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A Urysohn lemma for regular spaces

Ankit Gupta [a] and Ratna Dev Sarma [b]

[a] Department of Mathematics, Bharati College, (University of Delhi), Delhi 110058, India
(ankitsince1988@yahoo.co.in)
[b] Department of Mathematics, Rajdhani College (University of Delhi), Delhi 110015, India
(ratna_sarma@yahoo.com)

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Abstract

Using the concept of m-open sets, M-regularity and M-normality are introduced and investigated. Both these notions are closed under arbitrary product. M-normal spaces are found to satisfy a result similar to Urysohn lemma. It is shown that closed sets can be separated by m-continuous functions in a regular space.

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1. Introduction

Nowadays topological approaches are being investigated in various diverse field of science and technology such as computer graphics, evolutionary theory, robotics[4, 9, 10] etc. to name a few. In a finite topological space, the intersection of all open neighbourhoods of a point p is again an open neighbourhood of p, which is the smallest one. It is called the *minimal neighbourhood* of p. However, in general framework, we define the minimal open sets or m-open sets as the ones obtained by taking the arbitrary intersections of the open sets. They have been studied in the recent past by several researchers [2, 6]. In this paper, we further use them for construction of new notions in topology, namely M-regularity and M-normality. These two notions are distinct from the already existing notions of regularity and normality, and are found to have

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several interesting properties. Both of them are closed under arbitrary product. M-normal spaces are found to exhibit Urysohn lemma type property. Finally it is shown that even in regular spaces, disjoint pair of closed sets can be separated by mappings, the so called m-continuous mappings. In that sense, the last result of the paper may be treated as Urysohn lemma for regular spaces. Here it may be mentioned that the classical Urysohn lemma is unprovable in ZF[1, 5]. The usual proof of Urysohn lemma in Kelley [8] uses the axioms of dependent choice to successfully select open sets separating previously chosen sets [1]. Similar choices have been made in our proof also, making it valid only in ZFC.

2. Preliminaries

Definition 2.1 ([6])**.** Let (X, τ) be a topological space. A set $A \subseteq X$ is called *m-open* if A can be expressed as intersection of a subfamily of open sets.

The complement of an m-open set is called an *m-closed* set. The collection of m-open sets of a topological space (X, τ) is denoted by $\mathcal{M}$.

Clearly, every open set is m-open. In a finite space, open sets are the only m-open sets.

The following example gives an idea about the abundance of m-open sets.

Example 2.2. Let $X = \mathbb{N}$ be the set of natural numbers, equipped with the co-finite topology. Then every subset of X is m-open.

Proposition 2.3 ([6])**.** *For a topological space (X, τ), we have the following results:*

(i) $\varnothing, X \in \mathcal{M}$;
(ii) $\mathcal{M}$ *is closed under arbitrary union;*
(iii) $\mathcal{M}$ *is closed under arbitrary intersection.*

Definition 2.4. [6] Let (X, τ) and (Y, μ) be two topological spaces. Then a function $f\colon X \to Y$ is said to be *m-continuous at a point $x \in X$* if for every open neighbourhood V of $f(x)$ there exists an m-open set U containing x in X such that $f(U) \subseteq V$.

A function $f\colon X \to Y$ is said to be *m-continuous*[6] if it is m-continuous at each point x of X.
Since every open set is m-open. Therefore every continuous function is m-continuous.

But the converse need not be true.

Example 2.5. Let $X = \mathbb{N}$ be the set of natural numbers equipped with the co-finite topology and $Y = \{a, b, c\}$ with the topology $\mu = \{\varnothing, \{a\}, \{b, c\}, Y\}$. Then consider a function $f\colon X \to Y$ defined as:

$$f(x) = \begin{cases} a & \text{if } x < 10, \\ b & \text{if } 10 \leq x < 100, \\ c & \text{otherwise.} \end{cases}$$

Since every subset $A \subseteq \mathbb{N}$ is m-open under the co-finite topology, function f is m-continuous. But f is not a continuous function as $f^{-1}(\{a\}) = \{x \in \mathbb{N} \,|\, x < 10\}$ is not an open set in $\mathbb{N}$ under the co-finite topology.

