APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 23, no. 2 (2022), 303-314
doi:10.4995/agt.2022.16925

Cardinal invariants and special maps of quasicontinuous functions with the topology of pointwise convergence

Mandeep Kumar [a] and Brij Kishore Tyagi [b]

[a] Department of Mathematics, University of Delhi, Delhi - 110007, India. (mjakhar5@gmail.com)
[b] Department of Mathematics, Atma Ram Sanatan Dharma College, University of Delhi, New Delhi - 110021, India. (brijkishore.tyagi@gmail.com)

Communicated by D. N. Georgiou

Abstract

For topological spaces X and Y, let $Q_p(X,Y)$ be the space of all quasicontinuous functions from X to Y with the topology of pointwise convergence. In this paper, we study the cardinal invariants such as character, weight, density, pseudocharacter, spread and cellularity of the space $Q_p(X,Y)$. We also discuss the properties of the restriction and induced maps related to the space $Q_p(X,Y)$.

2020 MSC: *54C35; 54C08; 54C30.*

Keywords: *quasicontinuous functions; topology of pointwise convergence; character; weight; density; spread; cellularity; induced map; restriction map.*

1. Introduction

Kempisty [10] introduced a weaker form of continuity for real-valued functions, named as quasicontinuity. The properties of quasicontinuous functions are discussed in many papers, for example see [2, 13, 15, 16].

The quasicontinuous functions have various applications in different areas of mathematics; for instance topological groups [11], dynamical systems [4] and the study of minimal usco and minimal cusco maps [6]. Some examples [3] of

Received 23 December 2021 – Accepted 3 April 2022

quasicontinuous functions are the doubling function

$$D : [0, 1) \to [0, 1) \text{ defined by } D(x) = 2x \pmod 1,$$

the extended $\sin(1/x)$ function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

the floor function from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\},$$

and any monotonic left or right continuous function from $\mathbb{R}$ to $\mathbb{R}$ [15].

The set of all real-valued quasicontinuous maps on a topological space X with the topology of pointwise convergence, denoted by $Q_p(X, \mathbb{R})$, is studied in [7, 8, 9]. The pointwise convergence of real-valued quasicontinuous maps defined on a Baire space is examined in [7]. In [9] metrizability, first countability, closed and compacts subsets of the space $Q_p(X, \mathbb{R})$ are discussed. The cardinal functions of the space $Q_p(X, \mathbb{R})$ are studied in [8].

In this paper, we study the results about metrizability, first countability and cardinal functions of the space $Q_p(X, Y)$ along with the concept of induced and the restriction maps.

In a more detail, this paper is organized as follows: In Section 3, we define the topology of pointwise convergence on $Q(X, Y)$, the set of all quasicontinuous functions from a topological space X to a topological space Y. In Section 4, when X is Hausdorff and Y is a nontrivial T_1-space, we compare the cardinal functions π-character, character and weight of the space $Q_p(X, Y)$. Moreover, if Y is second countable, we characterize these cardinal functions. For a regular space X and a nontrivial T_1-space Y, we discuss pseudocharacter and spread of the space $Q_p(X, Y)$. We also show that $Q_p(X, Y)$ is dense in the space Y^X. In Section 5, we discuss the topological properties of induced maps and the restriction map related to the space $Q_p(X, Y)$.

2. Preliminaries

Throughout this paper, the symbols X, Y, Z are topological spaces unless otherwise stated, $\mathbb{R}$ is the space of real numbers with the usual topology, $\mathbb{N}$ is the set of positive integers, and $\mathbb{I}$ is the closed interval $[-1, 1]$. The topology of a space X is denoted by $\tau(X)$. By a nontrivial space we mean a topological space with at least two different points. The symbol A^o denotes the interior of A in X and the symbol $\overline{A}$ denotes the closure of A in X.

Definition 2.1. A map $f : X \to Y$ is quasicontinuous [15] at $x \in X$ if for every open set U containing x and every open set V containing $f(x)$, there exists a nonempty open set $G \subseteq U$ such that $f(G) \subseteq V$. If f is quasicontinuous at every point of X, we say that f is quasicontinuous.

Note that every continuous map is quasicontinuous. Conversely, for $X = [0, 1)$ with the usual topology and $Y = [0, 1)$ with the Sorgenfrey topology, the identity map from X to Y is quasicontinuous but nowhere continuous [12].

Levine [13] studied quasicontinuous maps under the name of semi-continuity using the terminology of semi-open sets. A subset A of a space X is said to be semi-open (or quasi-open [15]) if $A \subseteq \overline{A^o}$. A map $f : X \to Y$ is quasicontinuous if and only if for every open set V in Y, $f^{-1}(V)$ is semi-open in X.

3. Quasicontinuous functions and the topology of pointwise convergence

Let $F(X,Y)$ be the set of all functions and $C(X,Y)$ be the set of all continuous functions from X to Y. The function spaces $F(X,Y)$ and $C(X,Y)$ with the topology of pointwise convergence denoted by $F_p(X,Y)$ and $C_p(X,Y)$, respectively, are widely studied in the literature, for example [1, 5, 14, 17].

For $x \in X$ and $V \in \tau(Y)$, let $S(x,V) = \{f \in F(X,Y) : f(x) \in V\}$. Then $F_p(X,Y)$ has a subbase

$$\mathcal{S} = \{S(x,V) : x \in X, V \in \tau(Y)\}.$$

Note that $F(X,Y) = Y^X$ and the topology of pointwise convergence on $F(X,Y)$ is just the product topology on Y^X.

Let $Q(X,Y)$ be the set of all quasicontinuous functions in $F(X,Y)$. The space $Q(X,Y)$ with the topology of pointwise convergence is the subspace $Q(X,Y)$ of the space $F_p(X,Y)$ and is denoted by $Q_p(X,Y)$.

