UNIVERSITAT POLITÈCNICA DE VALÈNCIA

Dept. of Applied Mathematics

Linear chaos for non-local operators and numerical methods in PDEs

Master's Thesis

Master's Degree in Mathematical Research

AUTHOR: Vargas Moreno, Álvaro

Tutor: Peris Manguillot, Alfredo

Cotutor: Murillo Arcila, Marina

ACADEMIC YEAR: 2021/2022

Contents

Summary

In this work we analyze the dynamics of some fractional operators. The objective is to establish the conditions under which these operators are chaotic. For this purpose, we will rely on proving chaos for the Toeplitz operators associated to these fractional operators. Likewise, we will also establish a relationship between chaos for certain numerical methods and the chaotic dynamics of certain operators that define these schemes. The dynamics of such operators depend on the sampling in time and space of the numerical method.

Resumen

En este trabajo se analiza la dinámica de algunos operadores fraccionarios. El objetivo es establecer las condiciones sobre las que estos operadores son caóticos. Para ello nos basaremos en probar caos para los operadores de Toeplitz asociados a dichos operadores fraccionarios. Asimismo, en este trabajo estableceremos una relación entre el caos para ciertos métodos numéricos y la dinámica caótica que definen estos esquemas. La dinámica de estos últimos dependerá del paso espacio-temporal del método numérico.

Chapter 1

Preliminaries

In this section we recall some basic definitions and theorems that will be useful in this work. Some classical references where these results can be found are [22], [20], [6].

1.1 Metric, Banach, Fréchet and Hilbert spaces

The first basic definition is the notion of metric space, that will be of capital importance in order to define the Banach and Hilbert spaces.

Definition 1.1.1 *(**Metric space**). A real-valued function $d : X \times X \to \mathbb{R}$, defined for each pair of elements $x, y \in X$ is called a* metric *if it satisfies:*

(i) $d(x, y) \geqslant 0$, $d(x, x) = 0$ *and* $d(x, y) > 0$ *if* $x \neq y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, z) \leqslant d(x, y) + d(y, z)$, *the triangle inequality.*

A set X provided with a metric is called a metric space and $d(x, y)$ is called the distance between x and y.

We will understand by a neighborhood of a point $p \in X$ a set $\mathcal{U} \subset X$, which contains an open set $\mathcal{V}$ containing p.

A point x in a metric space X is called isolated if there exists some neighbourhood of x which does not contain any other point from X.

A metric space is said to be locally compact if each point has a compact neighbourhood. Finally, we say that a metric space is complete if every Cauchy sequence in X converges to an element of X.

Theorem 1.1.2 ***(Baire category theorem)***. *Let (X, d) be a complete metric space and $\{G_n\}_n$ a sequence of nonempty dense open sets. Then $G := \bigcap_{n=1}^{\infty} G_n$, is a dense G_δ-set in X.*

Definition 1.1.3 ***(Seminorm)***. *A functional $p : X \to \mathbb{R}_+$ on a vector space X over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is called a seminorm if it satisfies, for all $x, y \in X$ and $\lambda \in \mathbb{K}$,*

(i) $p(x + y) \leqslant p(x) + p(y)$

(ii) $p(\lambda x) = |\lambda| p(x)$.

If, in addition,

(iii) $p(x) = 0$ *implies that* $x = 0$,

then p is called a norm.

Definition 1.1.4 ***(Fréchet space)***. *A Fréchet space is a vector space X endowed with a separating increasing sequence $(p_n)_n$ of seminorms which is complete when endowed with the metric given by:*

$$d(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} min(1, p_n(x - y)), x, y \in X$$

Definition 1.1.5 ***(Normed space)***. *The pair $(X, || \bullet ||)$ is called a normed space where X is a vector space endowed with a norm $|| \bullet ||$.*
Every normed linear space may be regarded as a metric space, being $\|x - y\|$ the distance between x and y. A Banach space is a normed linear space which is complete when endowed with the metric defined by its norm.

Definition 1.1.6 ***(Hilbert space)*** *A Hilbert space H is a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product. We say that H is a complex inner product space if H is a complex vector space on which there is an inner product $\langle \bullet, \bullet \rangle : H \times H \to \mathbb{C}$ such that for every pair of elements $x, y \in H$ it is satisfied:*

(i) $\langle y, x \rangle = \overline{\langle x, y \rangle}$.

(ii) For all $a, b \in \mathbb{C}$:

$$\langle ax_1 + bx_2, y \rangle = a\langle x_1, y \rangle + b\langle x_2, y \rangle.$$

(iii) $\langle x, y \rangle \geqslant 0$, *and it is equal to 0 if and only if* $x = 0$.

The norm defined by the inner product $\langle \bullet, \bullet \rangle$ is the real-valued function:

$$\|x\| = \sqrt{\langle x, x \rangle}$$

and the distance between two points x,y $\in H$ is defined in terms of the norm by:

$$d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y \rangle}.$$

Proposition 1.1.7 *Let $(X, \|\|_X)$ and $(Y, \|\|_Y)$ be Banach spaces and let $T : X \to Y$ be a linear operator. The following four statements are equivalent:*

(i) T is continuous at 0.

(ii) T is continuous.

(iii) T is uniformly continuous.

(iv) T is bounded, i.e., there exists a constant $C > 0$ such that $\|Tx\|_Y \leqslant C\|x\|_X$ for all $x \in X$.

Definition 1.1.8 *Let X and Y be Banach spaces. We denote by $\mathcal{L}(X, Y)$ the space of continuous linear operators $T : X \to Y$ under the operator norm. The space $\mathcal{L}(X, Y)$ is a Banach space whenever Y is a Banach space. If $\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$, the* dual *$X^* = \mathcal{L}(X, \mathbb{K})$ of a Banach space X is the space of all continuous linear functionals on X. If $x^* \in X^*$ then we write,*

$$x^*(x) := \langle x, x^* \rangle, \quad x \in X.$$

The adjoint *$T^* : X^* \to X^*$ of an operator T on X is defined by $T^*x^* = x^* \circ T$, that is,*

$$\langle x, T^*x^* \rangle = \langle Tx, x^* \rangle, \quad x \in X, \ x^* \in X^*.$$

Theorem 1.1.9 ***(Hahn-Banach theorem)****. Let X be a vector space, M a subspace of X, p a seminorm on X and $u : M \to \mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) a linear functional such that $|u(x)| \leqslant p(x)$ for all $x \in M$. Then u has a linear extension $\hat{u}$ to X such that $|\hat{u}(x)| \leqslant p(x)$ for all $x \in X$.*

The next corollary is an immediate consequence of the Hahn-Banach theorem.

Corollary 1.1.10 *If p is a seminorm on X and $x_0 \in X$ then there exists a linear functional u on X such that $u(x_0) = p(x_0)$ and $|u(x)| \leqslant p(x)$ for all $x \in X$.*

Moreover, if X is a Frèchet space, then we have the following corollaries of the Hahn-Banach theorem.

Corollary 1.1.11

(i) Every continuous linear functional on a subspace of X extends to a continuous linear on X. Moreover, if X is a Banach space this extension preserves the norm.

(ii) If M is a closed subspace of X and $x \notin M$ then there exists a continous linear functional x^* on X that vanishes on M with $\langle x, x^* \rangle \neq 0$.

(iii) A subspace M is dense in X if and only if every continous linear functional that vanishes on M also vanishes on X.

(iv) For any $x \in X$, if $\langle x, x^* \rangle = 0$ for all $x^* \in X^*$ then $x = 0$.

1.2 Classical Banach and Hilbert spaces

In this section we recall some classical Banach and Hilbert spaces.

Definition 1.2.1 *Let $1 \leqslant p < \infty$. Then we define the space*

$$\ell^p := \{x = (x_n)_n \in \mathbb{K}^{\mathbb{N}_0} : \quad \sum_{n=0}^{\infty} |x_n|^p < \infty\}$$

of p-summable sequences. This space when endowed with the norm $\|x\| := \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}$ is a Banach space.

One particularly important ℓ^p space for our work is ℓ^2, which endowed with the inner product $\langle x, y\rangle := \sum_{n=1}^{\infty} x_n \overline{y_n}$ is a Hilbert space.

Definition 1.2.2 *The space*

$$\ell^\infty := \{x = (x_n)_n \in \mathbb{K}^{\mathbb{N}_0} : \quad \sup_{n\in\mathbb{N}_0} |x_n| < \infty\}$$

is a Banach space when endowed with the norm $\|x\| := \sup_{n\in\mathbb{N}_0} |x_n|$.

Definition 1.2.3 *Let $a < b$ and $1 \leqslant p < \infty$. Then we define*

$$L^p[a, b] := \left\{ f : [a, b] \to \mathbb{K} : \quad f \text{ is measurable and } \int_a^b |f(t)|^p dt < \infty \right\}$$

as the space of p-integrable functions which endowed with the norm $\|f\| := \left(\int_a^b |f(t)|dt\right)^{1/p}$ $L^p[a, b]$ is a Banach space.

In particular, $L^2[a, b]$ when endowed with the inner product $\langle f, g\rangle := \int_a^b f(t)\overline{g(t)}dt$ is a Hilbert space. In the proposition 1.3.2 we will use the fact that the functions $t \to \frac{1}{\sqrt{2\pi}} e^{int}, n \in \mathbb{Z}$, form an orthonormal basis in $L^2[0, 2\pi]$.

1.3 Hardy Spaces

In this section we recall the definition of the Hardy space H^2. This Hilbert space will be of capital importance in the development of the results of our work. A classical reference about Hardy spaces is [9].

Let $(a_n)_{n\geqslant 0}$ be a complex sequence such that $(a_n)_{n\geqslant 0} \in \ell^2(\mathbb{N}_0)$, the function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \ z \in \mathbb{C}, |z| < 1,$$

defines a holomorphic function on the complex open disk $\mathbb{D}$. With this construction in mind, we can define the Hardy space H^2 as follows.

Definition 1.3.1 *The Hardy Space H^2 is defined as the space of the holomorphic functions on the complex unit open disk, that is:*

$$H^2 = \{f : \mathbb{D} \to \mathbb{C} : \ f(z) = \sum_{n=0}^{\infty} a_n z^n, z \in \mathbb{D}, \ \textit{with } (a_n)_n \in \ell^2(\mathbb{N}_0)\}$$

The space H^2 is a Banach space when endowed with the norm:

$$\|f\| = \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2} \quad \text{when } f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

and it is a Hilbert space with the inner product:

$$\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n} \quad \text{when } f(z) = \sum_{n=0}^{\infty} a_n z^n \text{a and } g(z) = \sum_{n=0}^{\infty} b_n z^n.$$

The next proposition will present another equivalent representation of the Hardy space H^2 that will be useful in the development of our results.

Proposition 1.3.2 *A holomorphic function $f : \mathbb{D} \to \mathbb{C}$ belongs to H^2 if and only if*

$$\sup_{0 \leqslant r < 1} \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{it}\right) \right|^2 dt < \infty.$$

Proof. As $f(z)$ is a holomorphic function we can write $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then:

$$\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{it}\right) \right|^2 dt = \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=0}^{\infty} a_n \left(re^{it}\right)^n \right|^2 dt = \int_0^{2\pi} \left| \sum_{n=0}^{\infty} a_n r^n \frac{1}{\sqrt{2\pi}} e^{int} \right|^2 dt.$$

Using the Parseval's identity for the orthonormal basis $(\frac{1}{\sqrt{2\pi}} e^{int})_{n \in \mathbb{Z}}$ of $L^2([0, 2\pi])$:

$$\int_0^{2\pi} \left| \sum_{n=0}^{\infty} a_n r^n \frac{1}{\sqrt{2\pi}} e^{int} \right|^2 dt = \sum_{j=-\infty}^{\infty} \left| \int_0^{2\pi} \overline{\frac{1}{\sqrt{2\pi}} e^{ijt} \sum_{n=0}^{\infty} a_n r^n \frac{1}{\sqrt{2\pi}} e^{int} dt} \right|^2$$

$$= \frac{1}{(2\pi)^2} \sum_{j=-\infty}^{\infty} \left| \overline{\int_0^{2\pi} e^{-ijt} \sum_{n=0}^{\infty} a_n r^n e^{int} dt} \right|^2 = \frac{1}{(2\pi)^2} \sum_{n=0}^{\infty} \left| \overline{2\pi a_n r^n} \right|^2 = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

Finally, taking the supreme we obtain:

$$\sup_{0 \leqslant r < 1} \sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \lim_{r \to 1^-} \sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \|f\|. \quad \blacksquare$$

1.4 Spectral theory

In the present work we will use some results about hypercyclic operators in terms of its spectrum. In order to get a better understanding of these results, in this section we will present some basic results of functional analysis regarding the spectral theory.

Definition 1.4.1 *Let X be a complex Banach space X and let T be an operator on X. The spectrum $\sigma(T)$ of T is defined as*

$$\sigma(T) = \{\lambda \in \mathbb{C} \ ; \ \lambda I - T \text{ is not invertible}\}.$$

Moreover, each $0 \neq x \in X$ satisfying $Tx = \lambda x$ is an eigenvector for T corresponding to λ.
The point spectrum $\sigma_p(T)$ is the set of eigenvalues of T.
The number $r(T) := \sup_{\lambda \in \sigma(T)} |\lambda|$ is called the spectral radius of T.
For the spectral radius we have that

$$r(T) = \lim_{n \to \infty} \|T^n\|^{1/n}.$$

Theorem 1.4.2 *(**Riesz decomposition theorem**) If $\sigma(T) = \sigma_1(T) \cup \sigma_2(T)$, where σ_1 and σ_2 are two disjoint non-empty closed sets, then there are non-trivial T-invariant closed subspaces M_1 and M_2 of X such that $X = M_1 \oplus M_2$,*

$$\sigma(T \mid M_1) = \sigma_1 \quad \text{and} \quad \sigma(T \mid M_2) = \sigma_2.$$

Theorem 1.4.3 *(**Point spectral mapping theorem**) Let f be a holomorphic function on an open neighborhood O of $\sigma(T)$ that is not constant on any connected component of O. Then*

$$\sigma_p(T) = f(\sigma_p(T)).$$

Chapter 2

Linear Dynamical Systems

In this chapter we present some definitions and results about linear dynamical systems. The theory of dynamical systems studies the behaviour of evolving systems and is used in a wide variety of fields ranging from biological or medical modeling to engineering. All the results presented in this chapter can be found in [14].

Definition 2.0.1 ***(Discrete dynamical system)*** *A discrete dynamical system is a pair (X,T) consisting of a metric space X and a continous map $T : X \to X$.*

As we are interested in the evolution of the system starting with a certain initial vector x_0, we will define the iterates $T^n : X \to X, n \geqslant 0$ by the n-fold iteration of T:

$$T^n = T \circ ... \circ T \quad n \text{ times}$$

with $T^0 = I$ the identity operator on X.

Definition 2.0.2 *Let $T : X \to X$ be a dynamical system. For $x \in X$ we call:*

$$orb(x,T) := \{x, Tx, T^2x, ...\}$$

the orbit of x under T.

An interesting notion in the dynamical systems theory is the concept of conjugacy.

Definition 2.0.3 *Let $S : Y \to Y$ and $T : X \to X$ be dynamical systems.*

(a) Then T is called quasiconjugate to S if there exists a continuous map ϕ : $Y \to X$ with dense range such that $T \circ \Phi = \Phi \circ S$.

(b) If Φ can be chosen to be a homeomorphism then S and T are called conjugate.

Conjugacy is an equivalence between dynamical systems, and conjugate (or quasiconjugate) dynamical systems have the same dynamical behaviour. This motivates the following definition.

Definition 2.0.4 *We say that a property $\mathcal{P}$ for dynamical systems is preserved under (quasi)conjugacy if a dynamical system S has property $\mathcal{P}$ then every (quasi)conjugate dynamical system T has also property $\mathcal{P}$.*

Definition 2.0.5 *A dynamical system $T : X \to X$ is called topologically transitive if, for any pair U, V of nonempty open subsets of X, there exists some $n \geqslant 0$ such that $T^n(U) \cap V \neq \varnothing$.*

Proposition 2.0.6 *Topological transitivity is preserved under quasiconjugacy.*

Another important property in dynamical systems is the notion of mixing operators.