Theorem 2.6. *Let (X, τ) and (Y, μ) be two topological spaces. Then for a function $f: (X, \tau) \to (Y, \mu)$, the following are equivalent:*

(*i*) *f is m-continuous;*
(*ii*) *inverse image of every open subset of Y is m-open;*
(*iii*) *inverse image of every closed subset of Y is m-closed.*

Proof. **(*i*) ⇒ (*ii*):** Let U be any open subset of Y and let $x \in f^{-1}(U)$. Then $f(x) \in U$. Therefore there exists an m-open subset V in X such that $x \in V$ and $f(V) \subseteq U$. Thus $x \in V \subseteq f^{-1}(U)$, therefore $f^{-1}(U)$ is an m-open neighbourhood of x. Hence $f^{-1}(U)$ is m-open.
(*ii*) ⇒ (*iii*): Let A be any closed subset of Y. Then $Y \setminus A$ is open and therefore $f^{-1}(Y \setminus A)$ is m-open, that is, $X \setminus f^{-1}(A)$ is m-open. Hence $f^{-1}(A)$ is m-closed.
(*iii*) ⇒ (*i*): Let M be an open neighbourhood of $f(x)$, therefore $Y \setminus M$ is closed, and consequently $f^{-1}(Y \setminus M)$ is m-closed. Thus $f^{-1}(M)$ is m-open and hence $x \in f^{-1}(M) = N$ (say). Then, we have N is an m-open neighbourhood of x such that $f(N) \subseteq M$. □

In next result, we prove that arbitrary product of m-open sets is again m-open under the product topology. Here we use the fact that "An open set V of a product topology can be realized in the form $V = \prod_{i \in \Lambda} V_i$, where $V_i \in \tau_i$ and $V_i = X_i$ except for finitely many $i's$." This can be verified using the concept of basic open sets and the fact that

$$\left(\bigcup_{i \in I} A_i\right) \times \left(\bigcup_{j \in J} B_j\right) \times \ldots = \bigcup_{(i,j,\ldots) \in I \times J \times \ldots} (A_i \times B_j \times \ldots).$$

We also provide below the following results, which will be used in our paper.

Lemma 2.7 ([3, p. 28])**.** *Let $\{A_{\alpha,\beta}\}$ be an arbitrary family of non-empty sets. Then we have*

$$\bigcap_{\beta} \left(\prod_{\alpha} A_{\alpha,\beta}\right) = \prod_{\alpha} \left(\bigcap_{\beta} A_{\alpha,\beta}\right)$$

Lemma 2.8 ([7, p. 34])**.** *Let $\{(X_\alpha, \tau_\alpha) | \alpha \in \mathcal{A}\}$ be an arbitrary family of topological spaces and let $A_\alpha \subseteq X_\alpha$ for each $\alpha \in \mathcal{A}$. Then we have*

$$cl\left(\prod_{\alpha} V_\alpha\right) = \prod_{\alpha} (cl(V_\alpha)).$$

Theorem 2.9. *Let (X_α, τ_α) be topological spaces and U_α be an m-open set in (X_α, τ_α). Then the product of $U'_\alpha s$ is an m-open set in the product topology $\prod_{\alpha} \tau_\alpha$ of $X = \prod_{\alpha} X_\alpha$.*

Proof. Let $\{(X_\alpha, \tau_\alpha)\}_\alpha$ be a family of topological spaces and A be an m-open set in the product topology $X = \prod_\alpha X_\alpha$. Then there exists open sets U_i in X such that $A = \bigcap_i U_i$. Since U_i is an open set in the product topology of X, therefore there exists open set $U_{\alpha,i} \in \tau_\alpha$ with $U_{\alpha,i} = X_\alpha$ for all but finitely many α's such that $U_i = \prod_\alpha U_{\alpha,i}$. Hence $A = \bigcap_i \left(\prod_\alpha U_{\alpha,i}\right)$. Using the fact that $\bigcap_\beta \left(\prod_\alpha A_{\alpha,\beta}\right) = \prod_\alpha \left(\bigcap_\beta A_{\alpha,\beta}\right)$, in view of Lemma 2.7, we have $A = \bigcap_i \left(\prod_\alpha U_{\alpha,i}\right) = \prod_\alpha \left(\bigcap_i U_{\alpha,i}\right)$. Hence the proof. □

Now, we will show that every subset of a T_1-topological space (X, τ) is m-open.

