Dynamic response of periodic infinite structure to arbitrary moving load based on the Finite Element Method

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Abstract: *A common problem in railway engineering is the dynamic of repetitive structures subject to moving loads. Bridges, rails or catenaries are the most representative periodic structures, over which the train acts as a moving exciter. Usually, these structures are long enough to consider that their dynamic response is in permanent regime. To assume the steady-state regime some features have to be considered: infinite length structure, perfect periodicity and constant velocity of the moving load. This paper adopts these assumptions and provides the steady-state solution of a generic periodic structure subject to an arbitrary and also periodic moving load.*

The structure is divided into repetitive blocks modelled by the Finite Element Method. By applying the periodicity condition it is possible to consider the entire structure dynamics with only one block. The problem is stated in the frequency domain and moved back to time domain by means of Discrete Fourier Transform.

1 INTRODUCTION

The study of periodic structures subject to moving loads has a great relevance thanks to the wide use of high-speed trains. Rails, overhead contact lines or bridges are periodic structures whose dynamic response produced by the train has been studied under different approaches. The authors who consider an infinite periodic structure focus on the steady-state solution of the problem. The early analytic models found in the literature are based on an infinite continuous periodically supported string/beam [1, 2, 3]. In [1] an infinite periodic Euler-Bernoulli beam subject to a uniform moving harmonic pressure field is solved. The differential equation is solved in the domain between two supports and four boundary conditions allow to determine the coefficients of the solution. Boundary conditions are obtained from the periodicity condition of two consecutive supports and the momentum and shear equilibrium at these supports. In [2] a similar model subject to a constant moving load is solved using the modal method. A finite periodic supported beam is defined by N uncoupled differential equations based on a modal representation. The limit of the previous solution when $N \to \infty$ is computed for a moving constant load. The same problem is solved in [3], in which the Fourier Transform is used to shift to the frequency domain where the periodicity condition is easily formulated. The solution is obtained in the frequency domain and the Inverse Fourier Transform allows to obtain the response in the time domain. The presented approaches have in common the consideration of a periodic solution which allows considering only a single period or block of the string/beam between two consecutive supports.

The limitation of the previous references is their inability of modelling more complex structures. Some solutions have been found, for example in [4], in which an extension of the approach proposed in [3] is presented to solve a catenary model, including two strings and two spatial periods, one for supports and another for droppers. In [5], the beam is modelled by a two-and-a-half dimensional (2.5D) Finite Element model which allows to model any cross section of the

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beam. The solution is divided into the response produced by the external load and the response produced by the reactions of the supports. Fourier Transform respect to position x and time t is performed to solve the differential equation and the periodicity condition is applied to the reactions of the supports in the frequency domain. The same authors presented an improved model in [6] in which the dynamic interaction of multiple wheels with the periodic model is computed by means of the Fourier Series decomposition of the contact force.

The Finite Element Method (FEM) can be used to model any periodic structure by means of the so-called Wave Finite Element Method (WFEM). This method allows to compute the frequency response of finite or infinite periodic structures [7, 8]. The frequency response of a periodic infinite structure obtained by WFEM can be used to compute the response under a moving load by means of the Fourier Transform [9]. WFEM makes possible to model finite-length structures and even structures with transition zones [8], but for periodic infinite structures we present an alternative in which some inconveniences of WFEM are avoided. For example, some slender structures (as catenaries) present ill conditioning behaviour in WFEM.

In this paper, the periodicity condition is applied on FEM models to obtain the frequency response of any generic periodic infinite structure. Then, the response to a temporal excitation is obtained by means of the Discrete Fourier Transform (DFT). Finally, the pantograph-catenary dynamic interaction is solved with this method.