Definition 2.0.7 *A dynamical system $T : X \to X$ is called mixing if, for any pair U, V of nonempty open subsets of X there exists some $N \geqslant 0$ such that*

$$T^n(U) \cap V \neq \varnothing, \text{ for all } n \geqslant N.$$

Proposition 2.0.8 *The mixing property is preserved under quasiconjugacy.*

The following classical theorem due to G.Birkhoff states the equivalence between topological transitivity and the existence of a dense orbit.

Theorem 2.0.9 ***(Birkhoff transitivity theorem)****. Let T be a continuous map on a separable complete metric space X without isolated points. Then the following assertions are equivalent:*

(i) T is topologically transitive;

(ii) there exists some $x \in X$ such that $orb(x, T)$ is dense in X.

If one of these conditions holds then the set of points in X with dense orbit is a dense G_δ-set.

There exist different notions of chaos. In this work, we will consider Devaney chaos [8]. Before introducing the definition of a Devaney chaotic dynamical system we will define the property of sensitive dependence on the initial conditions.

Definition 2.0.10 ***(Sensitive dependence on initial conditions)*** *Let (X, d) be a metric space without isolated points. Then a dynamical system $T : X \to X$ is said to have sensitive dependence on initial conditions if there exists some $\delta > 0$ such that, for every $x \in X$ and $\epsilon > 0$, there exists some $y \in X$ with $d(x, y) < \epsilon$ such that, for some $n \geqslant 0$, $d(T^n x, T^n y) > \delta$. The number δ is called the sensitivity constant for T.*

Definition 2.0.11 ***(Devaney chaos-preliminary version)*** *Let (X, d) be a metric space without isolated points. Then a dynamical system $T : X \to X$ is said to be chaotic (in the sense of Devaney) if it satisfies the following conditions:*

(i) T has sensitive dependence on initial conditions.

(ii) T is topologically transitive.

(iii) T has a dense set of periodic points.

In 1992 Banks et al [3] demonstrated that sensitive dependence on initial condition in the Devaney's definition of chaos is implied by the other two conditions.

Theorem 2.0.12 *(**Banks-Brook-Cairns-Davis-Stacey**). Let X be a metric space without isolated points. If a dynamical system $T : X \to X$ is topologically transitive and has a dense set of periodic points then T has sensitive dependence on initial conditions with respect to any metric defining the topology of X.*

The previous theorem allows to drop the condition of sensitive dependence from the definition of Devaney chaos.

Definition 2.0.13 *(**Devaney chaos**) A dynamical system $T : X \to X$ is said to be chaotic (in the sense of Devaney) if it satisfies the following conditions:*

(i) T is topologically transitive.

(ii) T has a dense set of periodic points.

Proposition 2.0.14 *Devaney chaos is preserved under quasiconjugacy.*

2.1 Linear dynamics

In this work we will focus our study in linear dynamical systems, that is, dynamical systems that are defined by linear maps.

Definition 2.1.1 *(**Linear dynamical system**). A linear dynamical system is a pair (X, T) consisting of a separable Fréchet space X and a linear operator $T : X \to X$.*

In the context of the dynamics of linear operators, the property of having a dense orbit has its own name.

Definition 2.1.2 *(**Hypercyclicity**). A linear operator $T : X \to X$ is called hypercyclic if there is some $x \in X$ whose orbit under T is dense in X. In such a case, x is called a hypercyclic vector for T. The set of hypercyclic vectors for T is denoted by $HC(T)$.*

Proposition 2.1.3 *Hypercyclicity is preserved under quasiconjugacy.*

In the next result, we reformulate Birkhoff's transitivity theorem in the context of linear dynamics.

Theorem 2.1.4 *(**Birkhoff transitivity theorem in linear dynamics**). A linear operator T on a separable Fréchet space X is hypercyclic if and only if it is topologically transitive. In that case, the set $HC(T)$ of hypercyclic vectors is a dense G_δ-set.*

This allows us to rephrase the definition of chaos in the sense of Devaney in the context of linear dynamics.

Definition 2.1.5 ***(Linear chaos)*** *An operator T on a separable Fréchet space X is said to be chaotic (in the sense of Devaney) if it satisfies the folloeing conditions:*

(i) T is hypercyclic.

(ii) T has a dese set of periodic points.

2.1.1 Chaos criteria

In the following lines we recall some classical criteria for Devaney chaos. If nothing else is said we will assume $T : X \to X$ to be a linear operator on a separable Fréchet space. The following lemma will be useful in order to achieve one of the most important chaos criteria, the Godefroy-Shapiro criterion.

Lemma 2.1.6 *Let T be a linear operator on a separable Fréchet space X. Then the set of periodic points of T is given by*

$$Per(T) = span\{x \in X : Tx = e^{\alpha\pi i}x \text{ for some } \alpha \in \mathbb{Q}\}.$$

Proof. If $Tx = e^{\alpha\pi i}x$ with $\alpha = p/q$, $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. This implies that $T^{2q}x = e^{\frac{p\pi i}{q}2q}x = e^{2p\pi i}x = x$, so $x \in \operatorname{Per}(T)$ and $\operatorname{span}\{x \in X : Tx = e^{\alpha\pi i}x \text{ for some } \alpha \in \mathbb{Q}\} \subset \operatorname{Per}(T)$.
Now for the other inclusion let us suppose that $x \in \operatorname{Per}(T)$ such that $T^n x = x$. We can decompose the polynomial $z^n - 1$ into a product of monomials;

$$z^n - 1 = (z - \lambda_1)(z - \lambda_2)...(z - \lambda_n)$$

where $\lambda_i, i = 1, 2, ..., n$ are the roots of unity and therefore for $i = 1, 2, ..., n$, $\lambda_i = e^{\alpha_i \pi i}$ for some $\alpha_i \in \mathbb{Q}$. Since all the roots are different, we can define a basis of the space of polynomials of degree strictly less than n as $\{p_1, p_2, ..., p_n\}$. Where $p_i(z) := \prod_{j \neq i}(z - \lambda_j)$, $1 \leqslant i \leqslant n$. In particular, there are $\beta_i \in \mathbb{C}$, $i = 1, 2, ..., n$, such that:

$$1 = \sum_{i=1}^{n} \beta_i p_i(z).$$

Since T is a linear operator when we replace z by T we get:

$$I = \sum_{i=1}^{n} \beta_i p_i(T).$$

Therefore we have that $x = \sum_{i=1}^{n} \beta_i y_i$, where $y_i = p_i(T)x$ for $i = 1, 2, ..., n$. Now since $(z - \lambda_i)p_i(z) = z^n - 1$, we have that $(T - \lambda_i)y_i = (T^n - I)x = 0$. So $Ty_i = \lambda_i y_i = e^{\alpha_i \pi i}y_i$ and therefore $x \in \operatorname{span}\{x \in X : Tx = e^{\alpha\pi i}x \text{ for some } \alpha \in \mathbb{Q}\}$. ■

Theorem 2.1.7 ***(Godefroy-Shapiro Criterion)***. *Let T be an operator. If the subspaces*

$$X_0 := span\{x \in X : Tx = \lambda x \text{ with } \lambda \in \mathbb{D}\}$$
$$Y_0 := span\{x \in X : Tx = \lambda x \text{ with } \lambda \in \mathbb{C}\backslash\overline{\mathbb{D}}\}$$
$$Z_0 := span\{x \in X : Tx = e^{\alpha\pi i}x \text{ with } \alpha \in \mathbb{Q}\}$$

are dense in X then T is chaotic.

Proof In this proof we first show that if the subspaces X_0 and Y_0 are dense in X then the operator T is hypercyclic. For that purpose let U, V be a pair of nonempty open sets of X. Assuming X_0 and Y_0 being dense in X then there exist $x_0 \in X_0$ and $y_0 \in Y_0$ such that $x_0 \in X_0 \bigcap U$ and $y_0 \in Y_0 \bigcap V$. Thereby these vectors can be expressed as:

$$x_0 = \sum_{k=1}^{m} a_k x_k$$

$$y_0 = \sum_{k=1}^{J} b_k y_k$$

where $Tx_k = \lambda_k x_k$ with $|\lambda_k| < 1$ for $k = 1, ..., m$ and $Ty_k = \mu_k y_k$ with $|\mu_k| > 1$ for $k = 1, ..., J$. Now let us define the sequence $\{u_n\}_n$ as:

$$u_n := \sum_{k=1}^{J} b_k \frac{1}{\mu_k^n} y_k.$$

It is clear that $\{u_n\}_n \subset Y_0$ and also that $T^n u_n = y_0$. Furthermore, we observe that since $|\mu_k| > 1$, the sequence $\{u_n\}_n$ converges to 0 in the norm of X. So as U is a nonempty open set of X and $x_0 \in U$, there exists $n_1 \in \mathbb{N}$ such that $x_0 + u_n \in U$ for all $n \geqslant n_1$. It is also easy to observe that:

$$T^n x_0 = \sum_{k=1}^{m} a_k \lambda_k^n x_k.$$

Due to the fact that $|\lambda_k| < 1$ for $k = 1, ..., m$, then the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges also to 0. Now since V is a nonempty open set of X and $y_0 \in V$ there exists some $n_2 \in \mathbb{N}$ such that $T^n x + y_0 \in V$ for all $n \geqslant n_2$. So taking $N = \max\{n_1, n_2\}$ we have that for all $n \geqslant N$:

$$x_0 + u_n \in U \quad \text{and} \quad T^n(x + u_n) = T^n x + y \in V.$$

This shows that T is mixing and therefore hypercyclic. Now by lemma 2.1.6, the set Z_0 is the set of periodic points of T, so if X_0 and Y_0 are dense and furthermore Z_0 is dense, T is chaotic. ∎

The following theorem provides another criterion for chaos that will be based on the Godefroy-Shapiro criterion. Nevertheless, two definitions are required. The first one refers to a function that links the eigenvalues of an operator T with its eigenvectors.

Definition 2.1.8 ***(Eigenvector field)****. Given a lineal operator $T : X \to X$ on a complex Banach space X, a function $E : A \to X$, $A \subset \mathbb{C}$ is an eigenvector field of T if $E(\lambda) \in \ker(\lambda I - T)$ for any $\lambda \in A$ and*

$$\mathit{span}\{E(\lambda) : \lambda \in A\}$$

is dense in X.

The second one is the definition of a weakly holomorphic map.

Definition 2.1.9 ***(Weakly holomorphic map)****. Given a non-empty open set $U \subset \mathbb{C}$, the map $G : U \to X$ is said to be weakly holomorphic on U if for any $y \in X^*$, the composition $y \circ G : U \to \mathbb{C}$ is holomorphic.*

Theorem 2.1.10 ***(Eigenvalue criterion)****. Given an operator $T : X \to X$ on a complex Banach space X, if $U \subset \mathbb{C}$ is a connected nonempty open set such that $U \bigcap \mathbb{T} \neq \varnothing$ and $G : U \to X$ is a weakly holomorphic eigenvector field, then T is chaotic.*

Proof. By the Godefroy-Shapiro criterion 2.1.7 we need to show that:

$$\begin{aligned}
X_1 &:= \text{span}\{x \in X : Tx = \lambda x \text{ with } \lambda \in \mathbb{D}\} \\
X_2 &:= \text{span}\{x \in X : Tx = \lambda x \text{ with } \lambda \in \mathbb{C}\backslash\overline{\mathbb{D}}\} \\
X_3 &:= \text{span}\{x \in X : Tx = e^{\alpha\pi i}x \text{ with } \alpha \in \mathbb{Q}\}
\end{aligned}$$

are dense in X. In order to do this, since U is a nonempty open connected subspace of $\mathbb{C}$ and $U \cap \mathbb{T} \neq 0$, let us define the sets:

$$\begin{aligned}
U_1 &= \{U \cap \mathbb{D}\}, \\
U_2 &= \left\{U \cap \mathbb{C}\backslash\overline{\mathbb{D}}\right\}, \\
U_3 &= U \cap \left\{e^{\alpha\pi i}, \alpha \in \mathbb{Q}\right\}.
\end{aligned}$$

By the corollary of the Hahn-Banach theorem 1.1.11, a subspace M of a Banach space X is dense in X if and only if any continuous linear functional x^* that vanishes on M also vanishes on X. In the context of our problem, the subspaces X_1, X_2 or X_3 are dense in X if and only if given $j \in \{1, 2, 3\}$ and for any $y \in X^*$, the equality $\langle x, y\rangle = 0$ for every $x \in X_j$ implies $\langle x, y\rangle = 0$ for all $x \in X$. Now, if $y \in X^*$ is a functional that vanishes on X_1, X_2 or X_3 the holomorphic map $y \circ G$ vanishes on U_1, U_2 or U_3 that are sets with accumulation points in U. This implies that the composition vanishes on all the domain U so $y \circ G = 0$. Furthermore, as G is an eigenvector field,

$$\text{span}\{G(\lambda) : \lambda \in U\}$$

is dense in X. So y is a functional that annihilates in a dense subspace of X and therefore $y = 0$ and finally X_1, X_2 or X_3 are dense in X. ■

The following theorem is a criterion of chaos for operators in the complex sequence space ℓ^p.

Theorem 2.1.11 *Let X be one of the complex sequences spaces ℓ^p, $1 \leqslant p < \infty$, or c_0. Moreover, let φ be a nonconstant holomorphic function on a neighborhood $A \subset \overline{\mathbb{D}}$, then the following equivalence holds:*

(i) $\varphi(B)$ *is chaotic.*

(ii) $\varphi(\mathbb{D}) \bigcap \mathbb{T} \neq \varnothing$.

(iii) $\varphi(B)$ *has a nontrivial periodic point.*

Proof. (ii) $\Longrightarrow$ (i). Let us observe that the eigenvectors of B are the nonzero multiplies of the sequences of the form

$$e_\lambda = (\lambda, \lambda^2, \lambda^3, ...)$$

with $|\lambda| < 1$ being the condition that ensures that $e_\lambda \in X$.
Furthermore, for any $\Lambda \subset \mathbb{D}$ that has an accumulation point on the unit disc, the set

$$\operatorname{span}\{e_\lambda : \lambda \in \Lambda\}$$

is dense in X. To prove this claim, it is useful to use the corollary of the Hahn-Banach theorem 1.1.11. In the context of our claim, $\operatorname{span}\{e_\lambda : \lambda \in \Lambda\}$ is dense in X if and only if any continuous linear functional that vanishes on each $e_\lambda, \lambda \in \Lambda$ also vanishes on X. Now given a linear functional $x^* \in X^*$ that vanishes on each $e_\lambda, \lambda \in \Lambda$, via the canonical representation, there exists a sequence $(y_n)_n \in \ell^q$ with $\frac{1}{q} + \frac{1}{p} = 1$ such that:

$$x^*(e_\lambda) = \langle e_\lambda, x^* \rangle = \sum_{n=1}^{\infty} y_n \lambda^n, \qquad \text{for all } \lambda \in \mathbb{D}.$$

Nevertheless, since $(y_n)_n \in \ell^q$ then it is clear that the sequence is bounded so $x^*(e_\lambda)$ defines a holomorphic function on the unit disk which vanishes on a subset with an accumulation point. Via the identity theorem for holomorphic functions the holomorphic function $x^*(e_\lambda)$ vanishes also on the unit disk $\mathbb{D}$, which implies that each y_n is zero and therefore $x^* = 0$. So every functional x^* that vanishes on each $e_\lambda, \lambda \in \Lambda$ also vanishes in X and therefore $\operatorname{span}\{e_\lambda, \lambda \in \Lambda\}$ is dense in X.
Now, for any $\lambda \in \mathbb{D}$ we have that

$$\varphi(B)e_\lambda = \sum_{n=0}^{\infty} a_n B^n e_\lambda = \sum_{n=0}^{\infty} a_n \lambda^n e_\lambda = \varphi(\lambda)e_\lambda$$

so each e_λ is also an eigenvector of $\varphi(B)$ associated with the eigenvalue $\varphi(\lambda)$. By the Godefroy-Shapiro criterion if the subspaces:

$$\begin{aligned} X_0 &= \text{span}\{x \in X : \varphi(B)x = \lambda x \quad \text{with } |\lambda| < 1\} \\ Y_0 &= \text{span}\{x \in X : \varphi(B)x = \lambda x \quad \text{with } |\lambda| > 1\} \\ Z_0 &= \text{span}\{x \in X : \varphi(B)x = e^{\alpha\pi i}x \quad \text{for some } \alpha \in \mathbb{Q}\} \end{aligned}$$

are dense in X then $\varphi(B)$ is chaotic. Nevertheless, to show the implication (ii) $\Longrightarrow$ (i) we will show that the subspaces:

$$\begin{aligned} X_0' &= \text{span}\{e_\lambda : \varphi(B)e_\lambda = \varphi(\lambda)e_\lambda \quad \text{with } |\varphi(\lambda)| < 1\} \subset X_0 \\ Y_0' &= \text{span}\{e_\lambda : \varphi(B)e_\lambda = \varphi(\lambda)e_\lambda \quad \text{with } |\varphi(\lambda)| > 1\} \subset Y_0 \\ Z_0' &= \text{span}\{e_\lambda : \varphi(B)e_\lambda = \varphi(\lambda)e_\lambda \quad \text{with } \varphi(\lambda) \text{ being a root of unity }\} \subset Z_0 \end{aligned}$$

are dense in X and therefore $\varphi(B)$ is chaotic. Now since nonconstant holomorphic functions are open mappings, the condition (ii) shows that $\{\lambda \in \mathbb{D} : |\varphi(\lambda)| < 1\}$ and $\{\lambda \in \mathbb{D} : |\varphi(\lambda)| > 1\}$ are nonempty and open and therefore contain an accumulation point in $\mathbb{D}$. So both are subsets of $\mathbb{D}$ containing an accumulation point in $\mathbb{D}$ and by the claim they are dense in X. Now to show that Z_0' is dense in X let us observe that condition (ii) ensures that there exists a sequence $(\lambda_n)_n \subset Y \subset \mathbb{D}$ being Y a relatively compact subset of $\mathbb{D}$ such that $\varphi(\lambda_n)$ is a root of unity for all $n \in \mathbb{N}$. As every sequence on a relatively compact in a metric space converges in the space, then it is clear that the set $\{\lambda_n, n \in \mathbb{N}\}$ has an accumulation point in $\mathbb{N}$ and therefore Z_0' is dense in X.