Theorem 2.10. *Every subset of a T_1-space is m-open.*

Proof. Let (X, τ) be a topological space, which is T_1. Let A be a non-empty subset of X. Then every singleton $\{x\} \subseteq X$ is a closed set. Therefore, consider $A = \bigcap_{x \in X \setminus A} X \setminus \{x\}$. Since every singleton is closed therefore $X \setminus \{x\}$ is an open set in X. Hence the arbitrary intersection of open sets is m-open, thus A is m-open. □

3. M-Regular Spaces

Definition 3.1. A topological space (X, τ) is said to be *M-regular* if for each pair consisting of a point x and an m-closed set B not containing x, there exists a disjoint pair of an m-open set U and an open set V containing x and B respectively. In other words,
For every pair, x and B with $x \notin B$, where B is an m-closed set, there exist an m-open set U and an open set V such that $x \in U$, $B \subseteq V$ and $U \cap V = \varnothing$.

Now, we provide a characterization for M-regularity.

Theorem 3.2. *Let (X, τ) be a topological space. Then X is M-regular if and only if for a given point $x \in X$ and an m-open neighbourhood U of x, there exists an m-open neighbourhood V of x such that $x \in V \subseteq cl(V) \subseteq U$.*

Proof. Suppose that X is M-regular. Let $x \in X$ and $U \subseteq X$, an m-open neighbourhood of x, be given. Then $B = X \setminus U$ is an m-closed set disjoint from x. By the given hypothesis, there exists a disjoint pair of an m-open set V and an open set W containing x and B respectively. Thus we have, $x \in V$ and $X \setminus U = B \subseteq W$, that is, $x \in V \subseteq X \setminus W \subseteq U$. Hence we have, $x \in V \subseteq cl(V) \subseteq cl(X \setminus W) = X \setminus W \subseteq U$. Therefore we have, $x \in V \subseteq cl(V) \subseteq U$.

Conversely, suppose that a point $x \in X$ and an m-closed set B not containing x are given. Then $U = X \setminus B$ is an m-open set containing x. Then, by the hypothesis, there exists an m-open neighbourhood V of x such that $x \in V \subseteq cl(V) \subseteq U$. Thus we have, a disjoint pair of m-open set V and an open set $X \setminus cl(V)$ which contains x and B respectively. Therefore (X, τ) is M-regular. $\square$

With the help of following example, we show that an M-regular space need not be regular.

Example 3.3. Let $X = \mathbb{N}$ be the set of natural numbers, equipped with the co-finite topology. Then every subset A of X is m-open. Therefore X is a M-regular space but not a regular space.

But, every regular topological space is M-regular. For this, we have the following result:

Theorem 3.4. *Every regular space is M-regular.*

Proof. Let (X, τ) be a topological space which is regular. We have to show that X is M-regular. For this, let V be any m-open subset of X and let $x \in V$. Then, we have $V = \bigcap_{j \in J} V_j$, where V_j's are open sets in X. Therefore, as $x \in V = \bigcap_{j \in J} V_j$, we have $x \in V_j$ for all $j \in J$. Since the space X is given to be regular, thus there exists open set W_j such that $x \in W_j \subseteq cl(W_j) \subseteq V_j$, for all $j \in J$. Now, consider $W = \bigcap_{j \in J} W_j$ an m-open set in X, we have $x \in W \subseteq \bigcap_{j \in J} cl(W_j) \subseteq \bigcap_{j \in J} V_j = V$. Thus we have, $x \in W \subseteq cl(W) \subseteq V$, where W is an m-open subset of X. Hence X is M-regular.

$\square$

From the Theorem 2.10, one can conclude that every T_1-space is m-regular.

But the converse of the above statement is not true. That is, an M-regular space need not be T_1. For this, we have the following example:

Example 3.5. Let $X = \{a, b, c, d\}$ be a non-empty set equipped with a topology $\tau = \{\varnothing, \{a, b\}, \{c, d\}, X\}$. Then (X, τ) is an M-regular space but it is not a T_1-space.

Next we will provide the decomposition of T_1-space with the help of M-regularity.