For $x \in X$ and $V \in \tau(Y)$, denote $[x,V] = \{f \in Q(X,Y) : f(x) \in V\}$. Then $\mathcal{S}' = \{[x,V] : x \in X, V \in \tau(Y)\}$ is a subbase for the space $Q_p(X,Y)$. Observe that for $x \in X$ and $V_1, V_2 \in \tau(Y)$, we have $[x,V_1] \cap [x,V_2] = [x, V_1 \cap V_2]$. For $x_1, \ldots, x_n \in X$ and $V_1, \ldots, V_n \in \tau(Y)$, denote

$$[x_1, \ldots, x_n; V_1, \ldots, V_n] = \{f \in Q(X,Y) : f(x_i) \in V_i, 1 \le i \le n\}.$$

Clearly the family

$$\mathcal{B} = \{[x_1, \ldots, x_n; V_1, \ldots, V_n] : x_i \in X, V_i \in \tau(Y), 1 \le i \le n, n \in \mathbb{N}\}$$

is a base for the space $Q_p(X,Y)$. If $\mathcal{V}$ is a basis for Y then the family

$$\mathcal{B}' = \{[x_1, \ldots, x_n; V_1, \ldots, V_n] : x_i \in X, V_i \in \mathcal{V}, 1 \le i \le n, n \in \mathbb{N}\}$$

is also a basis for the space $Q_p(X,Y)$. If (Y,d) is a metric space then for $f \in Q(X,Y)$; $x_1, \ldots, x_n \in X$ and $\epsilon > 0$, denote

$$O(f, x_1, \ldots, x_n, \epsilon) = \{g \in Q(X,Y) : d(g(x_i), f(x_i)) < \epsilon, 1 \le i \le n\}.$$

It is easy to see that the family

$$\mathcal{B}_f = \{O(f, x_1, \ldots, x_n, \epsilon) : x_1 \ldots, x_n \in X, n \in \mathbb{N}, \epsilon > 0\}$$

is a local base at $f \in Q_p(X,Y)$.

4. Cardinal functions and the space $Q_p(X,Y)$

In this section, we discuss first countability, metrizability and cardinal functions of the space $Q_p(X,Y)$. Before generalizing some results obtained in [8, 9], first we recall definitions of the cardinal functions for a topological space [8, 14].

A collection $\mathcal{V}$ of nonempty open subsets of X is called a local π-base at $x \in X$ if for each open set U containing x, there exists $V \in \mathcal{V}$ such that $V \subseteq U$. The π-character of a point $x \in X$ is $\pi_\chi(x,X) = \aleph_0 + \min\{|\mathcal{V}| : \mathcal{V}$ is a local π-base at $x\}$. The π-character of a space X is defined as $\pi_\chi(X) = \sup\{\pi_\chi(x,X) :$

$x \in X\}$. The character of a space X is $\chi(X) = \sup\{\chi(x, X) : x \in X\}$, where $\chi(x, X) = \aleph_0 + \min\{|\mathcal{B}_x| : \mathcal{B}_x$ is a base at $x\}$. The weight of a space X is defined by $\omega(X) = \aleph_0 + \min\{|\mathcal{B}| : \mathcal{B}$ is a base for $X\}$. A collection β of nonempty subsets of a space X is called a π-base for X provided that every nonempty open subset of X contains some member of β. The π-weight of a space X is defined by $\pi\omega(X) = \aleph_0 + \min\{|\beta| : \beta$ is a π-base for $X\}$. The density of a space X is $d(X) = \aleph_0 + \min\{|D| : D$ is a dense subset of $X\}$. The pseudocharacter of a point $x \in X$ is $\psi(x, X) = \aleph_0 + \min\{|\gamma| : \gamma$ is a family of open sets in X such that $\cap\gamma = \{x\}\}$. The pseudocharacter of a space X is defined as $\psi(X) = \sup\{\psi(x, X) : x \in X\}$. The spread of a space X is defined as $s(X) = \aleph_0 + \sup\{|D| : D \subseteq X$ is discrete$\}$. The cellularity or Souslin number of a space X is defined by $c(X) = \aleph_0 + \sup\{|\mathcal{U}| : \mathcal{U}$ is a family of pairwise disjoint nonempty open subsets of $X\}$. A space X is said to have Souslin property if $c(X) = \aleph_0$.

Lemma 4.1 ([8, Lemma 4.2])**.** *Let X and Y be topological spaces and $f : X \to Y$ be a map such that for any $x \in X$, there exists an open set G in X such that $x \in \overline{G}$ and $f(y) = f(x)$ for all $y \in G$. Then f is quasicontinuous.*

Lemma 4.2 ([8, Lemma 4.3])**.** *Let X and Y be topological spaces such that X is Hausdorff. For given $x_1, \dots, x_n \in X$ and (not necessarily distinct) $y_1, \dots, y_n \in Y$, there exists a quasicontinuous map $f : X \to Y$ such that $f(x_i) = y_i$ for each $i \in \{1, \dots, n\}$.*

Theorem 4.3. *Let X and Y be topological spaces such that X is uncountable Hausdorff and Y is a nontrivial T_1-space. Then for any $f \in Q_p(X, Y)$, f does not have a countable local π-base.*

Proof. Suppose $\{U_n : n \in \mathbb{N}\}$ be a countable local π-base at some $f \in Q_p(X, Y)$. So for each n, there is a basic open set W_n such that $W_n \subseteq U_n$. Then $\{W_n : n \in \mathbb{N}\}$ is also a countable local π-base at f. Let $W_n = [x_1^n, \dots, x_{k_n}^n; V_1^n, \dots, V_{k_n}^n]$ for each $n \in \mathbb{N}$. The set $A = \{x_j^i : 1 \le j \le k_i, i \in \mathbb{N}\}$ is countable.