(i) $\Longrightarrow$ (iii) is trivial by definition of Devaney chaos.

(iii) $\Longrightarrow$ (ii). By condition (iii) there is some point $x \neq 0$ from X and some $N \geqslant 1$ such that $\varphi^N(B)x = \varphi(B)^N x = x$. This implies that $1 \in \sigma_P(\varphi^N(B))$, the point spectrum of $\varphi(B)^N$. Since φ^N is a nonconstant holomorphic function, by the point spectrum theorem 1.4.3 it follows that $\sigma_P(\varphi^N(B)) = \varphi^N(\sigma_P(B))$. So $1 = \varphi^N(\lambda)$ for some $\lambda \in \sigma_P(B)$. Recall that $\sigma_P(B) = \mathbb{D}$ and let us assume by reductio ad absurdum that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then it is clear that $\varphi^N(\mathbb{D}) \subset (\mathbb{D})$, leading us to a contradiction with the fact that $\varphi^N(\lambda) = 1$. So $\varphi(\mathbb{D}) \not\subset \mathbb{D}$ and by the open mapping theorem for nonconstant holomorphic functions it follows that $\varphi(\mathbb{D}) \bigcap \mathbb{T}$. ■

As a consequence of this last theorem, the next criterion for chaos will be useful in the analysis of many non-local difference operators.

Theorem 2.1.12 *Let $b \in \ell^1(\mathbb{N}_0)$ and $T_b : \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0)$ be given by*

$$T_b u(n) = \sum_{j=0}^{\infty} b(j) B u(n), \quad n \in \mathbb{N}_0,$$

where B denotes the backward shift operator. Let also $\varphi_b : \mathbb{C} \to \mathbb{C}$ be given by

$$\varphi_b(z) := \sum_{j=0}^{\infty} b(j) z^j .$$

Then the following assertions are equivalent.

(i) T_b *is chaotic.*

(ii) $\varphi_b(\mathbb{D}) \bigcap \mathbb{T} \neq \varnothing$.

Proof. Let us observe that $\varphi_b(B) = T_b$. Since $b \in \ell^1(\mathbb{N}_0)$ it is clear that φ_b is holomorphic on a neighborhood of $\overline{\mathbb{D}}$ and therefore the assertions are equivalent as a direct consequence of theorem 2.1.11. ■

Chapter 3

Toeplitz operators

3.1 Toeplitz operators

Toeplitz operators were introduced by Otto Toeplitz and they are one of the most studied operators in the Hardy space H^2. On this space, via the identification of H^2 with ℓ^2, the Toeplitz operators can be represented as matrices that have constant diagonals, the so called Toeplitz matrices. A classical reference where all the following results can be found is [5].

Definition 3.1.1 *The Toeplitz operator with symbol $\phi \in L^\infty(\mathbb{T})$ is defined as the operator in H^2, $T_\phi : H^2 \to H^2$, such that $T_\phi(f) = P(M_\phi(f))$, $f \in H^2$, where M_ϕ is the multiplication operator by ϕ ($M_\phi(f) = \phi \cdot f$) and $P : L^2(\mathbb{T}) \to H^2$ is the Riesz projection.*

If we write

$$\Phi(z) = \sum_{n \in \mathbb{Z}} a_n z^n \in L^\infty(\mathbb{T}),$$

given $f(z) = \sum_{n=0}^\infty b_n z^n \in H^2(\mathbb{D})$, we can write $(T_\Phi f)(z) = \sum_{n=0}^\infty c_n z^n$, where the sequence $(c_n)_n$ is obtained as the convolution of $a = (a_n)_n$ with $(b_n)_n$ as follows:

$$c_n = (a * b)_n = \sum_{j=-\infty}^{n} a_j b_{n-j}, \quad n \in \mathbb{N}_0.$$

Considering the equivalency between $H^2(\mathbb{D})$ and ℓ^2, the Toeplitz operator can be represented in matrix form as an infinite matrix with constant diagonals:

$$(c_n)_n = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & \cdots \\ a_3 & a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \end{bmatrix}. \tag{3.1}$$

In case that $\Phi(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ is such that $(a_n)_n \in \ell^1(\mathbb{Z})$, then by the Young's convolution inequality:

$$\|a * b\|_2 \leqslant \|a\|_1 \cdot \|b\|_2 < \infty$$

and it follows that T_Φ is a well-defined bounded operator on $\ell^2(\mathbb{N}_0)$.
As we are interested in the dynamical behaviour of these operators, the study of the chaos for Toeplitz operators will be always referred to the study of the conditions on the symbol of the Toeplitz operator that leads to a chaotic behaviour. The next proposition will show a family of Toeplitz operators that are not chaotic. In order to prove this result, we will first need to introduce the following lemma.

Lemma 3.1.2 *Let T be a hypercyclic operator on a Banach complex vector space X, then its adjoint operator T^* has no eigenvalues. Equivalently, if the adjoint T^* of an operator has eigenvalues then the operator T cannot be hypercyclic.*

Proof Let $x \in X$ be a hypercyclic vector for T. By reductio ad absurdum let suppose that T^* has an eigenvalue $\lambda \in \mathbb{C}$ associated to the eigenvector $x^* \in X^*$

$$T^* x^* = \lambda x^*$$

with $x^* \neq 0$. Then by definition of adjoint operator it is verified that

$$\langle x, T^* x^* \rangle = \langle Tx, x^* \rangle$$

and therefore for any $n \geqslant 0$:

$$\langle T^n x, x^* \rangle = \langle x, (T^*)^n x^* \rangle = \lambda^n \langle x, x^* \rangle.$$

Now since x is an hypercyclic vector of T, by definition we get that the set $\{T^n x\}_{n\in\mathbb{N}}$ is dense in X. Moreover since $x^* \neq 0$ is a continuous functional it is clear that

$$\{\langle T^n x, x^* \rangle\}_{n\in\mathbb{N}}$$

is dense in $\mathbb{C}$. Nevertheless the set

$$\{\lambda^n \langle x, x^* \rangle\}_{n\in\mathbb{N}}$$

is not dense in $\mathbb{C}$, leading us to a contradiction. So if T is an hypercyclic operator on X its adjoint operator has no eigenvalues. ■

Proposition 3.1.3 *If the symbol of the Toeplitz operator is analytic and bounded on $\mathbb{D}$ then the Toeplitz operator is not hypercyclic and therefore is not chaotic.*

Proof Let the symbol of the Toeplitz operator T_ϕ be:

$$\phi(z) = \sum_{n=0}^{\infty} a_n z^n$$

with $\sup_{n\geqslant 0}|a_n| < \infty$. Since the Hardy space H^2 is a Hilbert space then there exists some $x^* \in X^*$ such that $x^* \hookrightarrow 1 + 0z + 0z^2 + ... = 1 \in H^2$. Now given $f \in H^2$ such that $f = \sum_{n\geqslant 0} b_n z^n$, it follows that

$$\langle f, T_\phi^* x^* \rangle = \langle T_\phi f, x^* \rangle = \langle T_\phi f, 1 \rangle = a_0 b_0 = \langle f, \overline{a_0} \rangle = \langle f, \overline{a_0} \cdot 1 \rangle = \langle f, \overline{a_0} \cdot x^* \rangle$$

so $T_\phi^* x^* = \overline{a_0} x^*$ and therefore $\overline{a_0}$ is an eigenvalue of T_ϕ^* associated to the eigenvector $x^* \hookrightarrow 1$ and by lemma 3.1.2, T_ϕ cannot be hypercyclic. ■

3.1.1 Tridiagonal Toeplitz Operators

In this section we will characterize chaos and hypercyclicity for a particular Toeplitz operator, the tridiagonal one. Tridiagonal Toeplitz operators are studied as generators of chaotic semigrups associated to birth and death processes in [2] and [1].

Definition 3.1.4 *The tridiagonal Toeplitz operator is defined as the Toeplitz operator $T_\phi : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ with symbol $\phi(z) = a_1 z + a_0 + \frac{a_{-1}}{z}$, where $a_1, a_0, a_{-1} \in \mathbb{C}$.*

Note that if a_1 is zero, the Toeplitz operator T_ϕ is an anti-analytic operator and if a_{-1} is zero the tridiagonal Toeplitz operator is an analytic operator and we have proven that these operators are not chaotic. In this section we characterize chaos for tridiagonal Toeplitz operators when a_1 and a_{-1} are not zero. For this purpose it will be useful the eigenvalue criterion 2.1.10. To do this we will have to solve the equation $T_\phi z = \lambda z$ in order to find the eigenvalues of the operator. Now let us note that for $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2(\mathbb{D})$:

$$T_z f(z) = z f(z).$$

On the other hand,

$$T_{\frac{1}{z}} f(z) = P\left(\frac{1}{z} f(z)\right) = P\left(\frac{1}{z}\sum_{n=0}^{\infty} a_n z^n\right) = \sum_{n=1}^{\infty} a_n z^n = \frac{1}{z}(f(z) - f(0)).$$

Note also that $T_\phi = a_{-1} T_{\frac{1}{z}} + a_0 I + a_1 T_z$, so $T_\phi f = \lambda f$ is equivalent to

$$a_{-1}\frac{f(z) - f(0)}{z} + a_0 f(z) + a_1 z f(z) = \lambda f(z).$$

Therefore:

$$f(z) = \frac{a_{-1} f(0)}{a_1 z^2 + (a_0 - \lambda) z + a_{-1}}.$$

Now let us assume that $f(0) = 0$, this implies that $f(z)(a_1 z^2 + (a_0 - \lambda) z + a_{-1}) = 0$ and knowing that $f(z)$ is an analytic function this will lead us to $f(z) = 0$ for all $z \in \mathbb{D}$. Nevertheless, by definition of eigenvector the function $f(z)$ must be non-zero and therefore for being an eigenvector $f(0) \neq 0$. Assuming this

last condition we can also consider without loss of generality that $f(0) = 1$, otherwise if an eigenvector f such that $f(0) = a$ with $0 \neq a \neq 1$ is associated to an eigenvalue λ every proportional vector is also an eigenvector associated to λ and in particular the proportional vector $f_1(z) = \frac{1}{a} f(z)$ is an eigenvector such that $f_1(0) = 1$.
Another condition for $f(z)$ in order to be an eigenvector is that $f(z)$ must belong to $H^2(\mathbb{D})$, so necessarily the polynomial $q_\lambda(z) = a_1 z^2 + (a_0 - \lambda)z + a_{-1}$ must have its roots in $\mathbb{C}\backslash\mathbb{D}$. Otherwise the eigenvector function $f(z)$ would have a singularity in $\mathbb{D}$ and would not be analytic. This condition is equivalent to the roots of the polynomial:

$$p_\lambda(z) := z^2 q_\lambda(1/z) = a_{-1} z^2 + (a_0 - \lambda)z + a_1$$

being in $\mathbb{D}$. And to do so we will consider the following criterion.

Lemma 3.1.5 ***(Jury test)*** *Consider the family of the equations for $z \in \mathbb{C}$:*

$$z^2 + wz + r = 0,$$

where $w \in \mathbb{C}$ and $r \in (-1, 1) \subset \mathbb{R}$. For a fixed r let us denote:

$$W_r = \left\{ w \in \mathbb{C} : |z| < 1, \quad \textit{whenever} \quad z^2 + wz + r = 0 \right\}$$

then

$$W_r = E_r := \left\{ w \in \mathbb{C} : \left(\frac{Re(w)}{1 + r} \right)^2 + \left(\frac{Im(w)}{1 - r} \right)^2 < 1 \right\}.$$

Proof. Let us consider z_1 and z_2 the roots of the equation for a fixed r, so $z^2 + wz + r = (z - z_1)(z - z_2)$.
The proof is subdivided in three cases.

- The first one is when $r = 0$. This implies that the equation is $z^2 + wz = 0$, so the roots will be $z_1 = 0$, $z_2 = -w$. Therefore $\left(\frac{\mathrm{Re}(w)}{1+r}\right)^2 + \left(\frac{\mathrm{Im}(w)}{1-r}\right)^2 = \mathrm{Re}(w)^2 + \mathrm{Im}(w)^2 = |z_2|^2$. So it is clear that:

  $$|z_2| < 1 \iff |z_2|^2 < 1 \iff \mathrm{Re}(w)^2 + \mathrm{Im}(w)^2 < 1$$

  and this implies $E_0 = W_0$.

- The second case corresponds to $r \in (0, 1)$. We can assume without loss of generality that $z_1 < z_2$. An easy computation shows that when $r \neq 0$ we have that $r = z_1 \cdot z_2$ and $z_1 + z_2 = -w$. Therefore there exist r_1, r_2 and θ such that $z_1 = r_1 \cdot e^{i\theta}$ and $z_2 = r_2 \cdot e^{-i\theta}$. We can assume r_1 and r_2 to be positive (otherwise another θ can be chosen in order to verify this condition). Now we can observe that $|z_1| < 1$ and $|z_2| < 1$ if and only if $r_1 \in (r, 1)$ and $r_2 \in (r, 1)$. Now since:

  $$-w = z_1 + z_2 = (r_1 + r_2)\cos(\theta) + i(r_1 - r_2)\sin(\theta)$$

and

$$1 + r = 1 + r_1 r_2 = r_1 + [(1 - r_1) + r_1 r_2] > r_1 + [(1 - r) + r_1 r_2] =$$

$$= r_1 + [(1 - r) + r] = 1 + r_1 > r_1 + r_2$$

we obtain that if $|z_1| < 1$ and $|z_2| < 1$ (so $w \in E_r$) then:

$$\left(\frac{\mathrm{Re}(w)}{1+r}\right)^2 + \left(\frac{\mathrm{Im}(w)}{1-r}\right)^2 = \left(\frac{r_1+r_2}{1+r}\right)^2 \cos^2(\theta) + \left(\frac{r_1-r_2}{1-r}\right)^2 \sin^2(\theta) <$$

$$< \left(\frac{1+r}{1+r}\right)^2 \cos^2(\theta) + \left(\frac{1-r}{1-r}\right)^2 \sin^2(\theta) = \cos^2(\theta) + \sin^2(\theta) = 1$$

and therefore $w \in W_r$.
Conversely, if $w \in W_r$ then

$$\left(\frac{\mathrm{Re}(w)}{1+r}\right)^2 + \left(\frac{\mathrm{Im}(w)}{1-r}\right)^2 = \left(\frac{r_1+r_2}{1+r}\right)^2 \cos^2(\theta) + \left(\frac{r_1-r_2}{1-r}\right)^2 \sin^2(\theta) < 1.$$

Now let us assume without loss of generality $r_1 \leqslant r_2$. By reductio ad absurdum, if $r_2 \geqslant 1$ then $r_1 \leqslant r < 1$ and this implies

$$r_2 - r_1 \geqslant 1 - r_1 \geqslant 1 - r,$$

or reformulating

$$1 - r_1 \leqslant r_2 - r.$$

Also it is verified that

$$1 + r = r_1 + [(1 - r_1) + r] \leqslant r_1 + r_2 - r + r = r_1 + r_2,$$

so with the above inequalities it is clear that:

$$\begin{aligned} &\left(\frac{r_1+r_2}{1+r}\right)^2 \cos^2(\theta) + \left(\frac{r_1-r_2}{1-r}\right)^2 \sin^2(\theta) \\ &= \left(\frac{r_1+r_2}{1+r}\right)^2 \cos^2(\theta) + \left(\frac{-(r_2-r_1)}{1-r}\right)^2 \sin^2(\theta) \\ &\geqslant \left(\frac{1+r}{1+r}\right)^2 \cos^2(\theta) + \left(\frac{1-r}{1-r}\right)^2 \sin^2(\theta) = 1, \end{aligned}$$

which leads to a contradiction, so $r_2 < 1$ and $w \in E_r$. We have demonstrated that in this case $E_r = W_r$.