Theorem 3.6. *Every M-regular T_0-space is T_1.*

Proof. Let (X, τ) be a topological space. Let $x, y \in X$ be a pair of distinct points of an M-regular T_0-space X. Let there exists an open set V in X such that $x \in V$ but $y \notin V$. Since every open set is m-open and the space X is given to be M-regular, therefore there exists an m-open set U in X such that

$x \in U \subseteq cl(U) \subseteq V$. Then consider, $W = X \setminus cl(U)$ is open in X such that $x \notin W$ and $y \in W$. Thus X is T_1. □

From the Theorem 3.6, one can state the following:

Theorem 3.7. *Every T_0-space is M-regular if and only if it is T_1.*

However a T_0-space need not be M-regular.

Example 3.8. Let X be a Sierpiński space, that is, $X = \{a, b\}$ with topology $\tau = \{\varnothing, \{a\}, X\}$. Then (X, τ) is a T_0-space but not M-regular.

In our next result, we show that Hausdorffness is a sufficient condition for M-regularity.

Theorem 3.9. *Every Hausdorff space is M-regular.*

Proof. Let (X, τ) be a Hausdorff space. Let x and B be a pair of a point and an m-closed set such that $x \notin B$. Then for every $y \in B$, we have $x \neq y$. Therefore by the given hypothesis, there exists a disjoint pair of open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. Then consider $V = \bigcup\limits_y \{V_y \,|\, y \in B\}$, an open cover of B and $U = \bigcap\limits_y \{U_y \,|\, y \in B\}$ is an m-open set containing x. Thus we have a disjoint pair consisting an open set V and an m-open set U containing B and x respectively. Therefore X is M-regular. □

The converse, however need not be true.

Example 3.10. Let $X = \{a, b, c, d\}$ be a non-empty set equipped with a topology $\tau = \{\varnothing, \{a, b\}, \{c, d\}, X\}$. Then (X, τ) is an M-regular space but it is not a Hausdorff space.

Our next result is on the product of M-regular spaces.

Theorem 3.11. *Any arbitrary product of M-regular spaces is again M-regular.*

Proof. Let $\{(X_\alpha, \tau_\alpha)\}_\alpha$ be a family of M-regular spaces and $X = \prod\limits_\alpha X_\alpha$. Let $x = (x_\alpha) \in X$ be a point and U be an m-open neighbourhood of $x \in X$. Since U is an m-open set in X, therefore $U = \bigcap\limits_i U_i$, where U_i is an open set in X under the product topology. Therefore, we have $U_i = \prod\limits_\alpha U_{\alpha,i}$, where $U_{\alpha,i} \in \tau_\alpha$. Hence, we have $U = \bigcap\limits_i \left(\prod\limits_\alpha U_{\alpha,i} \right)$. We use the fact that $\bigcap\limits_\beta \left(\prod\limits_\alpha A_{\alpha,\beta} \right) = \prod\limits_\alpha \left(\bigcap\limits_\beta A_{\alpha,\beta} \right)$ in view of Lemma 2.7. Therefore, we have $U = \bigcap\limits_i \left(\prod\limits_\alpha U_{\alpha,i} \right) = \prod\limits_\alpha \left(\bigcap\limits_i U_{\alpha,i} \right)$ and $x \in U$. Hence we have $x_\alpha \in \bigcap\limits_i U_{\alpha,i}$, where $\bigcap\limits_{i} U_{\alpha,i} = W_\alpha$ (say) is an m-open set in (X_α, τ_α). Since (X_α, τ_α) is M-regular, therefore there exists an m-open set V_α of (X_α, τ_α) such that $x_\alpha \in V_\alpha \subseteq cl_\alpha(V_\alpha) \subseteq W_\alpha$. Now, $V = \prod\limits_{\alpha} V_\alpha$ is an m-open set in X in view of Theorem 2.9. Again $cl(V) = cl\left(\prod\limits_{\alpha} V_\alpha\right) = \prod\limits_{\alpha} cl_\alpha(V_\alpha)$, in view of Lemma 2.8. Thus, we have an m-open set V in X such that $x \in V \subseteq cl(V) \subseteq \prod\limits_{\alpha} W_\alpha \subseteq U$. Hence X is M-regular. □

4. M-Normal Spaces

Definition 4.1. A topological space (X, τ) is said to be *M-normal* if for each disjoint pair consisting of a closed set A and an m-closed set B, there exists a disjoint pair consisting of an m-open set U and an open set V in X containing A and B respectively.