Since X is uncountable, choose $x \in X \setminus A$. Because Y is a nontrivial T_1-space, choose $y \in Y$ such that $y \notin V$ for some open set V containing $f(x)$. Then $W = [x, V]$ is an open set containing f. Suppose $W_n \subseteq W$ for some $n \in \mathbb{N}$. By Lemma 4.2, let $g : X \to Y$ be a quasicontinuous function such that $g(x_i^n) \in V_i^n$ for each $i \in \{1, \dots, k_n\}$ and $g(x) = y$. Then $g \in W_n \setminus W$, a contradiction. □

Corollary 4.4. *Let X and Y be topological spaces such that X is Hausdorff and Y is a nontrivial T_1-space. If the space $Q_p(X, Y)$ has a countable local π-base at some $f \in Q_p(X, Y)$, then X is countable.*

Corollary 4.5. *Let X be an uncountable Hausdorff space and Y be a nontrivial T_1-space. Then for any $f \in Q_p(X, Y)$, f does not have a countable local base.*

Using Corollary 4.4, a more general result than [9, Theorem 3.2] is the following.

Theorem 4.6. *Let X and Y be spaces such that X is Hausdorff and Y is a nontrivial metrizable space. Then the following are equivalent:*

(a) $F_p(X,Y)$ *is metrizable.*
(b) $F_p(X,Y)$ *is first countable.*
(c) $Q_p(X,Y)$ *is metrizable.*
(d) $Q_p(X,Y)$ *is first countable.*
(e) For any $f \in Q_p(X,Y)$*,* f *has a countable local* π*-base.*
(f) X *is countable.*

Proof. Clearly (d) implies (e) holds. The assertion (e) implies (f) follows from Corollary 4.4. Using the facts that $Q_p(X,Y)$ is a subspace of $F_p(X,Y)$ and a countable product of metrizable spaces is metrizable, the rest of the implications can be verified easily. □

The result obtained in Theorem 4.3 can also be deduced from the following result about the cardinal functions related to the space $Q_p(X,Y)$.

Theorem 4.7. *Let X and Y be topological spaces such that X is Hausdorff and Y is a nontrivial T_1-space. Then $|X| \leq \pi_\chi(Q_p(X,Y)) \leq \chi(Q_p(X,Y)) \leq \omega(Q_p(X,Y))$. Moreover, if X is infinite and Y is second countable, we have $|X| = \pi_\chi(Q_p(X,Y)) = \chi(Q_p(X,Y)) = \pi\omega(Q_p(X,Y)) = \omega(Q_p(X,Y))$.*

Proof. To show $|X| \leq \pi_\chi(Q_p(X,Y))$, let $y_1 \in Y$ and $f \in Q_p(X,Y)$ be the constant function such that $f(x) = y_1$ for each $x \in X$. Let $\{U_t : t \in T\}$ be a local π-base at f with $|T| \leq \pi_\chi(Q_p(X,Y))$. Since each U_t is a nonempty open subset of $Q_p(X,Y)$, there exists a basic open set $B_t = [x_1^t, \dots, x_{n_t}^t; V_1^t, \dots, V_{n_t}^t] \subseteq U_t$ for each $t \in T$. Then the collection $\mathcal{B}_f = \{B_t : t \in T\}$ is also a local π-base at f. For each $t \in T$, let $A_t = \{x_1^t, \dots, x_{n_t}^t\}$. We claim that $\bigcup_{t \in T} A_t = X$.

Let $x \in X$. Since Y is a nontrivial T_1-space, choose $y_2 \in Y$ such that $y_2 \notin V_1$ for some open set V_1 in Y containing y_1. Because $[x, V_1]$ is an open set containing f and $\mathcal{B}_f$ is a local π-base at f, there exists $t \in T$ such that $B_t = [x_1^t, \dots, x_{n_t}^t; V_1^t, \dots, V_{n_t}^t] \subseteq [x, V_1]$. We claim that $x \in A_t = \{x_1^t, \dots, x_{n_t}^t\}$.

Suppose $x \notin A_t$. Since X is Hausdorff, there exists an open set U such that $x \in U$ and $\overline{U} \cap A_t = \varnothing$. Because B_t is nonempty, let $s_t \in B_t$. Then $s_t(x_i^t) \in V_i^t$ for each $i \in \{1, \dots, n_t\}$. By Lemma 4.2, there is a quasicontinuous map $h_t : X \to Y$ such that $h_t(x_i^t) = s_t(x_i^t)$ for each $i \in \{1, \dots, n_t\}$. Let us define $g : X \to Y$ such that

$$g(z) = \begin{cases} y_2 & \text{if } z \in \overline{U} \\ h_t(z) & \text{if } z \in X \setminus \overline{U} \end{cases}$$

By Lemma 4.1, g is a quasicontinuous map such that $g \in B_t$, but $g \notin [x, V_1]$, which contradicts $B_t \subseteq [x, V_1]$. So $\bigcup_{t \in T} A_t = X$. Hence $|X| \leq \pi_\chi(Q_p(X,Y)) \leq \chi(Q_p(X,Y)) \leq \omega(Q_p(X,Y))$.