- The last case would be when $r \in (-1, 0)$. However, this case can be reduced to the previous one just taking into account the observation that $iW_r = W_{-r}$ and $iE_r = E_{-r}$. ■

Nevertheless, as $a_1, a_{-1} \in \mathbb{C}$, then the Jury test must be generalized.

Lemma 3.1.6 ***(Generalized Jury test).*** *The roots of the equation $z^2+wz+re^{i\theta}$, with $w \in \mathbb{C}$, $\theta \in [0,2\pi)$ and $r \geqslant 0$ belong to $\mathbb{D}$ if and only if $r<1$ and*

$$\left(\frac{Re(we^{-i\frac{\theta}{2}})}{1+r}\right)^2+\left(\frac{Im(we^{-i\frac{\theta}{2}})}{1-r}\right)^2<1.$$

Proof Let us consider the polynomial $p(z)=z^2+(we^{-i\frac{\theta}{2}})z+r$. Let us call $q(z)=z^2+wz+re^{i\theta}$, and note that $q(ze^{i\frac{\theta}{2}})=p(z)e^{i\theta}$, so $p(z)=0$ if and only if $q(ze^{i\frac{\theta}{2}})=0$. Applying the Jury test to $p(z)$ and taking into account that $q(ze^{i\frac{\theta}{2}})=p(z)=0$ with $|z|<1$ if and only if $r<1$ and

$$\left(\frac{\mathrm{Re}(we^{-i\frac{\theta}{2}})}{1+r}\right)^2+\left(\frac{\mathrm{Im}(we^{-i\frac{\theta}{2}})}{1-r}\right)^2<1,$$

the lemma holds. $\blacksquare$

Now we can apply the generalized Jury test to the polynomial:

$$\frac{1}{a_{-1}}p_\lambda(z)=z^2+\frac{(a_0-\lambda)}{a_{-1}}z+\frac{a_1}{a_{-1}}.$$

Note that for $\frac{1}{a_{-1}}p_\lambda(z)$ having roots in $\mathbb{D}$ it is necessary (by the generalized Jury test) that $|\frac{a_1}{a_{-1}}|<1$. Therefore a necessary condition for the tridiagonal Toeplitz operator in order to be chaotic is that $|a_{-1}|>|a_1|>0$.
Let us consider the following ellipse

$$E:=\left\{z\in\mathbb{C}:\frac{\mathrm{Re}(z)^2}{(|a_{-1}|+|a_1|)^2}+\frac{\mathrm{Im}(z)^2}{(|a_{-1}|-|a_1|)^2}=1\right\}$$

and its interior

$$E_0:=\left\{z\in\mathbb{C}:\frac{\mathrm{Re}(z)^2}{(|a_{-1}|+|a_1|)^2}+\frac{\mathrm{Im}(z)^2}{(|a_{-1}|-|a_1|)^2}<1\right\}.$$

Also let us consider the interior of the outer parallel at distance one of this ellipse:

$$A_0:=\{z\in\mathbb{C}:d(z,E_0)<1\}.$$

In the case that $|a_{-1}|+|a_1|<1$ the ellipse would have its major axis less than one. Consequently, in this case we would also consider F the inner parallel curve at distance one of E. This set F is defined through a bijection. So each point x of E corresponds to a point y of F, where y corresponds to the point that is at a unit distance from x in the direction of the interior normal vector to the ellipse at x. In figure 3.1.1 we can see an example of an ellipse with major axis less than 1 and its inner parallel F at distance 1. As we observe in figure 3.1.1 there is only a connected component of the interior of F that contains 0, whose closure will be denoted as A_0' and it is represented in the figure as the blue region.

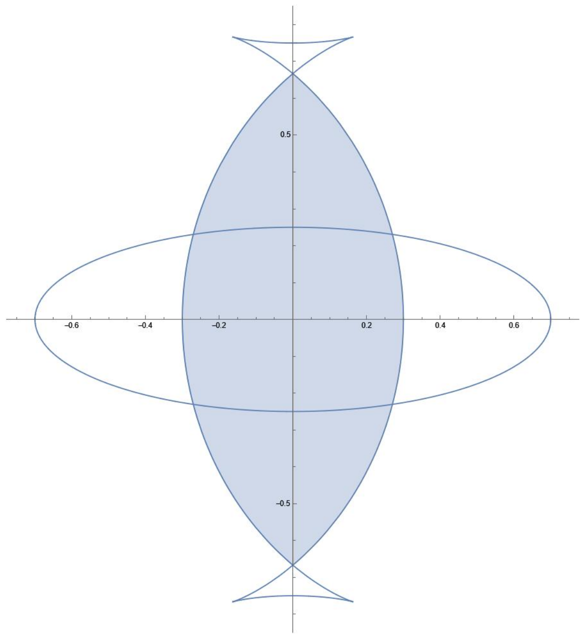

Figure 3.1: Example of an ellipse with its major axis less than 1, inner parallel curve and A_0' set.

Lemma 3.1.7 *Let $a_1, a_{-1} \in \mathbb{C}$ with $a_1 = |a_1|e^{i\theta_1}$, $a_{-1} = |a_{-1}|e^{i\theta_{-1}}$, and $\theta_1, \theta_{-1} \in [0, 2\pi)$, $\theta = \frac{\theta_1+\theta_{-1}}{2}$. Then there exists λ with $|\lambda| = 1$ such that $p_\lambda(z) = a_{-1}z^2 + (a_0 - \lambda)z + a_1$ has its roots in $\mathbb{D}$ if and only if a_0 satisfies one of the following cases:*

1. *If $|a_{-1}| + |a_1| > 1$ then $a_0 e^{-i\theta} \in A_0$.*
2. *If $|a_{-1}| + |a_1| = 1$ then $a_0 e^{-i\theta} \in A_0 \backslash \{0\}$.*
3. *If $|a_{-1}| + |a_1| < 1$ then $a_0 e^{-i\theta} \in A_0 \backslash A_0'$.*

Proof. By applying the Jury test to the polynomial

$$\frac{1}{a_{-1}} p_\lambda(z) = z^2 + \frac{(a_0 - \lambda)}{a_{-1}} z + \frac{a_1}{a_{-1}}$$

we know that its roots are in $\mathbb{D}$ if and only if the next inequality holds:

$$\frac{\operatorname{Re}\left(\frac{a_0-\lambda}{a_{-1}} e^{-i\frac{\theta_1-\theta_{-1}}{2}}\right)^2}{\left(1 + \frac{|a_1|}{|a_{-1}|}\right)^2} + \frac{\operatorname{Im}\left(\frac{a_0-\lambda}{a_{-1}} e^{-i\frac{\theta_1-\theta_{-1}}{2}}\right)^2}{\left(1 - \frac{|a_1|}{|a_{-1}|}\right)^2} < 1.$$

Now by multiplying $|a_1|^2$ in the numerator and denominator and simplifying we obtain:

$$\frac{\operatorname{Re}\left(a_0e^{-i\theta}-\lambda e^{-i\theta}\right)^2}{(|a_1|+|a_{-1}|)^2}+\frac{\operatorname{Im}\left(a_0e^{-i\theta}-\lambda e^{-i\theta}\right)^2}{(|a_1|-|a_{-1}|)^2}<1.$$

Now let us call $b_0 = e^{-i\theta}$ so the inequality will be

$$\frac{\operatorname{Re}\left(b_0-\lambda e^{-i\theta}\right)^2}{(|a_1|+|a_{-1}|)^2}+\frac{\operatorname{Im}\left(b_0-\lambda e^{-i\theta}\right)^2}{(|a_1|-|a_{-1}|)^2}<1.$$

Furthermore, let us call $\lambda' = \lambda e^{-i\theta}$, so the previous inequality is satisfied for $\lambda \in \mathbb{T}$ if and only if $\lambda' \in \mathbb{T}$, and then we obtain the following inequality:

$$\frac{\operatorname{Re}\left(b_0-\lambda'\right)^2}{(|a_1|+|a_{-1}|)^2}+\frac{\operatorname{Im}\left(b_0-\lambda'\right)^2}{(|a_1|-|a_{-1}|)^2}<1.$$

Note that this last inequation is verified if and only if $(b_0-\lambda') \in E_0$. This, in turn is equivalent to having $z_1, z_2 \in E_0$ such that $|b_0 - z_1| < 1$ and $|b_0 - z_2| > 1$. To check this equivalence let assume that there exists $\lambda' \in \mathbb{T}$ such that $(b_0-\lambda') \in E_0$. Now since E_0 is open, there exist z_1' and z_2' such that $(b_0 - z_1'), (b_0 - z_2') \in E_0$, $|z_1'| < 1$ and $z_2' > 1$. Now, if we call $z_1 = (b_0 - z_1')$ and $z_2 = (b_0 - z_2')$ one implication is proven. On the other hand, let us assume that there exist $z_1, z_2 \in E_0$ such that $|b_0 - z_1| < 1$ and $|b_0 - z_2| > 1$ and consider the function $f : E_0 \to \mathbb{R}$ defined by:

$$f(z) = |b_0 - z|.$$

Since E_0 is a connected set, the image of the function $f(E_0)$ is an interval in $\mathbb{R}$ that has points greater and smaller than one, so necessarily contains one. So there exists some $z \in E_0$ such that $|b_0 - z| = 1$ and taking $z = b_0 - \lambda'$ the other implication holds.
Therefore, there exists $\lambda \in \mathbb{T}$ such that $p_\lambda(z)$ has its roots in $\mathbb{D}$ if and only if there exist some $z_1, z_2 \in E_0$ satisfying $|b_0 - z_1| < 1$ and $|b_0 - z_2| > 1$. The equivalence with the cases of a_0 will be shown:

1. If $|a_1| + |a_{-1}| > 1$ then $b_0 \in A_0$.
   In this case the major axis of the ellipse E is greater than one so by the definition of A_0 (interior of the outer parallel at distance one to E) it is clear that there exist $z_1, z_2 \in E_0$ such that $|b_0 - z_1| < 1$ and $|b_0 - z_2| > 1$. In figure 3.2 we see in blue an ellipse with major axis greater than one and in green the outer parallel curve.

2. If $|a_1| + |a_{-1}| = 1$ then $b_0 \in A_0 \backslash \{0\}$.
   In this case the major axis of the ellipse E_0 is equal to one so by the same reason as in the previous case there exist some $z_1, z_2 \in E_0$ such that $|b_0 - z_1| < 1$ and $|b_0 - z_2| > 1$. In figure 3.3 we see one example of an ellipse (in blue) with major axis equal to 1 and its outer parallel curve.

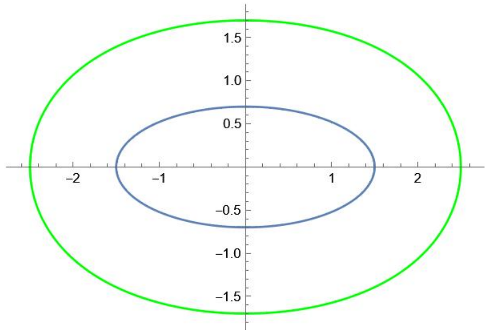

Figure 3.2: Example of case 1.

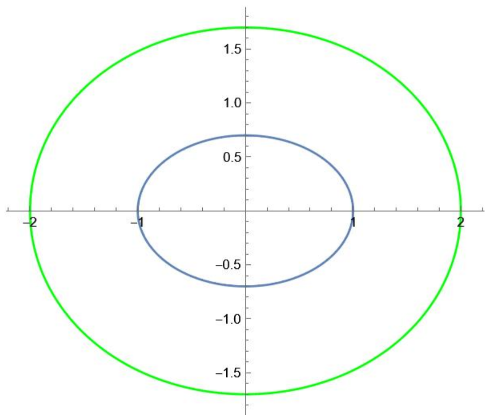

Figure 3.3: Example of case 2

3. If $|a_1| + |a_{-1}| < 1$ then $b_0 \in A_0 \backslash A_0'$.
   Since $b_0 \in A_0$ and A_0 is the outer parallel of the ellipse at distance one, then we can find $z_1 \in E_0$ such that $|b_0 - z_1| < 1$. Furthermore as $b_0 \notin A_0'$, we can find some $z_2 \in E_0$ such that $|b_0 - z_2| > 1$. In figure 3.4 we can see an example of an ellipse (in blue) with major axis strictly less than one and its outer (red) and inner (green) parallels. ■

So far, it has been characterised the existence of $\lambda \in \mathbb{T}$ such that the λ-eigenvector of T_ϕ belongs to $H^2(\mathbb{D})$. Nevertheless, we will see in proposition 3.1.9 that this is equivalent to the existence of an open subset $B \subset \mathbb{C}$ with non empty intersection with $\mathbb{T}$ such that $f_\lambda \in H^2(\mathbb{D})$ for any $\lambda \in B$, where f_λ are the eigenvectors of the tridiagonal Toeplitz operator T_ϕ. The next lemma will be useful for the achievement of this result.

Lemma 3.1.8 *Let $p_b : \mathbb{C} \to \mathbb{C}$ be the polynomial given by*

$$p_b(z) = az^2 + bz + c$$

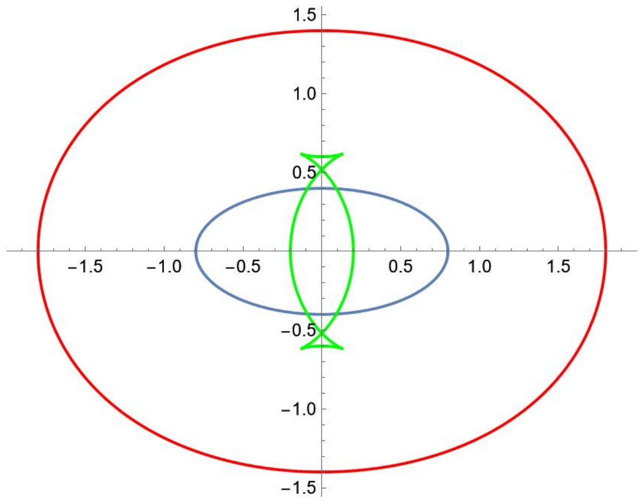

Figure 3.4: Example of case 3

with roots z_1 and z_2. Then for each $\epsilon > 0$ there exists $\delta > 0$ such that every polynomial of the form $p_{b_\delta} = az^2 + b_\delta z + c$ with $|b - b_\delta| < \delta$ have their roots z_{1_δ} and z_{2_δ} such that $|z_1 - z_{1_\delta}| < \epsilon$ and $|z_2 - z_{2_\delta}| < \epsilon$.