Remark 4.2. An M-normal space need not be M-regular.

For this, consider the Sierpiński space mentioned in the Example 3.8. The space (X, τ) is M-normal but not M-regular.

In our next result, we provide a characterization for M-normality.

Theorem 4.3. *Let (X, τ) be a topological space. Then (X, τ) is M-normal if and only if for a given closed set C and an m-open set D such that $C \subseteq D$, there is an m-open set G such that $C \subseteq G \subseteq cl(G) \subseteq D$.*

Proof. Let C and D be the closed and m-open sets respectively such that $C \subseteq D$. Then $X \setminus D$ is an m-closed set such that $C \cap (X \setminus D) = \varnothing$. Then, from the M-normality, there exist an m-open set G and an open set V such that $C \subseteq G$, $X \setminus D \subseteq V$ and $G \cap V = \varnothing$. Therefore $X \setminus V \subseteq D$ and hence $C \subseteq G \subseteq X \setminus V \subseteq D$, where $X \setminus V$ is a closed set. Hence $C \subseteq G \subseteq cl(G) \subseteq cl(X \setminus V) = X \setminus V \subseteq D$. Conversely, consider D and C as closed and m-closed sets respectively such that $C \cap D = \varnothing$. Then $X \setminus C$ is m-open set containing D. Then by the given hypothesis, there exist an m-open set G such that $D \subseteq G \subseteq cl(G) \subseteq X \setminus C$. Thus, we have $D \subseteq G$, $C \subseteq V$ and $G \cap V = \varnothing$, where $V = X \setminus cl(G)$, an open set. Hence X is M-normal. □

From the Example 3.3, one can easily verify that M-normality doesn't imply Normality. Here the space X is M-normal but it is not normal.

In the following result, we show that every normal space is M-normal.

Theorem 4.4. *Every normal space is M-normal.*

Proof. Let (X, τ) be a topological space which is normal. We have to show that X is M-normal. For this, let A be any closed subset of X and let V be an m-open subset of X such that $A \subseteq V$. Then, we have $V = \bigcap\limits_{j \in J} V_j$, where V_j's are open sets in X. Therefore, we have $A \subseteq V = \bigcap_{j\in J} V_j$, that is, $A \subseteq V_j$ for all $j \in J$. Since the space X is given to be normal, thus there exists open set W_j such that $A \subseteq W_j \subseteq cl(W_j) \subseteq V_j$, for all $j \in J$. Now, consider $W = \bigcap_{j\in J} W_j$, an m-open set in X, we have $A \subseteq W \subseteq \bigcap_{j\in J} cl(W_j) \subseteq \bigcap_{j\in J} V_j = V$. Thus we have, $A \subseteq W \subseteq cl(W) \subseteq V$, where W is an m-open subset of X. Hence X is M-normal. □

One specialty of M-normality is that it is preserved under arbitrary product.