If $\mathcal{B}_Y$ is a countable base for Y then $\{[x_1, \dots, x_n; V_1, \dots, V_n] : x_i \in X, V_i \in \mathcal{B}_Y, 1 \leq i \leq n\}$ is a base for the space $Q_p(X,Y)$. Thus $\omega(Q_p(X,Y)) \leq |X|$. □

Theorem 4.8. *Let X and Y be topological spaces such that X is infinite Hausdorff space and Y is a nontrivial metrizable space. Then $|X| = \pi_\chi(Q_p(X,Y)) = \chi(Q_p(X,Y))$.*

Proof. By Theorem 4.7, we have $|X| \leq \pi_\chi(Q_p(X,Y)) \leq \chi(Q_p(X,Y))$. To show $\chi(Q_p(X,Y)) \leq |X|$, let $f \in Q_p(X,Y)$. If (Y,d) is a metric space then the collection $\mathcal{B}_f = \{O(f, x_1, \ldots, x_k, \frac{1}{n}) : x_1, \ldots, x_k \in X, k, n \in \mathbb{N}\}$ is a local base at f. Thus $|X| = \pi_\chi(Q_p(X,Y)) = \chi(Q_p(X,Y))$. □

Lemma 4.9. *Let X and Y be topological spaces such that $U_1, \ldots, U_n$ are nonempty pairwise disjoint open subsets of X and $y_1, \ldots, y_n \in Y$. Then there exists a quasicontinuous map $g : X \to Y$ such that $g(U_i) = \{y_i\}$ for each $i \in \{1, \ldots, n\}$.*

Proof. Let $H = \overline{U_1} \cup \cdots \cup \overline{U_n}$ and $y_0 \in Y$. For $x \in H$, let $k = \min\{i \in \{1, \ldots, n\} : x \in \overline{U_i}\}$. Let us define $g : X \to Y$ such that

$$g(x) = \begin{cases} y_k & \text{if } x \in H \\ y_0 & \text{if } x \in X \setminus H \end{cases}$$

By Lemma 4.1, the map g is quasicontinuous and $g(U_i) = \{y_i\}$ for each $i \in \{1, \ldots, n\}$. □

Theorem 4.10. *Let X and Y be topological spaces such that X is Hausdorff. Then $d(Q_p(X,Y)) \leq \omega(X) \cdot d(Y)$.*

Proof. Let $\mathcal{B}$ be a base for X such that $|\mathcal{B}| \leq \omega(X)$ and $\mathcal{U}$ be the family of all finite pairwise disjoint nonempty members of $\mathcal{B}$. Let D be a dense set in Y such that $|D| \leq d(Y)$ and $\mathcal{V}$ be the family of all nonempty finite subsets of D. For each $U = \{U_1, \ldots, U_n\} \in \mathcal{U}$ and $y = \{y_1, \ldots, y_n\} \in \mathcal{V}$, by Lemma 4.9, there exists a quasicontinuous function $g_{U,y} : X \to Y$ such that $g_{U,y}(U_i) = y_i$ for each $i \in \{1, \ldots, n\}$. Then $G = \{g_{U,y} : U \in \mathcal{U}, y \in \mathcal{V}\}$ is dense set in $Q_p(X,Y)$ such that $|G| \leq \omega(X) \cdot d(Y)$.

Indeed, for any nonempty basic open set $H = [x_1, \ldots, x_n; V_1, \ldots, V_n]$ in $Q_p(X,Y)$, there exist $U = \{U_1, \ldots, U_n\} \in \mathcal{U}$ such that $x_i \in U_i$ and $y = \{y_1, \ldots, y_n\} \in \mathcal{V}$ such that $y_i \in V_i$ for each $i \in \{1, \ldots, n\}$. Thus there is $g_{U,y} \in G$ such that $g_{U,y}(x_i) \in V_i$ for each $i \in \{1, \ldots, n\}$ and hence $g_{U,y} \in H \cap G$. □

Corollary 4.11. *Let X and Y be topological spaces such that X is second countable Hausdorff and Y is separable. Then the space $Q_p(X,Y)$ is separable.*

Lemma 4.12. *Let X and Y be spaces such that X is regular. Then for any $x \in X$, any nonempty closed set $F \subseteq X$ such that $x \notin F$ and $y_1, y_2 \in Y$, there exists a quasicontinuous function $f : X \to Y$ such that $f(x) = y_1$ and $f(F) = \{y_2\}$.*

Proof. Since $x \notin F$ and X is regular, there exist open sets U and V such that $x \in U$, $F \subseteq V$ and $U \cap V = \varnothing$. Note that $x \notin \overline{V}$. Let us define $f : X \to Y$

such that

$$f(z) = \begin{cases} y_1 & \text{if } z \in X \setminus \overline{V} \\ y_2 & \text{if } z \in \overline{V} \end{cases}$$

By Lemma 4.1, f is a quasicontinuous map such that $f(x) = y_1$ and $f(F) = \{y_2\}$. □

Theorem 4.13. *Let X be a regular space and Y be any nontrivial space. Then $d(X) \leq \psi(Q_p(X, Y))$.*

Proof. Given a basic open set $U = [x_1, \ldots, x_n; V_1, \ldots, V_n]$ in $Q_p(X, Y)$, let $A_U = \{x_1, \ldots, x_n\}$. Let $f_0 \in Q_p(X, Y)$ be the constant function such that $f_0(x) = y_0$ for all $x \in X$ and γ be a family of open sets with $|\gamma| \leq \psi(Q_p(X, Y))$ such that $\cap\gamma = \{f_0\}$. For each $G \in \gamma$, there exists a basic open set $U_G = [x_1^G, \ldots, x_{n_G}^G; V_1^G, \ldots, V_{n_G}^G]$ such that $f_0 \in U_G \subseteq G$. We claim that the set $D = \bigcup\{A_{U_G} : G \in \gamma\}$ is dense in X.