Proof. It is well known that:

$$z_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$z_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

and also:

$$z_{1_\delta} = \frac{-b_\delta + \sqrt{b_\delta^2 - 4ac}}{2a}$$
$$z_{2_\delta} = \frac{-b_\delta - \sqrt{b_\delta^2 - 4ac}}{2a}.$$

We define the complex functions $f_i : \mathbb{C} \to \mathbb{C}$, $i = 1, 2$ as follows:

$$f_1(z) = \frac{-z + \sqrt{z^2 - 4ac}}{2a}$$

$$f_2(z) = \frac{-z - \sqrt{z^2 - 4ac}}{2a}.$$

With these definitions it is verified $f_1(b) = z_1$ and $f_2(b) = z_2$. Furthermore, since both f_1 and f_2 are continuous, given the open balls $B(f_1(b), \epsilon) = B(z_1, \epsilon)$ and $B(f_2(b), \epsilon) = B(z_2, \epsilon)$ there exist δ_1 and δ_2 such that $f_1(B(b, \delta_1)) \subset B(z_1, \epsilon)$ and $f_2(B(b, \delta_2)) \subset B(z_2, \epsilon)$. Taking now $\delta = \min\{\delta_1, \delta_2\}$ the lemma holds. ■

Proposition 3.1.9 *Let $\{p_\lambda\}_{\lambda \in \mathbb{C}}$ be the set of complex coefficient polynomials given by $p_\lambda(z) = a_{-1}z^2 + (a_0 - \lambda)z + a_1$. If there exists $\lambda_0 \in \mathbb{T}$ such that the roots of p_{λ_0} belong to $\mathbb{D}$, then there exists an open set $B \subset \mathbb{C}$ such that $B \cap \mathbb{T} \neq \varnothing$ and for every $\lambda \in B$ the roots of p_λ belong to $\mathbb{D}$.*

Proof. Let denote z_1 and z_2 the roots of the polynomial p_{λ_0}, and let ϵ be such that $B(z_i, \epsilon) \subset \mathbb{D}$ $i = 1, 2$. By the previous lemma 3.1.8 there exists δ such that $|\lambda - \lambda_0| < \delta$ and the roots of p_λ belong to $B(z_i, \epsilon)$ for $i = 1, 2$. So it is clear that there exists an open set $B = B(\lambda_0, \delta)$ such that for every $\lambda \in B$, the roots of p_λ belong to $\mathbb{D}$ as we wanted to prove. ■

With this last proposition we have demonstrated that the conditions in lemma 3.1.7 are equivalent to the existence of an open set $B \subset \mathbb{C}$ of eigenvalues with non empty intersection with $\mathbb{T}$. The next theorem shows that the eigenvectors associated to the set B will define an eigenvector field that satisfies the conditions of the eigenvector field criterion 2.1.10.

Theorem 3.1.10 *Let $B \subset \mathbb{C}$ be an open subset with non empty intersection with $\mathbb{T}$ and suppose that $f_\lambda \in H^2(\mathbb{D})$ for any $\lambda \in B$, where*

$$f_\lambda(z) := \frac{a_{-1}}{a_1 z^2 + (a_0 - \lambda)z + a_{-1}}.$$

Then the map $G : B \to H^2(\mathbb{D}), G(\lambda) := f_\lambda$, is weakly holomorphic and

$$span\{G(\lambda) : \lambda \in B\} \textit{ is dense in } H^2(\mathbb{D}).$$

Proof. Let g be a function in $H^2(\mathbb{D})$. Applying the Hahn-Banach theorem 1.1.9 in the form of the corollary 1.1.11, the subspace $\text{span}\{G(\lambda) : \lambda \in B\}$ is dense in $H^2(\mathbb{D})$ if and only if $\langle f_\lambda, g\rangle = 0$ for all λ in B implies $g = 0$ (recall that $H^2(\mathbb{D})$ is a Hilbert space and therefore $g \in (H^2(\mathbb{D}))^*$).

Let us consider $H : B \to \mathbb{C}$ defined by $H(\lambda) := \langle f_\lambda, g\rangle$. This defines an holomorphic function on B given by

$$H(\lambda) = \frac{1}{2\pi}\int_0^{2\pi} \frac{a_{-1}}{q_\lambda(e^{i\theta})}\overline{g(e^{i\theta})}d\theta, \quad \text{where } q_\lambda(z) = a_1 z^2 + (a_0 - \lambda)z + a_{-1}.$$

Now suppose that $H(\lambda) = 0$ for all $\lambda \in B$. This implies that all the derivatives will also vanish for all $\lambda \in B$ and in particular at a certain $e^{i\alpha} \in \mathbb{T} \bigcap B$, so we get:

$$\frac{d(H(\lambda))}{d\lambda}(e^{i\alpha}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}}{q_{e^{i\alpha}}(e^{i\theta})}h(\theta)d\theta = 0, \quad \text{where } h(\theta) := \frac{a_{-1}}{q_{e^{i\alpha}}(e^{i\theta})}\overline{g(e^{i\theta})},$$

and also:

$$\frac{d^n(H(\lambda))}{d\lambda^n}(e^{i\alpha}) = \frac{1}{2\pi}\int_0^{2\pi} (\Phi(\theta))^n\, h(\theta)d\theta = 0, \quad \text{where } \Phi(\theta) := \frac{e^{i\theta}}{q_{e^{i\alpha}}(e^{i\theta})}. \quad (3.2)$$

Now let us define $\Psi(z) := \frac{z}{q_{e^{i\alpha}}(z)}$. Since $q_{e^{i\alpha}}(z)$ is a polynomial with its roots in $\mathbb{C}\backslash\overline{\mathbb{D}}$, then $\Psi(z)$ is an analytic function in an open disc $U \supset \overline{\mathbb{D}}$. We state that

$\Psi(z)$ is univalent in a neighborhood of $\overline{\mathbb{D}}$. To prove this, let us take $z_1, z_2 \neq 0$ such that $\Psi(z_1) = \Psi(z_2)$. Then we have:

$$\frac{z_1}{q_{e^{i\alpha}}(z_1)} = \frac{z_2}{q_{e^{i\alpha}}(z_2)}$$

if and only if it is verified

$$z_1 \cdot q_{e^{i\alpha}}(z_2) = z_2 \cdot q_{e^{i\alpha}}(z_1),$$

and consequently

$$a_1 z_1 z_2^2 + (a_0 - e^{i\alpha}) z_1 z_2 + a_{-1} z_1 = a_1 z_2 z_1^2 + (a_0 - e^{i\alpha}) z_1 z_2 + a_{-1} z_2.$$

Developing this last identity we get

$$a_1 z_1 z_2^2 + a_{-1} z_1 = a_1 z_2 z_1^2 + a_{-1} z_2$$

and simplifying

$$\frac{a_1}{a_{-1}}(z_2 - z_1) = \frac{1}{z_1} - \frac{1}{z_2} = \frac{z_2 - z_1}{z_1 z_2},$$

so finally

$$\frac{a_1}{a_{-1}} = \frac{1}{z_2 z_1}.$$

Recall that $|a_{-1}| > |a_1|$, and then we have:

$$1 > \frac{|a_1|}{|a_{-1}|} = \frac{1}{|z_1 z_2|}.$$

It is clear that we can find a neighborhood A of $\overline{\mathbb{D}}$ in which for every pair $z_1, z_2 \in A$, it is verified that $\frac{1}{|z_1 z_2|} \geqslant 1$ and therefore $\Psi(z)$ must be univalent in A.

Since $\Psi(z)$ is univalent in A there exists $\Psi^{-1} : W \to A$, where $W := \Psi(A)$ is a simply connected open set. Let us set $H(M) := \{f : M \to \mathbb{C}; f \text{ is analytic}\}$ for an open set $M \subset \mathbb{C}$. The map $C_\Psi : H(W) \to H(A)$ defined by $f \to f \circ \Psi$ is an isomorphism since Ψ is univalent in A. It is known that the polynomials $\{1, z, z^2, ...\}$ are dense in $H(W)$ and therefore $C_\Psi\left(\text{span}\{1, z, z^2, ..., \}\right) = \text{span}\{1, \Psi(z), \Psi^2(z), ..., \}$ is dense in $H(A)$. Since $H^2(\mathbb{D}) \subset H(A)$, then $\mathcal{Y} := \text{span}\{1, \Psi(z), \Psi^2(z), ..., \}$ is dense in $H^2(\mathbb{D})$. Finally, since the identity (3.2) holds then $\overline{h(z)} \perp \mathcal{Y}$ and by the Hahn-Banach theorem $\overline{h(z)} = h(z) = 0$, leading to state that $g(z) = 0$, as we wanted to demonstrate. ■

All the previous results can be summarized in the following theorem.

Theorem 3.1.11 *Let $T : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ be a Toeplitz operator with symbol the function $\Phi(z) = \frac{a_{-1}}{z} + a_0 + a_1 z$, where $a_{-1} = |a_{-1}|e^{i\theta_{-1}}, a_1 = |a_1|e^{i\theta_1}$, with $\theta_1, \theta_{-1} \in [0, 2\pi)$, and $a_0 \in \mathbb{C}$. Set $\theta = \frac{\theta_1 + \theta_{-1}}{2}$, and let A_0, A_0' be the sets defined in lemma 3.1.7. Then the following assertions are equivalent:*

(i) $0 < |a_1| < |a_{-1}|$ *and* a_0 *satisfies one of the following conditions:*

(a) *If* $|a_{-1}| + |a_1| > 1$ *then* $a_0 e^{i\theta} \in A_0$.

(b) *If* $|a_{-1}| + |a_1| = 1$ *then* $a_0 e^{i\theta} \in A_0 \backslash \{0\}$.

(c) *If* $|a_{-1}| + |a_1| < 1$ *then* $a_0 e^{i\theta} \in A_0 \backslash A_0'$.

(ii) T *satisfies the Eigenfield criterion.*

(iii) T *satisfies the Godefroy-Shapiro criterion.*

(iv) T *is Devaney chaotic.*

We have just characterized Devaney Chaos for tridiagonal Toeplitz operators. Nevertheless, Baranov and Lishansky [4] proved chaos for Toeplitz operators with a more general symbol form.

3.1.2 Toeplitz operators with a more general symbol form

In [4], the authors provide sufficient and necessary conditions that ensure hypercyclicity for the Toeplitz operators with symbol $\Phi(z) = p(\frac{1}{z}) + \varphi(z)$, where p is a polynomial and φ is a bounded holomorphic function. One of the main results of their article corresponds to the specific case in which $p(z) = \frac{\gamma}{z}$ where $\gamma \in \mathbb{C}$.

Theorem 3.1.12 *Let* $\gamma \in \mathbb{C}$, *let* $\varphi \in H^\infty$ *and let* $\Phi(z) = \frac{\gamma}{z} + \varphi(z)$.

(a) *If* $T_\Phi : H^2 \to H^2$ *is hypercyclic then*

(i) *the function* Φ *is univalent in* $\mathbb{D}\backslash\{0\}$;

(ii) $\mathbb{D} \bigcap (\mathbb{C}\backslash\Phi(\mathbb{D})) \neq$ *and* $(\mathbb{C}\backslash\overline{\mathbb{D}}) \bigcap (\mathbb{C}\backslash\Phi(\mathbb{D})) \neq \varnothing$.

(b) *If* $\varphi \in A(\overline{\mathbb{D}})$, *that is, the space of bounded analytic functions in the disk that extends to continuous functions in the closure, and:*

(a) *the function* Φ *is univalent in* $\overline{\mathbb{D}}\backslash\{0\}$*;*

(b) $\mathbb{D} \bigcap (\mathbb{C}\backslash\Phi(\mathbb{D})) \neq$ *and* $(\mathbb{C}\backslash\overline{\mathbb{D}}) \bigcap (\mathbb{C}\backslash\Phi(\mathbb{D})) \neq \varnothing$;

then T_Φ *is hypercyclic.*

In [4] there are also results regarding the spectrum of the operator T_Φ linked to the concept of N-valence.

Definition 3.1.13 ***(N-valence of a function)****. Let* $A \subset \mathbb{C}$, *a function* $\Phi : A \to \mathbb{C}$ *is said to be N-valent in* A *if for all* $w \in A$ *the equation* $\Phi(z) = w$ *has at most* N *solutions in* A.

Proposition 3.1.14 *Assume that Φ is N-valent in $\mathbb{D}$. Then*

$$\sigma(T_\Phi) = \mathbb{C}\backslash\Phi(\mathbb{D}, N), \quad \mathbb{C}\backslash\overline{\Phi(\mathbb{D})} \subset \sigma(T_\Phi).$$

If $\lambda \in \mathbb{C}\backslash\overline{\Phi(\mathbb{D})}$ then the corresponding eigenspace has dimension N and the eigenvectors are given by

$$f_\lambda(z) = \frac{q(z)}{z^N\Phi(z) - \lambda z^N}$$

where q is an arbitrary polynomial of degree at most $N-1$.
In particular, for univalent Φ, we get that

$$f_\lambda(z) = \frac{1}{z\Phi(z) - \lambda z}$$

is a λ-eigenvector of T_Φ for any $\lambda \in \mathbb{C}\backslash\overline{\Phi(\mathbb{D})}$.

So far, hypercyclity on Toeplitz operators with symbol of the form $\Phi(z) = \varphi(z) + \frac{\gamma}{z}$, $\varphi \in H^\infty$, $\gamma \in \mathbb{C}$ has been characterized. Nevertheless, in [18] the authors reached chaos under the same hypothesis.

Theorem 3.1.15 *Let $\Phi(z) = \frac{\gamma}{z} + \varphi(z)$ with $\gamma \in \mathbb{C}$ and $\varphi \in A(\overline{\mathbb{D}})$ satisfying*

(i) the function Φ is univalent in $\overline{\mathbb{D}}\backslash\{0\}$;

(ii) $\mathbb{D}\bigcap(\mathbb{C}\backslash\Phi(\mathbb{D})) \neq \varnothing$ and $(\mathbb{C}\backslash\overline{\mathbb{D}})\bigcap(\mathbb{C}\backslash\Phi(\mathbb{D})) \neq \varnothing$.

Then the Toeplitz operator $T_\Phi : H^2 \to H^2$ is Devaney chaotic.

Proof. By the theorem 3.1.12, T_Φ is a well-defined hypercyclic operator in H^2. Moreover, this theorem shows that

$$f_\lambda(z) = \frac{1}{z\Phi(z) - \lambda z}$$

is a λ-eigenvector of T_Φ for any $\lambda \in \mathbb{C}\backslash\overline{\Phi(\mathbb{D})}$. By assumption we get that $G(\lambda) := f_\lambda$ is a weakly holomorphic map on an open set U that intersects $\mathbb{T}$. Baranov and Lishansky proved in [[4],theorem 1.1], that the map G satisfies the condition:

$$\text{span}\{G(\lambda) : \lambda \in U\} \quad \text{is dense in } H^2$$

and therefore by the eigenvector field criterion 2.1.10, T_Φ is Devaney chaotic. ■

Chapter 4

Non-local operators

In the last two decades there has been a growing interest in applying fractional or non-local operators in the field of mathematical modeling. That is why many researchers have focused their interest on the study of the dynamic behavior of such operators [13, 11]. In this chapter we will present some definitions from discrete fractional calculus. All the results presented throughout this chapter can be found in [18] and [12].

4.1 Nonlocal operators

For a real number a, we denote

$$\mathbb{N}_a := \{a, a+1, a+2, ...\},$$

and we write $\mathbb{N}_1 = \mathbb{N}$. Let us consider X a complex Banach space. We will denote by $s(\mathbb{N}_a, X)$ the vector space of all vector-valued sequences $f : \mathbb{N}_a \to X$.

Definition 4.1.1 *(**Forward Euler operator**). Let $f : \mathbb{N}_a \to X$ be a vector-valued sequence on a complex Banach space X. The forward Euler operator, denoted by Δ_a, is defined as the operator on $s(\mathbb{N}_a)$ given by the formula:*

$$\Delta_a f(t) := f(t+1) - f(t), \quad t \in \mathbb{N}_a.$$

Furthermore, for $m \in \mathbb{N}_2$, the m-th order forward difference operator $\Delta_a^m : s(\mathbb{N}_a, X) \to s(\mathbb{N}_a, X)$ is defined recursively by

$$\Delta_a^m := \Delta_a^{m-1} \circ \Delta_a.$$

For instance, for any $f \in (\mathbb{N}_0)$, we have

$$\Delta_0^m f(n) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} f(n+j), \quad n \in \mathbb{N}_0.$$

We will usually denote $\Delta_0 \equiv \Delta$ and $\Delta_a^0 \equiv I_a$, where $I_a : s(\mathbb{N}_a, x) \to s(\mathbb{N}_a, x)$ is the identity operator.