2 HARMONIC RESPONSE

In this section we obtain the harmonic response of the model as a tool for the computation of the steady-state response. Let consider an infinite structure with a periodic pattern along the longitudinal axis as in Fig. 1. The repeated block is called substructure and it is modelled by the FEM. The dynamic equation of the substructure for a harmonic load can be written as:

$$\mathbf{D}(\omega)\mathbf{u} = \mathbf{F} \tag{1}$$

in which $\mathbf{u}$ is the nodal displacement vector, $\mathbf{D}(\omega) = \mathbf{K} + i\omega\mathbf{C} - \omega^2\mathbf{M}$ is the dynamic stiffness matrix of the substructure and $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices, respectively. Note that the force vector $\mathbf{F}$ includes external forces and the reactions produced by the adjacent blocks. The nodes of the block can be divided into left (L) and right (R) boundary nodes and inner (I) nodes according to their positions. Thus, the previous equation can be split into:

$$\begin{bmatrix} \mathbf{D}_{\mathrm{LL}} & \mathbf{D}_{\mathrm{LI}} & \mathbf{D}_{\mathrm{LR}} \\ \mathbf{D}_{\mathrm{IL}} & \mathbf{D}_{\mathrm{II}} & \mathbf{D}_{\mathrm{IR}} \\ \mathbf{D}_{\mathrm{RL}} & \mathbf{D}_{\mathrm{RI}} & \mathbf{D}_{\mathrm{RR}} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_{\mathrm{L}} \\ \mathbf{u}_{\mathrm{I}} \\ \mathbf{u}_{\mathrm{R}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_{\mathrm{L}} \\ \mathbf{F}_{\mathrm{I}} \\ \mathbf{F}_{\mathrm{R}} \end{Bmatrix} \tag{2}$$

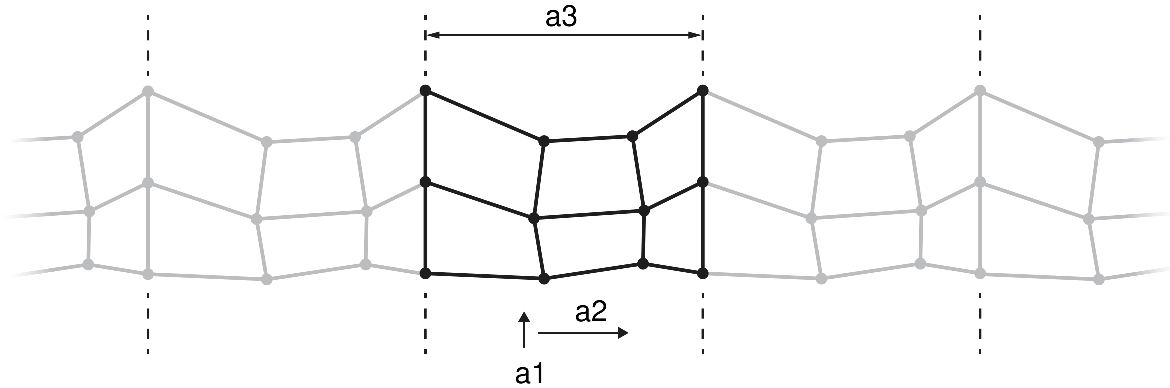

Figure 1: Periodic infinite FEM structure with moving load.

It is assumed that the load is repeated at every block so that the response of all blocks is identical but with a time lag that depends on the length L of the substructure and the velocity v of the moving load. This condition is called the periodicity condition and for the displacement u of any point it reads:

$$u(x,t) = u(x + nL, t + nL/v); \qquad n \in \mathbb{Z} \tag{3}$$

This condition allows to state the entire problem only in a single block of the structure, which is called the reference block. The periodicity condition can be moved to the frequency domain in which, the response of the next block to the reference one is:

$$\mathbf{u}^{next} = e^{-\frac{i\omega L}{v}}\mathbf{u} \tag{4}$$

Both blocks hold the following coupling condition in the common boundary:

$$\mathbf{u}_{\mathrm{R}} = \mathbf{u}_{\mathrm{L}}^{next} \tag{5}$$

so that the displacement of the left and right nodes of every substructure are related by:

$$\mathbf{u}_{\mathrm{L}} = e^{\frac{i\omega L}{v}}\mathbf{u}_{\mathrm{R}} \tag{6}$$