Suppose that $x \in X \setminus \overline{D}$ and $y_1 \in Y$ such that $y_1 \neq y_0$. By Lemma 4.12, there exists $f \in Q_p(X, Y)$ such that $f(x) = y_1$ and $f(\overline{D}) = \{y_0\}$. Then $f \in \cap\gamma$ and $f \neq f_0$, which is a contradiction. Thus D is dense in X. □

Note that if X is a Tychonoff space, then the result obtained in Theorem 4.13 for $Y = \mathbb{R}$ can be concluded from the results $d(X) = \psi(C_p(X, \mathbb{R}))$ [17, Problem 173] and $\psi(C_p(X, \mathbb{R})) \leq \psi(Q_p(X, \mathbb{R}))$ [17, Problem 159]. We cannot expect the equality in between $d(X)$ and $\psi(Q_p(X, \mathbb{R}))$ even for $X = \mathbb{R}$, because $d(\mathbb{R}) = \aleph_0$, while [8, Example 5.1] shows that $\psi(Q_p(\mathbb{R}, \mathbb{R})) = 2^{\aleph_0}$.

Theorem 4.14. *Let X be a regular space and Y be a nontrivial T_1-space. Then $s(X) \leq s(Q_p(X, Y))$.*

Proof. Let D be a discrete subspace of X and $\{V_d : d \in D\}$ be a family of open subsets of X such that $V_d \cap D = \{d\}$ for each $d \in D$. Choose $y_1, y_2 \in Y$ such that $y_1 \neq y_2$, by Lemma 4.12, there exists a quasicontinuous function $f_d : X \to Y$ such that $f_d(d) = y_1$ and $f_d(X \setminus V_d) = \{y_2\}$. Then the set $A = \{f_d : d \in D\}$ is discrete in $Q_p(X, Y)$. To see this, choose an open set G in Y such that $y_1 \in G$ but $y_2 \notin G$. Then $U_d = [d, G]$ is open in $Q_p(X, Y)$ and $U_d \cap A = \{f_d\}$. Thus $s(X) \leq s(Q_p(X, Y))$. □

If X is a Tychonoff space and $Y = \mathbb{R}$, then the result obtained in Theorem 4.14 can be obtained from the results $s(X) \leq s(C_p(X, \mathbb{R}))$ [17, Problem 176] and $s(C_p(X, \mathbb{R})) \leq s(Q_p(X, \mathbb{R}))$ [17, Problem 159].

Theorem 4.15. *Let X and Y be topological spaces such that X is Hausdorff. Then $Q(X, Y)$ is dense in $F_p(X, Y)$.*

Proof. Let $W = [x_1, \ldots, x_n; V_1, \ldots, V_n]$ be any nonempty basic open set in $F_p(X, Y)$ and $f \in W$. Then $f(x_i) \in V_i$ for each $i \in \{1, \ldots, n\}$. Since X is Hausdorff and $x_1, \ldots, x_n \in X$, by Lemma 4.2, there exists a quasicontinuous function $g : X \to Y$ such that $g(x_i) = f(x_i)$ for each $i \in \{1, \ldots, n\}$. Then $g \in W \cap Q(X, Y)$. □

Corollary 4.16. *Let X be a Hausdorff space and Y be a separable space. Then the space $Q_p(X,Y)$ has the Souslin property, that is, $c(Q_p(X,Y)) = \aleph_0$.*

Proof. It is known that if Y is separable then the space $F_p(X,Y)$ has the Souslin property [17, Problem 109]. Since $Q(X,Y)$ is dense in $F_p(X,Y)$, $c(Q_p(X,Y)) = c(F_p(X,Y))$ [17, Problem 110]. Thus the space $Q_p(X,Y)$ has the Souslin property. □

Note that if X is a Tychonoff space then $C(X,\mathbb{R})$ is a dense subset of the space $Q_p(X,\mathbb{R})$ [17, Problem 034]. Also $P(\mathbb{R},\mathbb{R})$, the set of all polynomials from $\mathbb{R}$ to $\mathbb{R}$ and $U(\mathbb{R},\mathbb{R})$, the set of all uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}$ are dense subsets of the space $Q_p(\mathbb{R},\mathbb{R})$ [17, Problem 041,043].

Proposition 4.17. *Let X and Y be topological spaces such that X is Hausdorff and Y is separable. If $\mathcal{F}$ is any locally finite family of nonempty open subsets of $Q_p(X,Y)$, then $\mathcal{F}$ is countable.*

Proof. If possible, suppose $\mathcal{F}$ is uncountable. Let $\mathcal{A}$ be a maximal disjoint family of nonempty open subsets of $Q_p(X,Y)$ such that each member of $\mathcal{A}$ meets at most finitely many members of $\mathcal{F}$. Because $\mathcal{F}$ is locally finite, the set $\bigcup\mathcal{A}$ is dense in $Q_p(X,Y)$. By Corollary 4.16, $c(Q_p(X,Y)) \leq \aleph_0$, which implies $\mathcal{A}$ is countable. Since $\bigcup\mathcal{A}$ is dense in $Q_p(X,Y)$, each $U \in \mathcal{F}$ intersect some $V \in \mathcal{A}$. But every member of $\mathcal{A}$ can intersect only finitely many members of $\mathcal{F}$. Since $\mathcal{A}$ is countable, this implies $\mathcal{F}$ is countable, a contradiction. □

5. Special maps and the space $Q_p(X,Y)$

The properties of induced maps related to the space $C(X,Y)$ with the topology of pointwise convergence and others are discussed in [14, Chapter II]. Before discussing the properties of induced maps related to the space $Q_p(X,Y)$, let us first define these maps in view of quasicontinuous maps.