Definition 4.1.2 *(**Translation operator**). The translation operator by $a \in \mathbb{R}$, denoted as $\tau_a : s(\mathbb{N}_a, X) \to s(\mathbb{N}_0, x)$ is defined as:*

$$\tau_a g(n) := g(a+n), \quad n \in \mathbb{N}.$$

Let us observe that $\tau^{-1} = \tau_{-a}$ and $\tau_{a+b} = \tau_a \circ \tau_b$. Furthermore, it is easy to check that:

$$\Delta_a^m \circ \tau_a^{-1} = \tau_a^{-1} \circ \Delta_0^m.$$

For any $\alpha \in \mathbb{R}\backslash\{0\}$, we also define the function $k^\alpha(n) : \mathbb{N}_0 \to \mathbb{R}$ as:

$$k^\alpha(n) = \frac{\alpha(\alpha+1)...(\alpha+n-1)}{n}.$$

In case $\alpha = 0$, we set $k^0(n) = e_0(n)$, being $e_i(j)$ the Kronecker delta. Note that using the properties of the Euler gamma function, for $\alpha \in \mathbb{R}\backslash\{-1,-2,..\}$, we have

$$\begin{aligned}\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n+1)} &= \frac{\alpha(\alpha+1)...(\alpha+n-1)\Gamma(\alpha)}{\Gamma(\alpha)n!} \\ &= \frac{\alpha(\alpha+1)...(\alpha+n-1)}{n!} = k^\alpha(n), \quad n \in \mathbb{N}_0.\end{aligned}$$

The following proposition is a useful property of the above function.

Proposition 4.1.3 *The following generation formula holds*

$$\sum_{j=0}^{\infty} k^\alpha(j) z^j = \frac{1}{(1-z)^\alpha}. \tag{4.1}$$

Proof. We know that for $q \in \mathbb{R}\backslash\mathbb{N}_0$ and $|z| < 1$:

$$(1+z)^q = \sum_{n=0}^{\infty} \binom{q}{n} z^n.$$

From the properties of the Euler Gamma function we also obtain for $\beta \in \mathbb{R}\backslash\mathbb{N}_0$ and $j \in \mathbb{N}_0$:

$$\begin{aligned}\frac{\Gamma(1+\beta)}{\Gamma(1+j)\Gamma(\beta-j+1)} &= \frac{\Gamma(\beta)\beta}{j!\Gamma(\beta-j)(\beta-j)} = \frac{\Gamma(\beta)\beta}{j!\frac{\Gamma(\beta)}{(\beta-1)(\beta-2)...(\beta-j)}(\beta-j)} \\ &= \frac{\beta(\beta-1)(\beta-2)...(\beta-j+1)}{j!} = \binom{\beta}{j}.\end{aligned}$$

So we can write:

$$(1+w)^\beta = \sum_{j=0}^{\infty} \binom{\beta}{j} z^n = \sum_{j=0}^{\infty} \frac{\Gamma(1+\beta)}{\Gamma(1+j)\Gamma(\beta-j+1)} w^j, \quad \beta \in \mathbb{R}\backslash\mathbb{N}_0, |w| < 1.$$

If we use the following expression with $\alpha = -\beta$ and $z = -w$ we obtain:

$$(1-z)^{-\alpha} = \frac{1}{(1-z)^{\alpha}} = \sum_{j=0}^{\infty} \frac{\Gamma(1-\alpha)}{\Gamma(1+j)\Gamma(-\alpha-j+1)}(-1)^j z^j$$

whenever $|z| < 1$ and $\alpha > 0$, so the coefficients in the development of $(1-z)^{-\alpha}$ are of the form:

$$\frac{\Gamma(1-\alpha)}{\Gamma(1+j)\Gamma(-\alpha-j+1)}(-1)^j.$$

And now using the identity $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$ for $z = \alpha + j$ and $z = \alpha$ we get:

$$\Gamma(1-\alpha)\Gamma(\alpha) = \frac{\pi}{\sin(\pi\alpha)}$$

and

$$\Gamma(1-\alpha-j)\Gamma(\alpha+j) = \frac{\pi}{\sin(\pi(\alpha+j))}.$$

Using the previous identities we obtain:

$$\begin{aligned}
\frac{\Gamma(1-\alpha)}{\Gamma(1+j)\Gamma(-\alpha-j+1)}(-1)^j &= \frac{\sin(\pi\alpha)}{\sin(\pi\alpha)(-1)^j}\frac{\Gamma(1-\alpha)}{\Gamma(1+j)\Gamma(-\alpha-j+1)} \\
&= \frac{\sin(\pi\alpha)\Gamma(1-\alpha)}{\pi}\frac{1}{\Gamma(1-\alpha-j)\Gamma(j+1)}\frac{\pi}{\sin(\pi(\alpha+j))} \\
&= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(1-\alpha-j)\Gamma(j+1)}\frac{\pi}{\sin(\pi(\alpha+j))} \\
&= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(1-\alpha-j)\Gamma(j+1)}\Gamma(1-\alpha-j)\Gamma(\alpha+j) \\
&= \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)\Gamma(j+1)} = k^{\alpha}(j),
\end{aligned}$$

so the generation formula (4.1) holds, as we wanted to demonstrate. ■

Definition 4.1.4 *(α-th fractional sum). Let $f : \mathbb{N}_0 \to X$ be a vector-valued sequence and $\alpha > 0$. The α-th fractional sum of f, denoted by $\Delta^{-\alpha} f$, is a vector valued sequence defined by the formula:*

$$\Delta^{-\alpha} f(n) := \sum_{j=0}^{n} k^{\alpha}(n-j)f(j), \quad n \in \mathbb{N}_0.$$

Recalling that the finite convolution $(*)$ between two sequences f and g is defined by:

$$(f * g)(n) := \sum_{j=0}^{n} f(n-j)g(j), \quad n \in \mathbb{N}_0$$

then the definition of α-th fractional sum is also equivalent to:

$$\Delta^{-\alpha} f(n) := (k^{\alpha} * f)(n), \quad n \in \mathbb{N}_0.$$

The next definition generalizes the previous α-th fractional sum.

Definition 4.1.5 *(**Nabla α-th fractional sum**). Let $\alpha > 0$. For any positive real number a, the nabla α-th fractional sum of a function f is:*

$$\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t} (t-s+1)^{\overline{\alpha-1}} f(s)$$

where $t \in \mathbb{N}_a$ and $t^{\overline{\alpha}} := \frac{\Gamma(t+\alpha)}{\Gamma(t)}$.

Definition 4.1.6 *(**Fractional difference operator in the sense of Riemann-Liouville**). The fractional difference operator in the sense of Riemann-Liouville $\Delta^\alpha : s(\mathbb{N}_0) \to s(\mathbb{N}_0)$ is defined by*

$$\Delta^\alpha f(n) := \Delta^m \circ \Delta^{-m-\alpha} f(n), \quad n \in \mathbb{N}_0,$$

where $m-1 < \alpha < m$, $m := \lceil \alpha \rceil$, the last integer that is greater than or equal to α.

For instance, for $0 < \alpha < 1$ we obtain:

$$\begin{aligned}
\Delta^\alpha f(n) &= \Delta \circ \Delta^{-(1-\alpha)} f(n) = \Delta \left(\sum_{j=0}^{n} k^{1-\alpha}(n-j) f(j) \right) \\
&= \sum_{j=0}^{n+1} k^{1-\alpha}(n+1-j) f(j) - \sum_{j=0}^{n} k^{1-\alpha}(n-j) f(j) \\
&= (1-\alpha) f(n+1) + \sum_{j=0}^{n} f(j) \left(k^{1-\alpha}(n+1-j) - k^{1-\alpha}(n-j) \right). \qquad (4.2)
\end{aligned}$$

Developing the expression $k^{1-\alpha}(n+1-j) - k^{1-\alpha}(n-j)$ we obtain:

$$\begin{aligned}
&k^{1-\alpha}(n+1-j) - k^{1-\alpha}(n-j) \\
&= \frac{(1-\alpha)(1-\alpha+1)\ldots(1-\alpha+n-j)}{(n+1-j)!} - \frac{(1-\alpha)(1-\alpha+1)\ldots(1-\alpha+n-j-1)}{(n-j)!} \\
&= \frac{(1-\alpha)\ldots(1-\alpha+n-j-1)}{(n-j)!} \left(\frac{1-\alpha+n-j}{n+1-j} - 1 \right) \\
&= \frac{(1-\alpha)\ldots(1-\alpha+n-j-1)}{(n-j)!} \left(\frac{-\alpha}{n+1-j} \right) \\
&= \frac{(-\alpha)(1-\alpha)\ldots(-\alpha+n-j)}{(n+1-j)!} = k^{-\alpha}(n+1-j).
\end{aligned}$$

Making the substitution of the previous development in equation (4.2) we get:

$$\Delta^\alpha f(n) = (1-\alpha) f(n+1) + \sum_{j=0}^{n} k^{-\alpha}(n+1-j) f(j) = \sum_{j=0}^{n+1} k^{-\alpha}(j) f(n+1-j) \qquad (4.3)$$

where in the last identity a change of variable has been made. This last identity (4.3) corresponds to the Grünwald-Letnikov scheme of approximation with unitary step for the one dimensional Caputo fractional derivative. In addition to the definition of fractional difference operator we also define the nabla fractional difference operator. Later, we will see that both the fractional difference operators are conjugated by the translation operator.

Definition 4.1.7 *(**Nabla fractional difference operator**) The nabla fractional difference operator $\nabla^{\alpha} : s(\mathbb{N}_a) \to s(\mathbb{N}_a)$ of order $\alpha > 0$ is defined by:*

$$\nabla_a^{\alpha} f(t) = \Delta_a^m \circ \nabla_a^{-(m-\alpha)} f(t), \quad t \in \mathbb{N}_a$$

where $m = \lceil \alpha \rceil$.

4.2 Transference principle

The following result known as transference principle shows that the fractional difference operator in the sense of Riemann-Liouville and the Nabla fractional difference operator are conjugated. This implies that both the operators are at the same time Devaney chaotic or not.

Theorem 4.2.1 *(**Transference principle**). Let $\alpha > 0$ and $a \in \mathbb{R}$ be given. Then we have*

$$\tau_a \circ \nabla_a^{\alpha} = \Delta^{\alpha} \circ \tau_a.$$

Proof. By the definition of Nabla α-th fractional sum, for $f \in s(\mathbb{N}_a)$ we have:

$$\tau \circ \nabla_a^{-\alpha} = \nabla_a^{-\alpha} f(n+a) = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} (n-j+1)^{\overline{\alpha-1}} f(a+j)$$

$$= \sum_{j=0}^{n} \frac{\Gamma(\alpha+n-j)}{\Gamma(\alpha)\Gamma(\alpha)\Gamma(n-j+1)} f(a+j) = \sum_{j=0}^{n} k^{\alpha}(n-j) f(a+j) = \Delta^{-\alpha} \circ \tau_a f(n),$$

for all $n \in \mathbb{N}_0$.

Let $f \in s(\mathbb{N}_a)$ be given. By the definition of Nabla fractional difference operator and the previous identities we have:

$$\begin{aligned}
\tau_a \circ \nabla_a^{\alpha} f(n) &= \tau_a \circ (\Delta_a^m \circ \nabla_a^{-(m-\alpha)}) f(n) = (\Delta_a^m \circ \nabla_a^{-(m-\alpha)}) f(n+a) = \\
&= \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} \nabla_a^{-(m-\alpha)} f(n+a+j) = \\
&= \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} \tau_a \circ \nabla_a^{-(m-\alpha)} f(n+j) = \\
&= \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} \Delta^{-(m-\alpha)} \circ \tau_a f(n+j) = \\
&= \Delta^m (\Delta^{-(m-\alpha)} \circ \tau_a f)(n) = \Delta^{\alpha} \circ \tau_a f(n),
\end{aligned}$$

for all $n \in \mathbb{N}_0$ and this proves the theorem. ■

Chapter 5

Chaos for non-local operators and numerical schemes

In this last chapter we will show the conditions under which Devaney chaos can be ensured for non-local operators and different numerical schemes. The main references among the chapter will be the articles [18] and [19].

5.1 Non-local operators are chaotic

Let $0 < \alpha < 1$. We recall equation (4.3):

$$\Delta^\alpha f(n) = (1-\alpha)f(n+1) + \sum_{j=0}^{n} k^{-\alpha}(n+1-j)f(j) = \sum_{j=0}^{n+1} k^{-\alpha}(j)f(n+1-j).$$

If we evaluate the previous operator on a generic canonical vector $e_l(n)$ we get:

$$\Delta^\alpha e_l(n) = \sum_{j=0}^{n+1} k^{-\alpha}(j)e_l(n+1-j).$$

The first observation we make is that if $n+1 < l$ then $n+1-j < l$ for all $j = 0, 1, ..., n+1$ so $\Delta^\alpha e_l(n) = 0$. Now if $n = l-1$ an easy computation shows that $\Delta^\alpha e_l(l-1) = k^{-\alpha}(0) = 1$. Finally if $n > l-1$ then we get that $n+1-j = l$ if and only if $j = n-l+1$ so $\Delta^\alpha e_l(n) = k^{-\alpha}(n-l+1)$. We can summarize these observations as follows:

$$\Delta^\alpha e_l(n) = \begin{cases} k^{-\alpha}(n+1-l) & \text{if } n \geqslant l \\ 1 & \text{if } n = l-1 \\ 0 & \text{if } n < l-1 \end{cases}.$$

Furthermore, we can also compute the representation of Δ^α in the canonical basis $\{e_l(n)\}_{n,l\in\mathbb{N}_0}$ as the following Toeplitz matrix:

$$\begin{bmatrix} k^{-\alpha}(1) & 1 & 0 & 0 & \dots \\ k^{-\alpha}(2) & k^{-\alpha}(1) & 1 & 0 & \dots \\ k^{-\alpha}(3) & k^{-\alpha}(2) & k^{-\alpha}(1) & 1 & \dots \\ k^{-\alpha}(4) & k^{-\alpha}(3) & k^{-\alpha}(2) & k^{-\alpha}(1) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \tag{5.1}$$

$$= \begin{bmatrix} -\alpha & 1 & 0 & 0 & \dots \\ \frac{-\alpha(-\alpha+1)}{2} & -\alpha & 1 & 0 & \dots \\ \frac{-\alpha(-\alpha+1)(-\alpha+2)}{3} & \frac{-\alpha(-\alpha+1)}{2} & -\alpha & 1 & \dots \\ \frac{-\alpha(-\alpha+1)(-\alpha+2)(-\alpha+3)}{4} & \frac{-\alpha(-\alpha+1)(-\alpha+2)}{3} & \frac{-\alpha(-\alpha+1)}{2} & -\alpha & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Theorem 5.1.2 will show that Δ^α is a well defined Toeplitz operator on $\ell^2(\mathbb{N}_0)$ and exhibits chaos for any $0 < \alpha < 1$. In order to achive it we will use the following lemma of univalence for meromorphic functions stated in [10].

Lemma 5.1.1 *Let M_n denote the class of functions of the form $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$ which are regular in $0 < |z| < 1$ and satisfy*

$$\mathscr{R}\left(\frac{D^{n+1}f(z)}{D^n f(z)} - 2\right) < -\frac{n}{n+1}, \quad |z| \leqslant 1$$

where $D^n f(z) = \frac{1}{z}\left(z^{n+1}\frac{f(z)}{n!}\right)^{(n)}$, $m \in \mathbb{N}_0$. Then $M_{n+1} \subset M_n$ for all $n \in \mathbb{N}_0$ and all functions in M_n are univalent.