Theorem 4.5. *Any arbitrary product of M-normal spaces is again M-normal.*

Proof. Let $\{(X_\alpha, \tau_\alpha)\}_\alpha$ be a family of M-normal spaces and $X = \prod_\alpha X_\alpha$. Let $A \subseteq X$ be a closed set and U be an m-open set such that $A \subseteq U$. Since U is an m-open set in X, therefore $U = \bigcap_i U_i$, where U_i is an open set in X under the product topology. Thus, we have $U_i = \prod_\alpha U_{\alpha,i}$, where $U_{\alpha,i} \in \tau_\alpha$ and $U_{\alpha,i} = X_\alpha$ for all but finitely many α's, as explained before Theorem 2.9 . Hence, we have $U = \bigcap_i \left(\prod_\alpha U_{\alpha,i}\right)$. We use the fact that $\bigcap_\beta \left(\prod_\alpha A_{\alpha,\beta}\right) = \prod_\alpha \left(\bigcap_\beta A_{\alpha,\beta}\right)$ following Lemma 2.7. Thus we have $U = \bigcap_i \left(\prod_\alpha U_{\alpha,i}\right) = \prod_\alpha \left(\bigcap_i U_{\alpha,i}\right)$. Similarly, we have $A = \prod_\alpha A_\alpha$, where A_α is a closed set in X_α. Since $A \subseteq U$, thus we have $A_\alpha \subseteq \bigcap_i U_{\alpha,i}$. Let $\bigcap_i U_{\alpha,i} = W_\alpha$, an m-open set in X_α. We have $A_\alpha \subseteq W_\alpha$ and since X_α is an M-normal space, therefore, there exists an m-open set V_α such that $A_\alpha \subseteq V_\alpha \subseteq cl(V_\alpha) \subseteq W_\alpha$. Thus we have, $\prod_\alpha A_\alpha \subseteq \prod_\alpha V_\alpha \subseteq \prod_\alpha cl_\alpha(V_\alpha) \subseteq \prod_\alpha W_\alpha$. Now, $\prod_\alpha V_\alpha$ is m-open in view of Theorem 2.9. Also, $cl\left(\prod_\alpha V_\alpha\right) = \prod_\alpha (cl(V_\alpha))$, in view of Lemma 2.8. Hence we have $V = \prod_\alpha V_\alpha$, an m-open set in X such that $A \subseteq V \subseteq cl(V) = \prod_\alpha (cl(V_\alpha)) = cl\left(\prod_\alpha V_\alpha\right)$. It follows that $A \subseteq V \subseteq cl(V) \subseteq \prod_\alpha W_\alpha \subseteq U$. Hence X is M-normal. □

Following result for M-normal spaces is analogous to the well known Urysohn lemma for normal spaces.

Theorem 4.6. *Let (X,τ) be a topological space. Then for each pair of disjoint subsets A and B of X, one of which is closed and the other is M-closed, there exists an m-continuous function f on X to $[0,1]$ (resp. $[a,b]$ for any real number $a,b,\ a<b$), such that $f(A)=\{0\}$ and $f(B)=\{1\}$ (resp. $f(A)=\{a\}$, and $f(B)=\{b\}$) provided X is M-normal.*

Proof. Let (X,τ) be an M-normal space. Let C_0 and C_1 be two disjoint sets, where C_0 is closed and C_1 is m-closed in X. Since $C_0\cap C_1=\varnothing$, therefore $C_0\subseteq X\setminus C_1$. Let P be the set of all dyadic rational numbers in $[0,1]$. We shall define for each p in P, an m-open set U_p of X, in such a way that whenever $p<q$, we have $cl(U_p)\subset U_q$.
First we define $U_1=X\setminus C_1$, an m-open set such that $C_0\subseteq U_1$. Since X is M-normal space, by Theorem 4.3, there exists an m-open set $U_{1/2}$ such that $C_0\subseteq U_{1/2}\subseteq cl(U_{1/2})\subseteq U_1$. Similarly, there also exist another m-open sets $U_{1/4}$ and $U_{3/4}$ such that $C_0\subseteq U_{1/4}\subseteq cl(U_{1/4})\subseteq U_{1/2}\subseteq cl(U_{1/2})\subseteq U_{3/4}\subseteq cl(U_{3/4})\subseteq U_1$, because $cl(U_{1/4})$ is again a closed set. Continuing the process, we define U_r, for each $r\in P$ such that $C_0\subseteq U_r\subseteq cl(U_r)\subseteq U_1$ and $cl(U_r)\subseteq U_s$ whenever $r<s$, for $r,s\in P$.
Let us define $\mathbf{Q}(x)$ to be the set of those dyadic rational numbers p such that the corresponding m-open sets U_p contains x:

$$\mathbf{Q}(x)\ =\ \{p\mid x\in U_p\}$$

Now we define a function $f\colon X\to[0,1]$ as

$$f(x)=\inf\mathbf{Q}(x)=\inf\{p\mid x\in U_p\}$$

Clearly, $f(C_0)=\{0\}$ and $f(C_1)=\{1\}$. Then we show that f is the desired m-continuous function. For a given point $x_0\in X$ and an open interval (c,d) in $[0,1]$ containing the point $f(x_0)$. We wish to find an m-open neighbourhood U of x_0 such that $f(U)\subseteq(c,d)$.
Let us choose rational numbers p and q such that $c<p<f(x_0)<q<d$. Then $U=U_q\setminus cl(U_p)=U_q\cap(X\setminus cl(U_p))$ is the desired m-open neighbourhood of x_0.