Note that the composition of two quasicontinuous maps need not be quasicontinuous [16]. However, if $f : X \to Y$ is quasicontinuous and $g : Y \to Z$ is continuous, then the composition map $gof : X \to Z$ is quasicontinuous. If $g : Y \to Z$ is continuous, then the induced map $g_* : Q(X,Y) \to Q(X,Z)$ is defined by $g_*(f) = g \circ f$ for all $f \in Q(X,Y)$. Also if $g \in Q(X,Y)$, then the induced map $g^* : C(Y,Z) \to Q(X,Z)$ is defined as $g^*(h) = h \circ g$ for all $h \in C(Y,Z)$.

Theorem 5.1. *For a given continuous map $g : Y \to Z$, the induced map $g_* : Q_p(X,Y) \to Q_p(X,Z)$ such that $g_*(f) = g \circ f$ is continuous. Moreover, if g is an embedding, then g_* is also an embedding.*

Proof. Let $f \in Q_p(X,Y)$ and U be any open set in $Q_p(X,Z)$ containing $g_*(f)$. There is a basic open set $V = [x_1,\ldots,x_n; V_1,\ldots,V_n]$ in $Q_p(X,Z)$ such that $g_*(f) \in V \subseteq U$. Now $W = [x_1,\ldots,x_n; g^{-1}(V_1),\ldots,g^{-1}(V_n)]$ is an open set in $Q_p(X,Y)$ such that $f \in W$ and $g_*(W) \subseteq V \subseteq U$. Thus the map g_* is continuous.

Note that if g is injective then g_* is also injective. Now to show $g_* : Q_p(X,Y) \to g_*(Q_p(X,Y))$ is an open map, let $[x,V]$ be any subbasic open set in $Q_p(X,Y)$. Since g is an embedding and V is open in Y, there exists an open set W in Z such that $g(V) = W \cap g(Y)$. We have $[x,V] = [x, g^{-1}(W)] = g_*^{-1}([x,W])$. Then $g_*([x,V]) = [x,W] \cap g_*(Q_p(X,Y))$ is open in $g_*(Q_p(X,Y))$. □

Proposition 5.2. *For any space X, there is a continuous map $h : Q_p(X,\mathbb{R}) \to Q_p(X,\mathbb{I})$ such that $h(f) = f$ for each $f \in Q_p(X,\mathbb{I})$.*

Proof. Consider the map $g : \mathbb{R} \to \mathbb{I}$ such that $g(t) = -1$ if $t < -1$, $g(t) = t$ if $t \in \mathbb{I} = [-1,1]$ and $g(t) = 1$ if $t > 1$. Clearly g is continuous. By Theorem 5.1, the map $h = g_* : Q_p(X,\mathbb{R}) \to Q_p(X,\mathbb{I})$ defined by $h(f) = gof$ is continuous. Also $h(f) = f$ for each $f \in Q_p(X,\mathbb{I})$. □

Theorem 5.3. *For a given quasicontinuous map $g : X \to Y$, the map $g^* : C_p(Y,Z) \to Q_p(X,Z)$ such that $g^*(h) = h \circ g$ is continuous. Moreover, if $g(X) = Y$, then g^* is an embedding.*

Proof. Let $h_0 \in C_p(Y,Z)$ and $V = [x_1,\dots,x_n; V_1,\dots,V_n]$ be any basic open set in $Q_p(X,Z)$ containing $g^*(h_0)$. Consider $U = [g(x_1),\dots,g(x_n); V_1,\dots,V_n]$ open in $C_p(Y,Z)$. Then $h_0 \in U$ and for any $h \in U$, we have $g^*(h) \in V$. Thus $g^*(U) \subseteq V$ and hence g^* is continuous.

Now suppose that $g(X) = Y$. To see g^* is an injection, let $h, h' \in C_p(Y,Z)$ such that $h \neq h'$. Then $h(y) \neq h'(y)$ for some $y \in Y$. Because $g(X) = Y$, let $x \in g^{-1}(y)$. Then $g^*(h)(x) = h(y) \neq h'(y) = g^*(h')(x)$. Hence $g^*(h) \neq g^*(h')$. To prove g^* is an embedding, it suffices to show that $(g^*)^{-1} : g^*(C_p(Y,Z)) \to C_p(Y,Z)$ is continuous. Let $g^*(f) \in g^*(C_p(Y,Z))$ and $U = [y_1,\dots,y_n; V_1,\dots,V_n]$ be any basic open set in $C_p(Y,Z)$ containing f. Choose $x_i \in g^{-1}(y_i)$ for each $i \in \{1,\dots,n\}$. Then $V = [x_1,\dots,x_n; V_1,\dots,V_n] \cap g^*(C_p(Y,Z))$ is open in $g^*(C_p(Y,Z))$ containing $g^*(f)$. To verify $(g^*)^{-1}(V) \subseteq U$, let $h \in V$. Then $h = g^*(h')$ for some $h' \in C_p(Y,Z)$. Since $h = g^*(h') = h' \circ g \in V$, we have $h' \circ g(x_i) \in V_i$ for each $i \in \{1,\dots,n\}$. This implies $h'(y_i) \in V_i$ for each $i \in \{1,\dots,n\}$ so that $h' \in U$. Hence $h' = (g^*)^{-1}(h) \in U$ and we have $(g^*)^{-1}(V) \subseteq U$. □

For any space X and maps $f, g : X \to \mathbb{R}$ such that f is continuous and g is quasicontinuous, it is easy to see that the map $f + g : X \to \mathbb{R}$ defined by $(f+g)(x) = f(x) + g(x)$ is quasicontinuous.