Theorem 5.1.2 *For any $0 < \alpha < 1$, the operator Δ^α defines a chaotic Toeplitz operator on $\ell^2(\mathbb{N}_0)$ with symbol $\Phi(z) = \frac{(1-z)^\alpha}{z}$.*

Proof. First we will prove that the fractional difference operator Δ^α is bounded in $\ell^2(\mathbb{N}_0)$. Given $u \in \ell^2(\mathbb{N}_0)$, we have by equation (4.3) that

$$\Delta^\alpha u = (1-\alpha)\tau_1 u + \tau_1 k^{-\alpha} * u,$$

where τ_1 is the translation operator by 1. By [23] and [12, Proposition 3.1 (viii)], we get that:

$$k^{-\alpha}(n) = \frac{1}{n^{\alpha+1}\Gamma(\alpha)}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right) \approx \frac{C}{n^{\alpha+1}}. \tag{5.2}$$

so the sequence $(k^{-\alpha}(n))_{n\in\mathbb{N}_0} \in \ell^1$.
Now using Young's inequality and the previous expression (5.2) we obtain:

$$\|\Delta^\alpha u\|_2 \leqslant \|(1-\alpha)\tau_1 u\|_2 + \|\tau_1 k^{-\alpha} * u\|_2 \leqslant |(1-\alpha)|\|\tau_1 u\|_2 + \|\tau_1 k^{-\alpha}\|_1\|u\|_2 < \infty$$

leading us to assert that Δ^α is a bounded operator in $\ell^2(\mathbb{N}_0)$. Now observing the expression (5.1) we get that Δ^α is a Toeplitz operator on $\ell^2(\mathbb{N}_0)$. Comparing the expression (5.1) with equation (3.1) we get that the coefficients of the Toeplitz operator symbol are given by:

$$\begin{cases} a_{-1} = 1 \\ a_n = k^{-\alpha}(n+1), & \text{if } n \geqslant 0 \\ a_n = 0, & \text{if } n < 0 \end{cases}$$

which leads us to state that the symbol of the operator is:

$$\Phi(z) = \frac{1}{z} + \sum_{j=0}^{\infty} k^{-\alpha}(j+1)z^j.$$

Let us denote $\varphi(z) = \sum_{j=0}^{\infty} k^{-\alpha}(j+1)z^j$. Using the generation formula (4.1) we have:

$$z \cdot \varphi(z) = \sum_{j=0}^{\infty} k^{-\alpha}(j+1)z^{j+1} = \sum_{j=0}^{\infty} k^{-\alpha}(j)z^j - k^{-\alpha}(0) = \frac{1}{(1-z)^{-\alpha}} - 1,$$

and therefore:

$$\varphi(z) = \frac{(1-z)^{\alpha}}{z} - \frac{1}{z}.$$

So the symbol of the Toeplitz operator Δ^{α} is:

$$\Phi(z) = \frac{1}{z} + \varphi(z) = \frac{(1-z)^{\alpha}}{z}.$$

Now we will prove the chaotic behaviour of the operator T_{Φ} and to do so we use Theorem 3.1.15. Let first check condition (i), that is, $\Phi(z) = \frac{(1-z)^{\alpha}}{z}$ is univalent in $\overline{\mathbb{D}}\backslash\{0\}$. Using Theorem 5.1.1 we have to show that

$$\mathscr{R}\left(\frac{D^{n+1}\Phi(z)}{D^n\Phi(z)} - 2\right) < 0, \quad |z| \leqslant 1.$$

An easy computation shows that $D^1\Phi(z) = \frac{(1-z)^{\alpha}}{z} - \alpha(1-\alpha)^{\alpha-1}$ and taking $z = a + ib$ with $-1 \leqslant 1 \leqslant 1$ and $-1 \leqslant b \leqslant 1$ then it follows that

$$\mathscr{R}\left(\frac{D^{n+1}\Phi(z)}{D^n\Phi(z)} - 2\right) = \mathscr{R}\left(-1 - \frac{\alpha z}{1-z}\right) = \frac{-(1-\alpha)^2 + (\alpha-1)b^2 - \alpha a(1-a)}{(1-a^2)+b^2}.$$

It is clear that $\mathscr{R}\left(\frac{D^{n+1}\Phi(z)}{D^n\Phi(z)} - 2\right) < 0$ if and only if $-(1-\alpha)^2 + (\alpha-1)b^2 - \alpha a(1-a) < 0$ and this last assertion holds since $0 < \alpha < 1$ and $-1 \leqslant a \leqslant 1$. It only remains to check condition (ii) in theorem 3.1.15. To do so, as $[-2^{\alpha}, 0]$ intersects $\mathbb{D}$ and $\mathbb{C}\backslash\overline{\mathbb{D}}$, we will check that $[-2^{\alpha}, 0] \subset \mathbb{C}\backslash\Phi(\mathbb{D})$. Let us first compute $\Phi(\mathbb{T})$:

$$\Phi(\mathbb{T}) = \left\{\frac{(1-e^{it})^{\alpha}}{e^{it}} : t \in [-\pi, \pi]\right\} = \left\{e^{i\left(t\alpha/2-1)-\frac{3\pi}{2}\right)}2^{\alpha}\sin(t/2)^{\alpha} : t \in [-\pi, \pi]\right\}.$$

If we represent the following set for $\alpha = 0.5$ we obtain the curve of figure 5.1. In this figure we have also represented the unit disk in shadowed blue. As we see in figure 5.1, the border between $\Phi(\mathbb{D})$ and $\mathbb{C}\backslash\Phi(\mathbb{D})$ is represented by a cardioid, so $\Phi(\mathbb{D})$ lies either in the interior of the cardiod or in the exterior. Nevertheless, observing that $\Phi(0) = \infty$ we get that $\Phi(\mathbb{D})$ corresponds to the exterior of the cardioid (in the figure $\Phi(\mathbb{D})$ is represented in shadowed orange). Let us also observe that $\Phi(1) = 0$ and $\Phi(-1) = -2^{\alpha}$ so it is clear by the shape of the cardioid that the interval $[0, -2^{\alpha}]$ belongs to its interior and therefore $[0, -2^{\alpha}] \cap \Phi(\mathbb{D}) = \varnothing$, also $[0, -2^{\alpha}] \subset \mathbb{C}\backslash\Phi(\mathbb{D})$ as we wanted to prove. ■

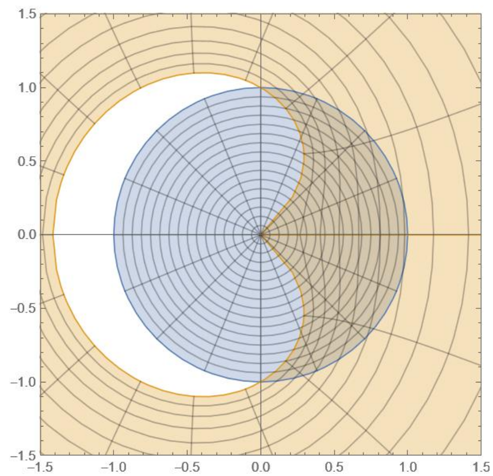

Figure 5.1: Representation of $\Phi(\mathbb{T})$ with $\alpha = 0.5$.

Let us observe that the symbol of the Toeplitz operator Δ^{α} coincides with the symbol of the explicit Euler approximation scheme for the Riemann-Liouville fractional difference operator [16].
The transference principle 4.2.1 ensures the following corollary.

Corollary 5.1.3 *For any $0 < \alpha < 1$ and $a > 0$, the Nabla difference operator ∇_a^{α} is chaotic in $\ell^2(\mathbb{N}_a)$.*

Proof. It is a direct consequence of theorem 5.1.2 throughout the transference principle 4.2.1 and the preservation of Devaney chaos under quasiconjugacy 2.0.14. ■

5.2 Chaos for operators related to fractional numerical schemes

In this section we will prove chaos for several operators that are related to fractional numerical schemes. Throughout the section we will implicitly use the notion of Gelfand transform.

Definition 5.2.1 ***(Gelfand transform).*** *Let $b \in \ell^1(\mathbb{N}_0)$ be a summable sequence, we define its Gelfand transform by:*

$$\delta(z) := \sum_{n=0}^{\infty} b(n) z^n, \quad z \in \mathbb{D}.$$

Time-stepping schemes for fractional operators [16] are defined by a convolution operator $\partial_b^\alpha : \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0)$ given by:

$$\partial_b^\alpha u(n) := (b * u), \quad n \in \mathbb{N}_0,$$

where $b \in \ell^1(\mathbb{N}_0)$ is a real valued sequence implicitly defined by the generating series:

$$\delta(\zeta) = \sum_{j=0}^{\infty} b(n) \zeta^n, \quad \zeta \in \mathbb{T},$$

and $\delta(\zeta)$ is called the symbol of the scheme. Evaluating the operator ∂_b^α on a generic canonical vector $e_l(n)$ we obtain:

$$\partial_b^\alpha e_l(n) = \sum_{j=0}^{n} b(n-j) e_l(j). \tag{5.3}$$

One easy observation is that if $l > n$ then $\partial_b^\alpha e_l(n) = 0$ and if $l \leqslant n$ then $\partial_b^\alpha e_l(n) = b(n-l)$. We can summarize this observation as follows:

$$\partial_b^\alpha e_l(n) = \begin{cases} 0 & \text{if } n < l \\ \\ b(n-l) & \text{if } n \geqslant l \\ \end{cases}.$$

Computing the representation of ∂_b^α in the canonical basis $\{e_l(n)\}_{n,l \in \mathbb{N}_0}$ we obtain the following matrix:

$$\begin{bmatrix} b(0) & 0 & 0 & 0 & \dots \\ b(1) & b(0) & 0 & 0 & \dots \\ b(2) & b(1) & b(0) & 0 & \dots \\ b(3) & b(2) & b(1) & b(0) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{5.4}$$

In the following paragraphs we will demonstrate that the adjoint operators $(\partial_b^\alpha)^*$ of some of the time-stepping schemes are chaotic. Nevertheless, to do so we will need the following lemma.

Lemma 5.2.2 *Let $\partial_b^\alpha : \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0)$ defined, as before, by a convolution operator:*

$$\partial_b^\alpha u(n) := (b * u), \quad n \in \mathbb{N}_0,$$

in this case the adjoint operator $(\partial_b^\alpha)^*u$, takes the form:

$$(\partial_b^\alpha)^*u(n) = \sum_{j=0}^{\infty} b(j)u(n+j) = \sum_{j=0}^{\infty} b(j)B^j u(n), \quad n \in \mathbb{N}_0, \tag{5.5}$$

where B denotes the backward operator.

Proof. Let us recall that the adjoint operator $(\partial_b^\alpha)^*$ is defined as the unique operator satisfying $\langle(\partial_b^\alpha)^*u, v\rangle = \langle u, \partial_b^\alpha v\rangle$ for all $u, v \in \ell^2(\mathbb{N}_0)$. Therefore evaluating $\langle u, \partial_b^\alpha v\rangle$ we obtain

$$\begin{aligned}
\langle u, \partial_b^\alpha v\rangle &= \sum_{n=0}^{\infty} u(n)\partial_b^\alpha v(n) = \sum_{n=0}^{\infty} u(n) \sum_{j=0}^{n} b(n-j)v(j) = \sum_{j=0}^{\infty}\sum_{j=n}^{\infty} u(n)b(n-j)v(j) \\
&= \sum_{j=0}^{\infty} v(j) \sum_{m=0}^{\infty} u(m+j)b(m) = \sum_{j=0}^{\infty} v(j) \sum_{m=0}^{\infty} (B^m u)(j)b(m) \\
&= \langle \sum_{m=0}^{\infty} (B^m u)(j)b(m), v\rangle = \langle(\partial_b^\alpha)^*u, v\rangle,
\end{aligned}$$

as we wanted to demonstrate. ■

Applying this last lemma, we obtain that the representation of the adjoint operator $(\partial_b^\alpha)^*$ in the canonical basis $\{e_l(n)\}_{n,l\in\mathbb{N}_0}$ corresponds to the following matrix:

$$\begin{bmatrix} b(0) & b(1) & b(2) & b(3) & \dots \\ 0 & b(0) & b(1) & b(2) & \dots \\ 0 & 0 & b(0) & b(1) & \dots \\ 0 & 0 & 0 & b(0) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{5.6}$$

Now let us observe that the symbol of the numerical scheme δ coincides with the Gelfand transform of the sequence b and also coincides with the holomorphic function that characterizes chaos in theorem 2.1.12. This observation allows us to consider condition (ii) in theorem 2.1.12 in order to prove chaos for the following operators we are going to consider.

Let us consider as a first example the backward Euler scheme, whose symbol is $\delta(\zeta) = 1 - \zeta$. Comparing with the series (5.3) we get that $b(n) = e_0(n) - e_1(n)$. And therefore:

$$\partial_b^\alpha u(n) = u(n) - u(n-1), \quad n \in \mathbb{N}_1.$$

We get that the associated matrix for this operator in the canonical basis is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & \dots \\ 0 & 0 & -1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Again by comparison with the series (5.5) and matrix (5.6) we obtain:

$$(\partial_b^\alpha)^* u(n) = u(n) - u(n+1), \quad n \in \mathbb{N}_0,$$

and the associated matrix is:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & \dots \\ 0 & 1 & -1 & 0 & \dots \\ 0 & 0 & 1 & -1 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

It is clear that the symbol $\delta(\zeta) = 1 - \zeta$ is such that $\delta(\mathbb{D}) \cap \mathbb{T} \neq 0$ so the adjoint operator of the backward Euler scheme is chaotic by theorem 2.1.12.
Let us now consider the fractional backward Euler scheme with symbol $\delta(\zeta) = \tau^{-\alpha}(1-\zeta)^\alpha$, for some $\alpha > 0$ and $\tau > 0$ the step size of the scheme. By the generation series (4.1) we get that the symbol of the scheme can be also written as follows:

$$\delta(\zeta) = \tau^{-\alpha}(1-z)^\alpha = \tau^{-\alpha}\left(\sum_{n=0}^{\infty} k^{-\alpha}(n) z^n\right),$$

so we can identify the sequence $b_\tau(n)$ as:

$$b_\tau = \tau^{-\alpha} k^{-\alpha}(n). \tag{5.7}$$

Let us observe that with the estimation (5.2), the sequence $(k^{-\alpha}(n))_{n\in\mathbb{N}_0}$ belongs to $\ell^1(\mathbb{N}_0)$, so it is clear that the sequence $(b_\tau(n))_{n\in\mathbb{N}_0}$ also belongs to $\ell^1(\mathbb{N}_0)$. With identity (5.7), the operator that defines the fractional backward Euler scheme is given by:

$$\partial_{b_\tau}^\alpha u(n) = (b_\tau * u)(n) = \sum_{j=0}^{n} \tau^{-\alpha} k^{-\alpha}(n-j) u(j)$$

and its dual operator is:

$$(\partial_{b_\tau}^\alpha)^* u(n) = \sum_{j=0}^{\infty} \tau^{-\alpha} k^{-\alpha}(j) B^j u(n).$$

It is surprising that the adjoint of the fractional backward Euler operator $(\partial_{b_\tau}^\alpha)^*$ correspond to the Weil fractional difference operator W_τ^α. The following result proves that chaos for the Weil fractional difference operator depends on the step size τ but not on the fractional order α.

Theorem 5.2.3 *For any $\alpha > 0$, the Weil fractional difference operator is chaotic on $\ell^2(\mathbb{N}_0)$ if and only if* $0 < \tau < 2$.

Proof. By theorem 2.1.12, W_τ^α is chaotic if and only if $\delta(\mathbb{D}) \cap \mathbb{T} \neq 0$, and this in turn is equivalent to the existence of $w \in \mathbb{T}$ such that $w = \tau^{-\alpha}(1-z)^\alpha$, with $|z| < 1$ and then $|1 - \tau w^{1/\alpha}| = |z|$. This last identity implies that W_τ^α is chaotic if and only if $\tau w^{1/\alpha}$ belongs to the unity disk with center 1. Nevertheless, this is satisfied if and only if $0 < \tau < 2$ as we wanted to demonstrate. ■

Let us now consider the fractional second order backward Euler difference, whose symbol is given by:

$$\delta(\zeta) = \tau^{-\alpha}\left(\frac{3}{2} - 2\zeta + \frac{1}{2}\zeta^2\right)^\alpha = \tau^{-\alpha}\left(\frac{3}{2}\right)^\alpha (1-\zeta)^\alpha \left(1 - \frac{\zeta}{3}\right)^\alpha. \tag{5.8}$$

It was proven in [17] that for this time-stepping scheme, the sequence $b(n)$ is

$$b(n) = \frac{3}{2}e_0(n) - 2e_1(n) + \frac{1}{2}e_2(n)$$

in case $\alpha = 1$, and

$$b(n) = \left(\frac{3}{2}\right)^\alpha \sum_{j=0}^{n} k^{-\alpha}(n-j)\frac{1}{3^j}k^{-\alpha}(j)$$

when $\alpha \neq 1$. Let us check that $b \in \ell^1(\mathbb{N}_0)$. In the case that $\alpha = 1$ trivially $b \in \ell^1(\mathbb{N}_0)$. When $\alpha \neq 1$ let us define $c(n) := \frac{1}{3^n}k^{-\alpha}(n)$. By the estimation (5.2) we have that $c(n) \approx \frac{C}{3^n n^{1+\alpha}}$, so $c \in \ell^1(\mathbb{N}_0)$. Now by the Young's convolution inequality we get:

$$\|b\|_1 = \left(\frac{3}{2}\right)^\alpha \|k^{-\alpha} * c\|_1 \leqslant \left(\frac{3}{2}\right)^\alpha \|k^{-\alpha}\|_1 \cdot \|c\|_1 < \infty,$$

so $b \in \ell^1(\mathbb{N}_0)$. We will consider for such a sequence $b(n)$ the following scheme:

$$\partial_b^\alpha u(n) = \tau^{-\alpha}(b * u)(n),$$

where $\tau > 0$ is the step size. The next result shows chaos for the adjoint of such an operator.