Applying this relation to Eq. (2):

$$\begin{bmatrix} \mathbf{D}_{\mathrm{LI}} & \mathbf{D}_{\mathrm{LR}} + e^{\frac{i\omega L}{v}}\mathbf{D}_{\mathrm{LL}} \\ \mathbf{D}_{\mathrm{II}} & \mathbf{D}_{\mathrm{IR}} + e^{\frac{i\omega L}{v}}\mathbf{D}_{\mathrm{IL}} \\ \mathbf{D}_{\mathrm{RI}} & \mathbf{D}_{\mathrm{RR}} + e^{\frac{i\omega L}{v}}\mathbf{D}_{\mathrm{RL}} \end{bmatrix} \left\{ \begin{array}{c} \mathbf{u}_{\mathrm{I}} \\ \mathbf{u}_{\mathrm{R}} \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{F}_{\mathrm{L}} \\ \mathbf{F}_{\mathrm{I}} \\ \mathbf{F}_{\mathrm{R}} \end{array} \right\} \tag{7}$$

The same procedure can be considered for the nodal forces:

$$\mathbf{F}^{next} = e^{-\frac{i\omega L}{v}}\mathbf{F} \tag{8}$$

which must satisfy the action-reaction principle in the boundary:

$$\mathbf{F}_{\mathrm{R}} = \mathbf{F}_{\partial\mathrm{R}} - \mathbf{F}_{\mathrm{L}}^{next} \tag{9}$$

in which $\mathbf{F}_{\partial\mathrm{R}}$ is the external load at the right boundary. By combining Eqs. (8) and (9) the left and right nodal forces of every substructure can be related by:

$$\mathbf{F}_{\mathrm{L}} = e^{\frac{i\omega L}{v}}\left(\mathbf{F}_{\partial\mathrm{R}} - \mathbf{F}_{\mathrm{R}}\right) \tag{10}$$

Introducing this constraint in Eq. (7) it becomes into:

$$\begin{bmatrix} \mathbf{D}_{\mathrm{LI}} & \mathbf{D}_{\mathrm{LR}} + e^{\frac{i\omega L}{v}}\mathbf{D}_{\mathrm{LL}} \\ \mathbf{D}_{\mathrm{II}} & \mathbf{D}_{\mathrm{IR}} + e^{\frac{i\omega L}{v}}\mathbf{D}_{\mathrm{IL}} \\ \mathbf{D}_{\mathrm{RI}} & \mathbf{D}_{\mathrm{RR}} + e^{\frac{i\omega L}{v}}\mathbf{D}_{\mathrm{RL}} \end{bmatrix} \left\{ \begin{array}{c} \mathbf{u}_{\mathrm{I}} \\ \mathbf{u}_{\mathrm{R}} \end{array} \right\} = \begin{bmatrix} \mathbf{0} & e^{\frac{i\omega L}{v}}\mathbf{I} \\ \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \left\{ \begin{array}{c} \mathbf{F}_{\mathrm{I}} \\ \mathbf{F}_{\partial\mathrm{R}} \end{array} \right\} + \begin{bmatrix} -e^{\frac{i\omega L}{v}}\mathbf{I} \\ \mathbf{0} \\ \mathbf{I} \end{bmatrix} \mathbf{F}_{\mathrm{R}} \tag{11}$$