Proposition 5.4. *For any space X, the map $s : C_p(X,\mathbb{R}) \times Q_p(X,\mathbb{R}) \to Q_p(X,\mathbb{R})$ defined by $s(f,g) = f + g$ is continuous.*

Proof. Let $(f_0, g_0) \in C_p(X,\mathbb{R}) \times Q_p(X,\mathbb{R})$ and U be any open set in $Q_p(X,\mathbb{R})$ containing $h_0 = f_0 + g_0$. There exist $x_1,\dots,x_n \in X$ and $\epsilon > 0$ such that $h_0 \in O(h_0, x_1,\dots,x_n,\epsilon) \subseteq U$. Then $V = O(f_0, x_1,\dots x_n, \frac{\epsilon}{2})$ and $W = O(g_0, x_1,\dots x_n, \frac{\epsilon}{2})$ are open in $C_p(X,\mathbb{R})$ and $Q_p(X,\mathbb{R})$, respectively. Therefore $V \times W$ is open in $C_p(X,\mathbb{R}) \times Q_p(X,\mathbb{R})$ containing (f_0, g_0). We claim that

$s(V \times W) \subseteq U$. For this, let $s(f,g) = f + g \in s(V \times W)$, then $|f(x_i) + g(x_i) - h_0(x_i)| \leq |f(x_i) - f_0(x_i)| + |g(x_i) - g_0(x_i)| < \epsilon$ for all $i \in \{1, \dots, n\}$. Thus $f + g \in O(h_0, x_1, \dots, x_n, \epsilon) \subseteq U$. □

Lemma 5.5. *For any $x \in X$, the evaluation map at x, $e_x : Q_p(X,Y) \to Y$ defined by $e_x(f) = f(x)$ is continuous.*

Proof. Let $f \in Q_p(X,Y)$ and V be any open set in Y containing $f(x)$. Then $U = [x, V]$ is an open set containing f such that $e_x(U) \subseteq V$. Thus e_x is continuous. □

For any space X and $A \subseteq X$, a family $\mathcal{B}_A$ of open subsets of X is called a base at A [5] if each member of $\mathcal{B}_A$ contains A and for any open set U containing A, there exists $B \in \mathcal{B}_A$ such that $B \subseteq U$. The character of A in X is defined as $\chi(A, X) = \aleph_0 + \min\{|\mathcal{B}_A| : \mathcal{B}_A \text{ is a base at } A\}$. Note that $\chi(\{x\}, X) = \chi(x, X)$.

Proposition 5.6. *Let X be a Hausdorff space. If there exists a compact subspace K of the space $Q_p(X,\mathbb{R})$ such that $\chi(K, Q_p(X,\mathbb{R})) \leq \aleph_0$, then X is countable.*

Proof. Given a basic open set $U = [x_1, \dots, x_n; V_1, \dots, V_n]$ in $Q_p(X,\mathbb{R})$, let $A_U = \{x_1, \dots, x_n\}$. Suppose that $\{B_n : n \in \mathbb{N}\}$ is a countable base at K in $Q_p(X,\mathbb{R})$. Fix $n \in \mathbb{N}$, for each $f \in K$, choose a basic open set U^n_f such that $f \in U^n_f \subseteq B_n$. For open cover $\{U^n_f : f \in K\}$ of K, choose a finite subcover $\{U^n_{f_1}, \dots, U^n_{f_{m_n}}\}$ for some $m_n \in \mathbb{N}$. Let $W_n = U^n_{f_1} \cup \dots \cup U^n_{f_{m_n}}$ and $A_n = A_{U^n_{f_1}} \cup \dots \cup A_{U^n_{f_{m_n}}}$, then $K \subseteq W_n \subseteq B_n$. Clearly $A = \bigcup\{A_n : n \in \mathbb{N}\}$ is countable. We claim that $A = X$.

Suppose that $x \in X \setminus A$. By Lemma 5.5, the map $e_x : Q_p(X,\mathbb{R}) \to \mathbb{R}$ defined by $e_x(f) = f(x)$ is continuous. Therefore the set $e_x(K)$ is bounded in $\mathbb{R}$. Choose $M > 0$ such that $|f(x)| < M$ for all $f \in K$. Since $W = [x, (-M, M)]$ is an open set containing K, there exists $k \in \mathbb{N}$ such that $K \subseteq B_k \subseteq W$ and hence $W_k = U^k_{f_1} \cup \dots \cup U^k_{f_{m_k}} \subseteq W$. Thus $U^k_{f_1} = [x_1, \dots, x_n; V_1, \dots, V_n] \subseteq W$ such that $x \notin \{x_1, \dots, x_n\}$. Since X is Hausdorff, by Lemma 4.2, choose $g \in Q_p(X,\mathbb{R})$ such that $g(x_i) \in V_i$ for each $i \in \{1, \dots, n\}$ and $g(x) = M$. Then $g \in W_k \setminus W$, which is a contradiction. □

The properties of the restriction map related to the space $C(X,\mathbb{R})$ with the topology of pointwise convergence are discussed in [1]. For $Y \subseteq X$, the restriction map is defined as $\pi_Y : F(X,Z) \to F(Y,Z)$ such that $\pi_Y(f) = f|_Y$ for all $f \in F(X,Z)$. Note that the restriction of a quasicontinuous map on an open or a dense subset is quasicontinuous [16]. A map $f : X \to Y$ is called almost onto if $f(X)$ is dense in Y.