Here, we will show that $x_0\notin cl(U_p)$. If $x_0\in cl(U_p)$, then for $s>p$, we have $x_0\in cl(U_p)\subseteq U_s$. Thus $x_0\in U_s$ for all $s>p$. This implies that $f(x_0)\leq p$, as $f(x_0)=\mathrm{Inf}\{s\,|\,x_0\in U_s\}$. This contradicts the fact that $f(x_0)>p$. Similarly, as $f(x_0)<q$, therefore $x_0\in U_q$ and hence $U=U_q\setminus cl(U_p)$ is the desired neighbourhood of x_0.
Hence f is an m-continuous function on X to $[0,1]$ with $f(C_0)=\{0\}$ and $f(C_1)=\{1\}$. This completes the proof. □

Since every closed set is m-closed, we get the following result:

Corollary 4.7. *Let (X,τ) be an M-normal topological space. Then for each pair of disjoint closed subsets A and B of X, there exists an m-continuous function f on X to $[0,1]$ (resp. $[a,b]$ for any real number $a,b,\ a<b$), such that $f(A)=\{0\}$ and $f(B)=\{1\}$ (resp. $f(A)=\{a\}$, and $f(B)=\{b\}$).*

Remark 4.8. From the proof of Theorem 4.6, it is clear that the proof is in line of the usual proof of Urysohn lemma in Kelley[8], wherein the choice function plays its role. Hence the proof is valid for ZFC. Again, it has been pointed out in [1] that the axiom of multiple choice also implies Urysohn lemma, since one can use the intersection of the finitely many separating open sets provided by MC. Essentially the same argument shows that DMC implies Urysohn lemma.

Since similar working is followed in our Theorem 4.6, hence the variant of Urysohn lemma in our paper is also valid for ZF with DMC.

Our next theorem provides the relation between regularity and M-normality.

Theorem 4.9. *Every regular space is M-normal.*

Proof. Let A and B be two disjoint sets such that A is closed and B is m-closed. Since $A \cap B = \varnothing$, therefore $B \subseteq X \setminus A$, where $X \setminus A$ is an open set containing the m-closed set B. Then, by the given hypothesis, for each $b \in B \subseteq X \setminus A$, there exists an open set U_b such that $b \in U_b \subseteq cl(U_b) \subseteq X \setminus A$. Thus, we have a collection $\mathcal{D} = \{U_b \,|\, b \in B\}$ which covers B. Further, if $D \in \mathcal{D}$, then $cl(D)$ is disjoint from A because $cl(D) \subseteq X \setminus A$.
Consider $V = \bigcup\{D \,|\, D \in \mathcal{D}\}$. Then V is an open set in X which contains B, an m-closed set. Since D lies in some U_b whose closure is disjoint from A , therefore $W = \bigcup\{cl(D) \,|\, D \in \mathcal{D}\}$ is disjoint from A. Therefore, $V = \bigcup\{D \,|\, D \in \mathcal{D}\}$ and $X \setminus W$ are two disjoint subsets of X. Now W, being union of closed sets, is m-closed. Thus we have one open set and one m-open set containing the sets B and A respectively. Hence X is M-normal. □

In view of Theorem 4.6 and 4.9, one can observe the following:

Theorem 4.10. *Let (X, τ) be a regular space. Then for every disjoint pair of sets consisting of a closed set A and an m-closed set B, there always exists an m-continuous mapping f from X to $[a, b]$ such that $f(A) = \{a\}$ and $f(B) = \{b\}$.*

The last and the final result of this section is a simple corollary of Theorem 4.10. However its importance lies in revealing the fact that even in regular spaces, closed sets can be separated by mappings, the so-called m-continuous mappings. In that sense, this result may be treated as Urysohn lemma for regular spaces.

Theorem 4.11. *Let (X, τ) be a regular space. Then for every disjoint pair of closed sets A and B, there always exists an m-continuous mapping f from X to $[a, b]$ such that $f(A) = \{a\}$ and $f(B) = \{b\}$.*

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