If all the unknowns are moved to the left-hand side,

$$\begin{bmatrix} \mathbf{D}_{\mathrm{LI}} & \mathbf{D}_{\mathrm{LR}} + e^{\frac{i\omega L}{v}}\mathbf{D}_{\mathrm{LL}} & e^{\frac{i\omega L}{v}}\mathbf{I} \\ \mathbf{D}_{\mathrm{II}} & \mathbf{D}_{\mathrm{IR}} + e^{\frac{i\omega L}{v}}\mathbf{D}_{\mathrm{IL}} & \mathbf{0} \\ \mathbf{D}_{\mathrm{RI}} & \mathbf{D}_{\mathrm{RR}} + e^{\frac{i\omega L}{v}}\mathbf{D}_{\mathrm{RL}} & -\mathbf{I} \end{bmatrix} \left\{ \begin{array}{c} \mathbf{u}_{\mathrm{I}} \\ \mathbf{u}_{\mathrm{R}} \\ \mathbf{F}_{\mathrm{R}} \end{array} \right\} = \begin{bmatrix} \mathbf{0} & e^{\frac{i\omega L}{v}}\mathbf{I} \\ \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \left\{ \begin{array}{c} \mathbf{F}_{\mathrm{I}} \\ \mathbf{F}_{\partial\mathrm{R}} \end{array} \right\} \tag{12}$$

the displacement and the nodal forces are given by:

$$\left\{ \begin{array}{c} \mathbf{u}_{\mathrm{I}} \\ \mathbf{u}_{\mathrm{R}} \\ \mathbf{F}_{\mathrm{R}} \end{array} \right\} = \hat{\mathbf{H}}(\omega) \left\{ \begin{array}{c} \mathbf{F}_{\mathrm{I}} \\ \mathbf{F}_{\partial\mathrm{R}} \end{array} \right\} \tag{13}$$

in which

$$\hat{\mathbf{H}}(\omega) = \begin{bmatrix} \mathbf{D}_{\mathrm{LI}} & \mathbf{D}_{\mathrm{LR}} + e^{\frac{i\omega L}{v}} \mathbf{D}_{\mathrm{LL}} & \mathbf{I} e^{\frac{i\omega L}{v}} \\ \mathbf{D}_{\mathrm{II}} & \mathbf{D}_{\mathrm{IR}} + e^{\frac{i\omega L}{v}} \mathbf{D}_{\mathrm{IL}} & \mathbf{0} \\ \mathbf{D}_{\mathrm{RI}} & \mathbf{D}_{\mathrm{RR}} + e^{\frac{i\omega L}{v}} \mathbf{D}_{\mathrm{RL}} & -\mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} & \mathbf{I} e^{\frac{i\omega L}{v}} \\ \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{14}$$

Eq. (13) can be rewritten in terms of displacements of the substructure:

$$\left\{ \begin{array}{c} \mathbf{u}_{\mathrm{L}} \\ \mathbf{u}_{\mathrm{I}} \\ \mathbf{u}_{\mathrm{R}} \end{array} \right\} = \mathbf{H}(\omega) \left\{ \begin{array}{c} \mathbf{F}_{\mathrm{I}} \\ \mathbf{F}_{\partial\mathrm{R}} \end{array} \right\} \tag{15}$$

in which

$$\mathbf{H}(\omega) = \begin{bmatrix} e^{\frac{i\omega L}{v}} \hat{\mathbf{H}}_{\mathrm{R}}(\omega) \\ \hat{\mathbf{H}}_{\mathrm{I}}(\omega) \\ \hat{\mathbf{H}}_{\mathrm{R}}(\omega) \end{bmatrix} \tag{16}$$

being $\hat{\mathbf{H}}_{\mathrm{I}}(\omega)$ and $\hat{\mathbf{H}}_{\mathrm{R}}(\omega)$ the two first rows of $\hat{\mathbf{H}}(\omega)$.

3 TIME-FREQUENCY ANALYSIS

To achieve the response to an arbitrary moving load, the excitation is moved to the frequency domain where the frequency response obtained in the previous section can be used. Then, the response in the frequency domain is moved back to the time domain.