Proposition 5.7. *Let X be a regular space and Y be an open subset of X. If the map $\pi_Y : Q(X,\mathbb{R}) \to Q(Y,\mathbb{R})$ such that $\pi_Y(f) = f|_Y$ is injective, then Y is dense in X.*

Proof. Let $h_0 \in Q(X,\mathbb{R})$ such that $h_0(x) = 0$ for all $x \in X$. Suppose that π_Y is injective but Y is not dense in X so that $z \in X \setminus \overline{Y}$. By Lemma 4.12, there exists

$h \in Q(X, \mathbb{R})$ such that $h(z) = 1$ and $h(\overline{Y}) = \{0\}$. We have $\pi_Y(h) = \pi_Y(h_0)$ but $h \neq h_0$, which is a contradiction. Hence Y is dense in X. $\square$

Theorem 5.8. *Let X be a Hausdorff space and $Y \subseteq X$ be open or dense in X. Then the restriction map $\pi_Y : Q_p(X, \mathbb{R}) \to Q_p(Y, \mathbb{R})$ such that $\pi_Y(f) = f|_Y$ is continuous and almost onto. Moreover, π_Y is a homeomorphism if and only if $Y = X$.*

Proof. Consider the natural projection $p_Y : \mathbb{R}^X \to \mathbb{R}^Y$ such that $p_Y(x) = x|Y$. Then p_Y is a continuous map [17, Problem 107] and $\pi_Y = p_Y|_{Q_p(X,\mathbb{R})}$. Therefore π_Y is continuous. By Theorem 4.15, $Q(X, \mathbb{R})$ is dense in $\mathbb{R}^X$. Since p_Y is continuous, $\mathbb{R}^Y = p_Y(\mathbb{R}^X) = p_Y(\overline{Q_p(X, \mathbb{R})}) \subseteq \overline{p_Y(Q_p(X, \mathbb{R}))}$. Thus $\pi_Y(Q_p(X, \mathbb{R})) = p_Y(Q_p(X, \mathbb{R}))$ is dense in $\mathbb{R}^Y$ and hence also dense in $Q_p(Y, \mathbb{R})$.

Now if π_Y is a homeomorphism and $Y \neq X$. For $x \in X \setminus Y$, the set $D = \{f \in Q_p(X, \mathbb{R}) : f(x) = 0\}$ is not dense in $Q_p(X, \mathbb{R})$, because $D \cap [x, (0, 1)] = \varnothing$. But $\pi_Y(D)$ is dense in $Q_p(Y, \mathbb{R})$. Let $G = [y_1, \ldots, y_n; V_1, \ldots, V_n]$ be any basic open set in $Q_p(Y, \mathbb{R})$ containing some g. By Lemma 4.2, there exists $f \in Q_p(X, \mathbb{R})$ such that $f(y_i) = g(y_i)$ and $f(x) = 0$. Then $f \in D$ such that $\pi_Y(f) \in G$. Hence $\pi_Y(D) \cap G \neq \varnothing$. Because the image of a dense set $\pi_Y(D)$ under the map $(\pi_Y)^{-1}$ is D, which is not dense. This implies that $(\pi_Y)^{-1}$ is not continuous, which is a contradiction. Finally, if $Y = X$ then π_Y is the identity map, and hence a homeomorphism. $\square$

Acknowledgements. *The first author acknowledges the fellowship grant of University Grant Commission, India with Student-ID DEC18-414765. The authors are thankful to the anonymous referee for his valuable comments and suggestions.*

References

[1] A. V. Arhangel'skii, Topological Function Spaces, Mathematics and its Applications (Soviet Series), vol. 78, Kluwer Academic Publishers Group, Dordrecht, 1992.

[2] J. Borsík, Points of continuity, quasicontinuity and cliquishness, Rend. Istit. Mat. Univ. Trieste 26 (1994), 5–20.

[3] R. Cazacu and J. D. Lawson, Quasicontinuous functions, domains, and extended calculus, Appl. Gen. Topol. 8 (2007), 1–33.

[4] A. Crannell, M. Frantz and M. LeMasurier, Closed relations and equivalence classes of quasicontinuous functions, Real Anal. Exchange 31 (2005/06), 409–424.

[5] R. Engelking, General Topology, Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989.

[6] Ľ. Holá and D. Holý, Minimal USCO maps, densely continuous forms and upper semi-continuous functions, Rocky Mountain J. Math. 39 (2009), 545–562.

[7] Ľ. Holá and D. Holý, Pointwise convergence of quasicontinuous mappings and Baire spaces, Rocky Mountain J. Math. 41 (2011), 1883–1894.

[8] Ľ. Holá and D. Holý, Quasicontinuous functions and the topology of pointwise convergence, Topology Appl. 282 (2020), Article No. 107301.

[9] D. Holý and L. Matejíčka, Quasicontinuous functions, minimal USCO maps and topology of pointwise convergence, Math. Slovaca 60 (2010), 507–520.

[10] S. Kempisty, Sur les fonctions quasicontinues, Fundamenta Mathematicae 19 (1932), 184–197.

[11] P. S. Kenderov, I. S. Kortezov and W. B. Moors, Topological games and topological groups, Topology Appl. 109 (2001), 157–165.

[12] P. S. Kenderov, I. S. Kortezov and W. B. Moors, Continuity points of quasi-continuous mappings, Topology Appl. 109 (2001), 321–346.

[13] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly 70 (1963), 36–41.

[14] R. A. McCoy and I. Ntantu, Topological Properties of Spaces of Continuous Functions, Lecture Notes in Mathematics, vol. 1315, Springer-Verlag, Berlin, 1988.

[15] T. Neubrunn, Quasi-continuity, Real Anal. Exchange 14 (1988/89), 259–306.

[16] Z. Piotrowski, A survey of results concerning generalized continuity of topological spaces, Acta Math. Univ. Comenian. 52/53 (1987), 91–110.

[17] V. V. Tkachuk, A C_p-Theory Problem Book, Topological and function spaces, Problem Books in Mathematics, Springer, New York, 2011.