f\|_{q,\eta_0} := \Big(\int_{B_{(X(\mu)_{[q]})^*}} |\langle |f|^q, x' \rangle| \, d\eta_0(x') \Big)^{1/q}, \quad f \in L^q(\mu).$$

Note that this function satisfies the left hand side inequality in (2). Indeed, a direct computation using two times Hölder's inequality shows that for all $f \in L^q(\mu)$,

$$\Big(\int_{B_{L^\infty(\mu)}} |\langle |f|^q, x' \rangle| \, d\eta_0(x') \Big)^{1/q} \leq \big\| |f|^q \big\|_{L^1(\mu)}^{1/q} = \|f\|_{L^q(\mu)}.$$

Thus, assume that $\|f\|_{q,\eta_0}$ is a norm and consider the function

$$\phi(f) := \frac{\|f\|^2_{L^q(\mu)}}{\|f\|_{q,\eta_0}}, \quad f \in L^q(\mu).$$

That is, we consider as F the completion of $L^q(\mu)$ with respect to the norm $\|\cdot\|_{q,\eta_0}$. For example, this holds if η_0 is defined as the Dirac's delta δ_{h_0} for a function $0 < h_0 \in L^\infty(\mu)$, since then

$$\int_{B_{L^\infty(\mu)}} |\langle |f|^q, x' \rangle| \, d\delta_{h_0}(x') = \int_\Omega |f|^q \, h_0 \, d\mu, \qquad f \in L^1(\mu).$$

Let us analyze this case: *when is $L^q(\mu)$ weakly weighted absolutely summable with respect to F?* We have to consider two situations.

a) If we have a function $h_0 \in L^\infty(\mu)$ such that $1/h_0 \in L^\infty(\mu)$, then there is a constant $\varepsilon > 0$ such that $h_0 \geq \varepsilon \, \chi_\Omega$ $\mu-$a.e. In this case we have $\|\cdot\|_{q,\eta_0} \geq \varepsilon \|\cdot\|_{L^q(\mu)}$, and so the function ϕ satisfies that $\phi(\cdot) \leq \|\cdot\|_{L^q(\mu)}/\varepsilon$. Therefore, this case is already covered by Remark 3.3. However, let us show how to argue this with Corollary 4.2 to use the same argument in b). We have that ϕ is weakly null bounded, and so if $L^q(\mu)$ is weakly weighted absolutely summable with respect to F, Corollary 4.2 gives that it is finite dimensional. The converse statement is obvious, since two norms in a finite dimensional space are always equivalent and the identity map in a finite dimensional space is absolutely summing. Thus, *$L^q(\mu)$ is weakly weighted absolutely summable with respect to F if and only if it has finite dimension.*

b) In the case that $0 < h_0 \in L^\infty(\mu)$ but $1/h_0$ is not in $L^\infty(\mu)$, we can also use the same ideas locally. Fix $\varepsilon > 0$. Then we have a measurable set $A_\varepsilon \in \Sigma$, $A_\varepsilon := \{w \in \Omega : 0 < h_0 < \varepsilon\}$. Then, under the assumption that $L^q(\mu)$ is weakly weighted absolutely summable with respect to F, the same argument used above shows that the L^q-space defined as $L^q(\mu|_{A_\varepsilon})$ is finite dimensional too. By taking a countable sequence defined by $\varepsilon := 1/n$, we obtain a sequence of finite dimensional spaces $(L^q(\mu|_{A_{1/n}}))_n$. It can be easily seen that the dimension of (a subsequence of) the corresponding spaces is strictly increasing: just take a new function with support in $A_{1/(n+j)} \setminus A_{1/n}$ for the first j such that $\mu(A_{1/(n+j)} \setminus A_{1/n}) \neq 0$, and note that for each

n we always find a $j \in \mathbb{N}$ fulfilling this requirement. We obtain in this way an increasing sequence of subspaces of strictly increasing dimension inside of $L^q(\mu)$, so it cannot has finite dimension. Of course, by Corollary 4.2 this implies that ϕ is not weakly null bounded, or $L^q(\mu)$ is not weakly weighted absolutely summable. But more can be said: *if there is any $A_{1/n}$ which is not finite dimensional, then $L^q(\mu)$ cannot be weakly weighted absolutely summable with respect to F.* This happens for example for non-atomic measures: the measure space defined by Lebesgue measure over $[0, 1]$ is a clear example.

The authors would like to thank the referee for her/his careful reading of the manuscript and for her/his suggestions.

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