The structure is excited by a moving load $\mathbf{f}(t)$ with period L/v. This moving load causes the nodal forces $\mathbf{F}_{\mathrm{I}}(t)$ and $\mathbf{F}_{\partial\mathrm{R}}(t)$ which are evaluated in N discrete times $t_n = n\Delta t$.

$$\left\{ \begin{array}{c} \mathbf{F}_{\mathrm{I}}(t_n) \\ \mathbf{F}_{\partial\mathrm{R}}(t_n) \end{array} \right\} = \begin{bmatrix} \mathbf{N}_{\mathrm{I}}^{\top}(t_n) \\ \mathbf{N}_{\mathrm{R}}^{\top}(t_n) \end{bmatrix} \mathbf{f}(t_n) \tag{17}$$

$\mathbf{N}_{\mathrm{I}}(t_n)$ and $\mathbf{N}_{\mathrm{R}}(t_n)$ are the shape functions of the inner and right nodes evaluated at time t_n. These functions are used in FEM to transform nodal displacements into point displacements and it can also be used to transform point forces to nodal equivalent forces.

The Discrete Fourier Transform (DFT) is used to obtain the frequency representation of the nodal forces:

$$\left\{ \begin{array}{c} \mathbf{F}_{\mathrm{I}}(\omega_k) \\ \mathbf{F}_{\partial\mathrm{R}}(\omega_k) \end{array} \right\} = \sum_{n=0}^{N-1} \left\{ \begin{array}{c} \mathbf{F}_{\mathrm{I}}(t_n) \\ \mathbf{F}_{\partial\mathrm{R}}(t_n) \end{array} \right\} e^{\frac{-i2\pi kn}{N}} \tag{18}$$

in which

$$\omega_k = k\frac{2\pi}{N\Delta t}; \qquad k \in [0, N-1] \tag{19}$$

The DFT considers that the temporal function is N-periodic, thus a long enough sequence (high N) is necessary to ensure a negligible influence of other periods. Note that the block periodicity of the moving load is different from the periodicity of nodal forces, which is fictitious, created by the discrete analysis with Fourier.

As the moving load is repeated in every block, Eq. (18) can be used with times t_n in which the moving load is acting on the reference block, from $t_n = 0$ to $t_n = t_{M-1}$ with $M = L/(v\Delta t)$.

At the instants in which $n \geq M$, the load $\mathbf{F}_{\mathrm{I}}(t_n) = 0$ and $\mathbf{F}_{\partial\mathrm{R}}(t_n) = \mathbf{F}_{\partial\mathrm{L}}(t_{n-M})$, being $\mathbf{F}_{\partial\mathrm{L}}$ the external load at the left boundary. Then, Eq. (18) can be written as:

$$\left\{ \begin{array}{c} \mathbf{F}_{\mathrm{I}}(\omega_k) \\ \mathbf{F}_{\partial\mathrm{R}}(\omega_k) \end{array} \right\} = \sum_{n=0}^{M-1} \left\{ \begin{array}{c} \mathbf{F}_{\mathrm{I}}(t_n) \\ \mathbf{F}_{\partial\mathrm{R}}(t_n) + \mathbf{F}_{\partial\mathrm{L}}(t_n) e^{\frac{-i2\pi kM}{N}} \end{array} \right\} e^{\frac{-i2\pi kn}{N}} \tag{20}$$

Eq. (15) allows to obtain the displacements of the substructure in the frequency domain. Now, the Inverse Discrete Fourier Transform (IDFT) is used to return to time domain, resulting in:

$$\left\{ \begin{array}{c} \mathbf{u}_{\mathrm{L}}(t_n) \\ \mathbf{u}_{\mathrm{I}}(t_n) \\ \mathbf{u}_{\mathrm{R}}(t_n) \end{array} \right\} = \frac{1}{N} \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} \left\{ \begin{array}{c} \mathbf{u}_{\mathrm{L}}(\omega_k) \\ \mathbf{u}_{\mathrm{I}}(\omega_k) \\ \mathbf{u}_{\mathrm{R}}(\omega_k) \end{array} \right\} e^{\frac{i2\pi kn}{N}} \tag{21}$$