Theorem 5.2.4 *The operator T_b, which is the dual of the operator that defines the fractional second order backward difference scheme with step size τ, is chaotic on $\ell^2(\mathbb{N}_0)$ if and only if $0 < \tau < 4$.*

Proof. By theorem 2.1.12, we have to prove that $\delta(\mathbb{D}) \cap \mathbb{T} \neq \varnothing$ if and only if $0 < \tau < 4$, where δ is given by (5.8). Since δ is a holomorphic function on $\mathbb{D}$, by the maximum principle we have $\sup_{z\in\mathbb{D}} |\delta(z)| = \max_{z\in\mathbb{T}} |\delta(z)| = 4^\alpha\tau^{-\alpha}$, whose maximum is attained at $z = -1$. This implies that $\delta(\mathbb{D}) \cap \mathbb{T} \neq \varnothing$ if and only if $4^\alpha\tau^{-\alpha} > 1$, and this in turn is equivalent to τ being in $(0, 4)$ as we wanted to prove. ■

Finally, we consider the fractional Crank-Nicholson stepping scheme, whose symbol is given by

$$\delta(\zeta) = \tau^{-\alpha}\frac{(1-\zeta)^{\alpha}}{1-\frac{\alpha}{2}+\frac{\alpha}{2}\zeta}, \tag{5.9}$$

where $0 < \alpha < 2$. By [17] we have that the sequence $b(n)$ is given by

$$b(n) = \tau^{-\alpha}\frac{2}{2-\alpha}\sum_{j=0}^{n} k^{-\alpha}(n-j)\left(\frac{\alpha}{\alpha-2}\right)^{j}.$$

For $0 < \alpha < 1$, let us define $c(n) := \left(\frac{\alpha}{\alpha-2}\right)^{n}$. Since $\alpha < 1$, we have $c \in \ell^1(\mathbb{N}_0)$ and by Young's convolution inequality

$$\|b\|_1 = \tau^{-\alpha}\frac{2}{2-\alpha}\|k^{-\alpha} * c\|_1 \leqslant \tau^{-\alpha}\frac{2}{2-\alpha}\|k^{-\alpha}\|_1 \cdot \|c\|_1 < \infty,$$

so $b \in \ell^1(\mathbb{N}_0)$. The next result shows conditions for chaos in the adjoint of the fractional Crank-Nicholson stepping scheme operator in the case $0 < \alpha < 1$.

Theorem 5.2.5 *Let $0 < \alpha < 1$. The operator T_b, which is the dual of the operator that defines the fractional Crank-Nicholson scheme with step τ, is chaotic on $\ell^2(\mathbb{N}_0)$ if and only if $0 < \tau < \frac{2}{(1-\alpha)^{1/\alpha}}$.*

Proof. By theorem 2.1.12, we have to prove that $\delta(\mathbb{D}) \cap \mathbb{T} \neq \varnothing$. By the maximum principle for holomorphic functions we have that $\sup_{z\in\mathbb{D}} \delta(z) = \max_{z\in\mathbb{T}} \delta(z)$, which observing the scheme (5.9) is attained at $z = -1$ with value $\delta(-1) = \tau^{-\alpha}\frac{2^{\alpha}}{1-\alpha}$. This implies that $\delta(\mathbb{D}) \cap \mathbb{T} \neq \varnothing$ if and only if $\tau^{-\alpha}\frac{2^{\alpha}}{1-\alpha} > 1$, which is in turn equivalent to $0 < \tau < \frac{2}{(1-\alpha)^{1/\alpha}}$ as we wanted to demonstrate. ■

5.3 Chaos for numerical schemes

Finite difference methods are one of the most used numerical methods in order to solve differential equations. A finite differences scheme consists on a system of equations that can be solved by basic linear algebra techniques. The derivatives in this numerical scheme are approximated by divided finite differences in a discrete set of points.

This transformation of a differential continuous problem into a system of algebraic equations makes possible to find the solution by iterative algorithms in computers, in fact, nowadays is one of the most used techniques for solving differentials equations. Many of these numerical schemes for differential equations can be regarded as dynamical systems. In this context, it is interesting the study of the dynamic behaviour and the conditions under which this system can exhibit chaos.

There is a variety of finite divided difference approximations of derivatives depending on the level of accuracy required. The general method to obtain these estimates is based in the Taylor series expansion of the function (see [7] for a complete derivation of these approximations), although there exist other non-standard approaches to derive divided finite difference approximations of derivatives [21]. The simplest examples of finite differences are the approximations of the derivative of a real function $f : \mathbb{R} \to \mathbb{R}$. With the usual notation, fixed $x_0 \in \mathbb{R}$ and $h > 0$, $x_n = x_0 + kh$, $k \in \mathbb{Z}$, $f_k = f(x_k)$ and $f'_k = f'(x_k)$, the well-known standard finite differences approximations for the first and the second derivatives are

$$
\begin{array}{lll}
f'_k \approx \dfrac{f_{k+1} - f_k}{h}\,, & f''_k \approx \dfrac{f_{k+2} - 2f_{k+1} + f_k}{h^2}\,, & \text{Forward} \\
f'_k \approx \dfrac{f_k - f_{k-1}}{h}\,, & f''_k \approx \dfrac{f_k - 2f_{k-1} + f_{k-2}}{h^2}\,, & \text{Backward} \\
f'_k \approx \dfrac{f_{k+1/2} - f_{k-1/2}}{h}\,, & f''_k \approx \dfrac{f_{k+1} - 2f_k + f_{k-1}}{h^2}\,, & \text{Centered}
\end{array}
$$

The next step is to construct finite difference schemes for differential equations using these approximate derivatives of divided differences. In this work, we consider finite difference schemes for the one-dimensional heat equation and other PDEs related with it, with the purpose of analyzing the conditions under which the numerical schemes are chaotic. Consider the heat (or diffusion) equation (HE) on a infinite thin rod that, in one dimension, is given by

$$
\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}\,, \tag{5.10}
$$

where $u(t, x)$ represents the temperature in the point x of the rod at time t, and $\alpha > 0$ is the thermal diffusivity. Herzog showed in [15] the chaotic behaviour of the solution semigroup to the HE on certain spaces of analytic functions with controlled growth.

We will present two examples of discretization of the heat equation (5.10) in order to prove its possibly chaotic behaviour.

Example 1. Centered space derivative scheme

One easy numerical scheme for (5.10) is obtained by using a forward discretization for the time derivative and a centered approximation for the space second derivative. We assume the thin rod to be infinite so that $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^+$. We denote

$$
t_n := n\Delta t\,, \quad x_k := k\Delta x\,, \quad u_k^n := u(t_n, x_k)\,,
$$

where Δt and Δx are the time and space steps respectively, and $n, k \in \mathbb{Z}_+$ and the common choice of $(0, 0)$ has been considered as the initial point. This way we obtain the finite difference equation

$$
\frac{u_k^{n+1} - u_k^n}{\Delta t} = \alpha \frac{u_{k+1}^n - 2u_k^n + u_{k-1}^n}{\Delta x^2}. \tag{5.11}
$$

If we let the sequence $u^n := (u_k^n)_{k\geqslant 0}$ to lie on a convenient sequence space X, the above difference equation reads as

$$u^{n+1} = Tu^n,$$

where $T : X \to X$ is a linear operator whose canonical matrix is an infinite tridiagonal matrix with constant diagonals. This means that the asymptotic behaviour of the numerical scheme (5.11) is given by the iterates of the linear operator T acting on the initial condition, that is, $u^{n+1} = T^{n+1}u^0$ for $n \geqslant 0$, we assume $u^0 \in X$.
Now, let us express equation (5.11) into the following explicit scheme

$$u_k^{n+1} = \lambda u_{k+1}^n + (1-2\lambda)u_k^n + \lambda u_{k-1}^n\,. \tag{5.12}$$

where $\lambda := \alpha\frac{\Delta t}{\Delta x^2}$. This way equation (5.12) can be written as $u^{n+1} = T_\Phi u^n$, where T_Φ is a tridiagonal Toeplitz operator with symbol

$$\Phi(z) = \lambda/z + (1-2\lambda) + \lambda z.$$

Since $a_{-1} = a_1$, by Theorem 3.1.11, we have that T_Φ is not chaotic.

Example 2. Forward space derivative scheme

Scheme (5.12) is not chaotic, nevertheless, it is possible to construct numerical schemes which exhibit chaos. Consider the numerical scheme for the HE obtained by applying a forward approximation of the space derivative in (5.10). We obtain the finite differences equation:

$$\frac{u_k^{n+1} - u_k^n}{\Delta t} = \alpha\frac{u_{k+2}^n - 2u_{k+1}^n + u_k^n}{\Delta x^2}. \tag{5.13}$$

This equation can be written as

$$u_k^{n+1} = \lambda u_{k+2}^n - 2\lambda u_{k+1}^n + (1+\lambda)u_k^n, \tag{5.14}$$

where $\lambda = \alpha\frac{\Delta t}{\Delta x^2}$. Scheme (5.14) reads as $u^{n+1} = T_\Phi u^n$, where $T_\Phi : X \to X$ is the Toeplitz operator

$$T_\Phi = \lambda B^2 - 2\lambda B + (\lambda+1)I = \lambda(B-I)^2 + I$$

and B denotes the backward shift operator. Taking $\varphi(z) = \lambda(z-1)^2 + 1$, we have that $\varphi(B) = T_\Phi$ is a Toeplitz operator whose symbol is the antianalytic function $\Phi(z) = \varphi(1/z)$ and the study of its chaotic behavior is addressed by theorem 2.1.11.

Proposition 5.3.1 *Let $\varphi(z) = \lambda(z-1)^2 + 1$ and set $\Phi(z) = \varphi(1/z)$. The Toeplitz operator $T_\Phi : X \to X$ is chaotic for each $\lambda > 0$.*

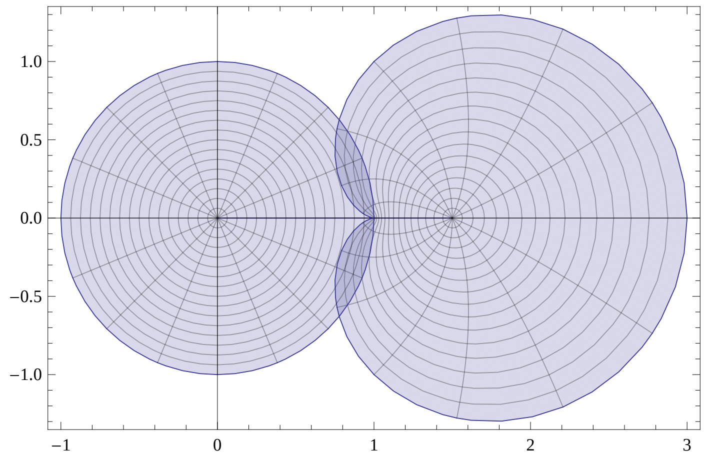

Figure 5.2: Example of the cardioid generated by the set $\varphi(\mathbb{T})$ with $\lambda = 0.5$ and its visual intersection with $\mathbb{D}$.

Proof. By Theorem 2.1.11 it is enough to check that $\varphi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$. Let us first compute $\varphi(\mathbb{T})$:

$$\begin{aligned}\varphi(e^{i\theta}) &= \lambda(e^{i\theta} - 1)^2 + 1 = \lambda(e^{2i\theta} - 2e^{i\theta} + 1) + 1\\ &= \lambda e^{i\theta}(e^{i\theta} - 2 + 1e^{-i\theta}) = 2\lambda(\cos\theta - 1)e^{i\theta} + 1, \quad \theta \in [0, 2\pi).\end{aligned}$$

By this last identity we get that $\varphi(\mathbb{T})$ is a cardioid such that its interior corresponds to $\varphi(\mathbb{D})$. As in the proof of theorem 5.1.2, we can represent the set $\varphi(\mathbb{D})$ with $\lambda = 0.5$ along with the complex unit disk $\mathbb{D}$, obtaining figure 5.2. Now, let us observe that the cardioid curve is symmetric with respect to the abscissas axis, and its cusp, the point where the tangent vector vanishes, is placed as $\varphi(1) = 1$, that is, whenever $\theta = 0$. So it is clear that for a small enough θ_0, it is verified $|\varphi(\theta_0)| < 1$, and therefore $\varphi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$ as we wanted to demonstrate. ■

Bibliography

[1] J. Banasiak and M. Lachowicz. "Topological chaos for birth-and-death-type models with proliferation". In: *Mathematical Models and Methods in Applied Sciences* 12(6) (2002), pp. 755–775.

[2] J. Banasiak and M. Moszyński. "Dynamics of birth-and-death processes with proliferation-stability and chaos". In: *Discrete & Continuous Dynamical Systems* 29(1) (2011), pp. 67–79.

[3] J. Banks et al. "On Devaney's definition of chaos". In: *The American mathematical monthly* 99(4) (1992), pp. 332–334.

[4] A. Baranov and A. Lishanskii. "Hypercyclic Toeplitz operators". In: *Results in Mathematics* 70(3) (2016), pp. 337–347.

[5] A. Böttcher, A.Y. Karlovich, and B. Silbermann. *Analysis of Toeplitz operators*. Springer Science & Business Media, 2013.

[6] H. Brézis. *Functional analysis, Sobolev spaces and partial differential equations*. Springer, 2011.

[7] S.C. Chapra and R.P. Canale. *Numerical Methods for Engineers*. McGraw-Hill Higher Education, 2006.

[8] R. Devaney. *An Introduction to Chaotic Dynamical Systems*. CRC press, 2018.

[9] P.L. Duren. *Theory of H^p Spaces*. Dover Publications, 2000.

[10] M. R. Ganingi, M. D. Ganigi, and B. A. Uralegaddi. "New criteria for meromorphic univalent functions". In: *Bulletin mathématique de la Société des Sciences Mathématiques de la République Socialiste de Roumanie* 33(81) (1989), pp. 9–13.

[11] C. Goodrich. "On discrete sequential fractional boundary value problems". In: *Journal of Mathematical Analysis and Applications* 385(1) (2012), pp. 111–124.

[12] C. Goodrich and C. Lizama. "A transference principle for nonlocal operators using a convolutional approach: fractional monotonicity and convexity". In: *Israel Journal of Mathematics* 236(2) (2020), pp. 533–589.

[13] C. Goodrich and A. C. Peterson. *Discrete Fractional Calculus*. Springer International Publishing, 2015.

[14] K.G. Grosse-Erdmann and A. Peris. *Linear Chaos*. Springer London, 2011.

[15] G. Herzog. “On a Universality of the Heat Equation”. In: *Mathematische Nachrichten* 188 (1997).

[16] B. Jin, B. Li, and Z. Zhou. “Discrete maximal regularity of time-stepping schemes for fractional evolution equations”. In: *Numerische mathematik* 138(1) (2018), pp. 101–131.

[17] C. Lizama and M. Murillo-Arcila. “Discrete maximal regularity for Volterra equations and nonlocal time-stepping schemes”. In: *Discrete and Continuous Dynamical Systems* 40(1) (2020), pp. 509–528.

[18] C. Lizama, M. Murillo-Arcila, and A. Peris. “Nonlocal operators are chaotic”. In: *Chaos: An Interdisciplinary Journal of Nonlinear Science* 30(10) (2020), pp. 103–126.

[19] S. Bartoll F. Martínez-Giménez, A. Peris, and F. Rodenas. *Chaos for numerical schemes of differential operators.* (preprint).

[20] R. Meise and D. Vogt. *Introduction to functional analysis.* Clarendon press, 1997.

[21] R. Mickens and P. Jordan. “A positivity-preserving nonstandard finite difference scheme for the damped wave equation”. In: *Numerical Methods for Partial Differential Equations* 20 (2004), pp. 639–649.

[22] W. Rudin. *Real and Complex Analysis*. McGraw-Hill, Inc., 1987.

[23] A. Zygmund and R. Fefferman. *Trigonometric Series*. Cambridge University Press, 2003.