In addition, if $\mathbf{F}(t_n)$ is a real function, $\mathbf{F}(\omega_k) = conj\,(\mathbf{F}(\omega_{-k}))$ and the same applies for $\mathbf{u}(\omega_k) = conj\,(\mathbf{u}(\omega_{-k}))$ because $\mathbf{H}(\omega)$ exhibits Hermitian symmetry. Then, the IDFT can be computed as:

$$\left\{ \begin{array}{c} \mathbf{u}_{\mathrm{L}}(t_n) \\ \mathbf{u}_{\mathrm{I}}(t_n) \\ \mathbf{u}_{\mathrm{R}}(t_n) \end{array} \right\} = \frac{1}{N} \left(\left\{ \begin{array}{c} \mathbf{u}_{\mathrm{L}}(\omega_0) \\ \mathbf{u}_{\mathrm{I}}(\omega_0) \\ \mathbf{u}_{\mathrm{R}}(\omega_0) \end{array} \right\} + 2\mathrm{Re} \left(\sum_{k=1}^{\frac{N}{2}} \left\{ \begin{array}{c} \mathbf{u}_{\mathrm{L}}(\omega_k) \\ \mathbf{u}_{\mathrm{I}}(\omega_k) \\ \mathbf{u}_{\mathrm{R}}(\omega_k) \end{array} \right\} e^{\frac{i2\pi kn}{N}} \right) \right) \tag{22}$$

It is also possible to truncate and consider only the N_c first frequencies if the effect of higher frequencies is negligible.

If Eq. (17) is introduced in Eq. (20) and the result in Eq. (15), then in Eq. (21) and finally in Eq. (22), a condensed formulation is obtained.

$$\mathbf{u}(t_n) = \sum_{\hat{n}=0}^{M-1} \mathbb{I}(n, \hat{n}) \mathbf{f}(t_{\hat{n}}) \tag{23}$$

in which

$$\mathbb{I}(n, \hat{n}) = \frac{1}{N} \sum_{k=0}^{N_c-1} a_k \mathrm{Re} \left(\mathbf{H}(\omega_k) \left[\begin{array}{ccc} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ e^{\frac{-i2\pi kM}{N}} \mathbf{I} & \mathbf{0} & \mathbf{I} \end{array} \right] e^{\frac{i2\pi k(n-\hat{n})}{N}} \right) \left[\begin{array}{c} \mathbf{N}_{\mathrm{L}}^{\top}(t_{\hat{n}}) \\ \mathbf{N}_{\mathrm{I}}^{\top}(t_{\hat{n}}) \\ \mathbf{N}_{\mathrm{R}}^{\top}(t_{\hat{n}}) \end{array} \right] \tag{24}$$

being $a_k = 2$ if $k \neq 0$ or $a_k = 1$ if $k = 0$. Note that the variable $\hat{n}$ is used to distinguish the instant of application of the load from the instant of evaluation of the displacement n.

To reduce the computational cost, $\mathbb{I}(n, \hat{n})$ can be written as:

$$\mathbb{I}(n, \hat{n}) = \mathbb{J}(\lambda) \mathbf{N}(t_{\hat{n}}) \tag{25}$$

in which

$$\mathbb{J}(\lambda) = \frac{1}{N} \sum_{k=0}^{N_c-1} a_k \mathrm{Re} \left(\mathbf{H}(\omega_k) \left[\begin{array}{ccc} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ e^{\frac{-i2\pi kM}{N}} \mathbf{I} & \mathbf{0} & \mathbf{I} \end{array} \right] e^{\frac{i2\pi k\lambda}{N}} \right) \tag{26}$$

and $\lambda = n - \hat{n}$.

4 NUMERICAL EXAMPLE

In this section, a numerical example of application of this method is analysed. The pantograph-catenary dynamic interaction is solved under the hypothesis of steady-state behaviour. The infinite catenary is composed of repetitive blocks as shown in Fig. 2 and the pantograph applies a vertical load f_c on the contact wire.

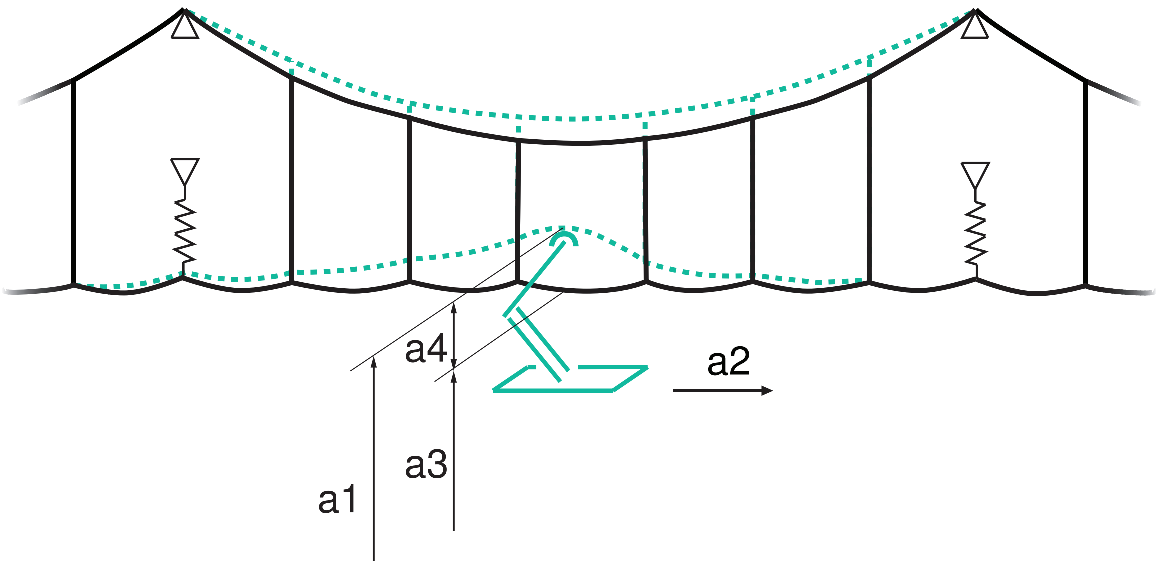

Figure 2: Pantograph interaction with the periodic catenary.

To solve the problem, Eq. (23) is only evaluated at the contact point instead of at all the degrees of freedom of the block. That is:

$$u_c(n) = \sum_{\hat{n}=0}^{M-1} \mathbb{I}_c(n,\hat{n}) f_c(t_{\hat{n}}) \tag{27}$$

in which $\mathbb{I}_c(n,\hat{n})$ is obtained from a simple transformation of $\mathbb{I}(n,\hat{n})$. This formulation provides a discrete matrix operator $\mathbb{I}_c(n,\hat{n})$ that relates the M values of the moving load with the M values of the contact point vertical displacement. The total height of the contact point is composed of the displacement produced by the load and the initial height profile of the contact wire:

$$z_c(n) = z_{cw}(n) + u_c(n) \tag{28}$$

A linear pantograph is considered whose dynamic response is defined by the frequency response function $H_p(\omega)$ of its contact point. This contact point is excited by an M-periodic force $-f_c(n)$ due to the action-reaction principle. The displacement $z_c(n)$ of the pantograph contact point can be also obtained by using the DFT:

$$z_c(n) = z_{ext} - \sum_{\hat{n}=0}^{M-1} \mathbb{I}_p(n,\hat{n}) f_c(t_{\hat{n}}) \tag{29}$$

in which z_{ext} is the displacement produced by a constant external load (produced by the bellow of the uplift mechanism) and:

$$\mathbb{I}_p(n,\hat{n}) = \frac{1}{M} \sum_{k=0}^{M/2} a_k \mathrm{Re}\left(H_p(\omega_k) e^{\frac{i2\pi k(n-\hat{n})}{M}} \right) \tag{30}$$

being:

$$\omega_k = k\frac{2\pi}{M\Delta t} \tag{31}$$

The contact force can be obtained imposing the same displacement $z_c(n)$ of the pantograph contact point (Eq. (29)) and the catenary contact point (Eq. (28)) at the M instants of time. That is:

$$z_{ext} - \sum_{\hat{n}=0}^{M-1} \mathbb{I}_p(n,\hat{n}) f_c(t_{\hat{n}}) = z_{cw}(n) + \sum_{\hat{n}=0}^{M-1} \mathbb{I}_c(n,\hat{n}) f_c(t_{\hat{n}}) \tag{32}$$

with $n = 0,..,M-1$. This system of M linear equations allows to compute the contact force $f_c(t_n)$.

As an example of a particular solution in which the dynamic interaction is produced at 300 km/h and the catenary model has 5 droppers per span, Fig. 3 gives a comparison between the solution obtained from the proposed method and the solution obtained from a FEM simulation performed with a long enough catenary to achieve the steady-state regime. The perfect coincidence shown gives validity to the proposed algorithm.

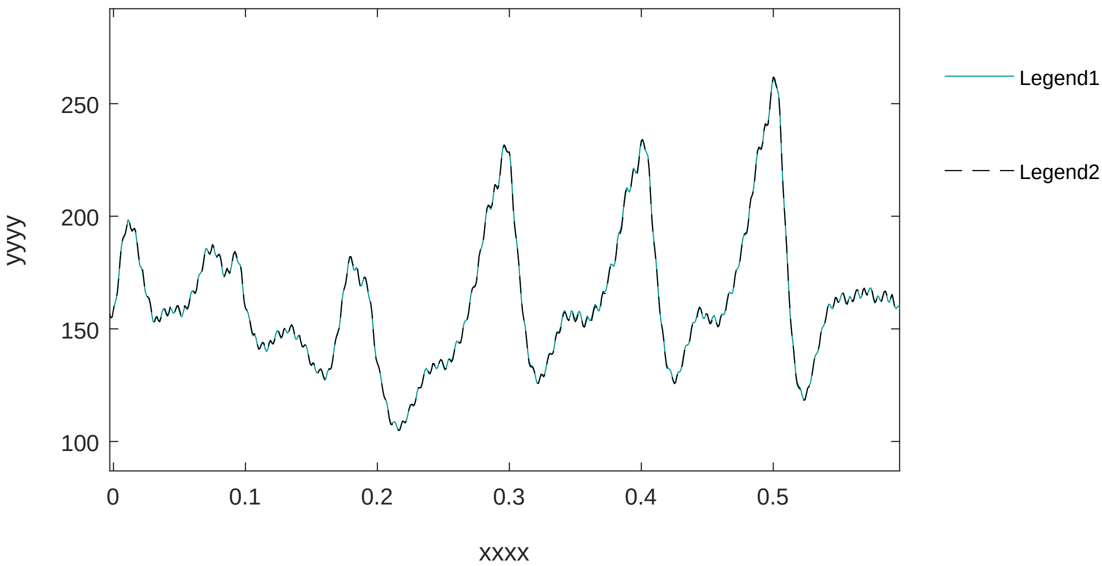

Figure 3: Solution of the infinite periodic model (PFEM) and a long conventional FEM model.

5 CONCLUSIONS

- The periodicity condition allows to compute the dynamic response of infinite periodic structures subject to a moving load combined with a FE model and discrete Fourier analysis.
- The repeated block is modelled by the Finite Element Method so that the proposed algorithm can be applied to any generic linear structure.
- The proposed method can be divided into two parts: the first devoted to compute the discrete operator that relates the load with the displacement of the structure and the second in which this operator is applied at different problems. The second part has a very low computational cost which makes it very suitable to perform (Hardware In the Loop) HIL tests or pantograph optimisation within the frame of pantograph-catenary dynamic interaction.

- A pantograph-catenary dynamic interaction problem is solved in this work to exemplify the proposed formulation. This has allowed us to validated the obtained results with a conventional FEM simulation, in which a long catenary must be used to ensure the steady-state interaction with the pantograph.

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