Fixed point theorems for F-contraction mapping in complete rectangular M-metric space

MOHAMMAD ASIM [a] ⓘ, SAMAD MUJAHID [b] ⓘ AND IZHAR UDDIN [b] ⓘ

[a] Department of Mathematics, Faculty of Science, Shree Guru Gobind Singh Tricentenary University, Gurugram, Haryana, India. (mailtoasim27@gmail.com)
[b] Department of Mathematics, Jamia Millia Islamia, New Delhi-110025, India.
(mujahidsamad721@gmail.com, izharuddin1@jmi.ac.in)

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ABSTRACT

In this paper, we prove a fixed point result for F-contraction principle in the framework of rectangular M-metric space. An example is also adopted to exhibit the utility of our result. Finally, we apply our fixed point result to show the existence of solution of Fredholm integral equation.

1. INTRODUCTION

Fixed point theory is an important and very active branch of functional analysis. It provides essential tools for solving problems arising in various branches of mathematical analysis. It guarantees the uniqueness and existence of the solution of integral and differential equations. In 1922, a polish mathematician Stefan Banach gives a contraction principle [10], which is one of the most well-known and important discovery in mathematics.

In the literature, there are two ways to generalized the Banach contraction principle either change the contraction condition or alter the metric space. In fixed point theory several contractions defined in metric space such that Boyd and Wong's nonlinear contraction principle [11], Meir-Keeler contraction [20, 1, 6], Suzuki contraction [33], Kannan contraction [17], Ćirić generalized contraction [14], Ćirić's quasi contraction [15], weak-contraction [29], Chatterjea contraction [13], Zamfirescu contraction [35] and F-Suzuki contraction [27] and many more [9, 25].

In 2012, Wardowski [34] introduced a new type of contraction for real-valued mapping F defined on positive real numbers and satisfying some conditions and obtained a fixed point theorem for it. After that several authors have worked on F-contraction mapping in different metric space. In 2014, Kumam and Piri [27] applied weaker condition on self map and extended the result of Wardowski [34]. In 2014, Minak et al. [21] obtained result for generalized F-contractions including Ćirić type generalized F-contraction and almost F-contraction on complete metric space. In 2017, Kumam et al. [28] introduced the F-contraction in the setting of complete asymmetric metric spaces and extend several results. In 2018, Kadelburg and Radenović [16] obtained the result on concerning F-contraction in b-metric space. In 2019, Luambano et al. [18] introduced the fixed point theorem for F-contraction in partial metric space and obtained certain results for it with suitable examples and many more [31, 30, 23, 22].

In 2014, Asadi et al. [4] introduced M-metric space, which extends the p-metric space given by Matthews [19] and proved the Banach contraction principle for it. Several authors have worked in this metric space [23, 3, 24, 2]. In 2000, Branciari [12] introduced rectangular metric space which is the another generalization of metric space. In 2018, Özgür et al. [26] introduced rectangular M-metric space. They were inspired by the work of Branciari [12] and Shukla [32], who defined partial rectangular metric spaces which is the generalization of rectangular metric space.

In 2019, Asim et al. [5, 7, 8] generalized the rectangular M-metric space as rectangular M_b-metric space, extended rectangular M_{r_ξ}-metric space and M_ν-metric space.

In this article, we establish the fixed point theorem for F-contraction in rectangular M- metric space. Throughout the article $\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+$ is the set of all positive real numbers and $\mathbb{N}$ is the set of all natural numbers.

2. Preliminaries

In this section, we collect some basic notions, definitions, examples and auxiliary results.

In 2000, Branciari [12] introduced rectangular metric space. The definition is as follows:

Definition 2.1 ([12])**.** Let X be a non-empty set. A function $r : X \times X \to \mathbb{R}_+$ is said to be a rectangular metric on X, if it satisfies the following (for all $x, y \in X$ and for all distinct point $u, v \in X \setminus \{x, y\}$):
(1) $r(x, y) = 0$, if and only if $x = y$,
(2) $r(x, y) = r(y, x)$ and
(3) $r(x, y) \leq r(x, u) + r(u, v) + r(v, y)$.
Then, the pair (X, r) is called a rectangular metric space.

After that, Shukla [32] introduced partial rectangular metric space. The definition is as follows:

Definition 2.2 ([32])**.** Let X be a non-empty set. A function $\rho : X \times X \to \mathbb{R}_+$ is said to be a partial rectangular metric on X, if it satisfies the following conditions (for any $x, y \in X$ and for all distinct point $u, v \in X \setminus \{x, y\}$):
(1) $x = y$ if and only if $\rho(x, y) = \rho(x, x) = \rho(y, y)$,
(2) $\rho(x, x) \leq \rho(x, y)$,
(3) $\rho(x, y) = \rho(y, x)$ and
(4) $\rho(x, y) \leq \rho(x, u) + \rho(u, v) + \rho(v, y) - \rho(u, u) - \rho(v, v)$.
Then, the pair (X, ρ) is called a partial rectangular metric space.

In 2014, Asadi et al. [4] generalized the partial metric space to M-metric space and obtained certain theorems related to M-metric space.
Notation: The following notations are useful in the sequel:
(i) $m_{xy} := m(x, x) \vee m(y, y) = \min\{m(x, x), m(y, y)\}$ and
(ii) $M_{xy} := m(x, x) \wedge m(y, y) = \max\{m(x, x), m(y, y)\}$.

Definition 2.3 ([4])**.** Let X be a non-empty set. A function $m : X \times X \to \mathbb{R}_+$ is called a m-metric, if it satisfying the following conditions:
(1) $m(x, x) = m(y, y) = m(x, y) \iff x = y$,
(2) $m_{xy} \leq m(x, y)$,
(3) $m(x, y) = m(y, x)$ and
(4) $(m(x, y) - m_{xy}) \leq (m(x, z) - m_{xz}) + (m(z, y) - m_{zy})$.
Then, the pair (X, m) is called an M-metric space.

In 2018, Özğur et al. [26] introduced rectangular M-metric space and definition are as follows:
Notation: The following notations are useful in the sequel:
(i) $m_{r_{xy}} := m_r(x, x) \vee m_r(y, y) = \min\{m_r(x, x), m_r(y, y)\}$ and
(ii) $M_{r_{xy}} := m_r(x, x) \wedge m_r(y, y) = \max\{m_r(x, x), m_r(y, y)\}$.

Definition 2.4 ([26])**.** Let X be a non-empty set. A function $m_r : X \times X \to \mathbb{R}_+$ is called m_r-metric, if it satisfying the following conditions:
(RM1) $m_r(x, x) = m_r(y, y) = m_r(x, y) \iff x = y$,
(RM2) $m_{r_{xy}} \leq m_r(x, y)$,
(RM3) $m_r(x, y) = m_r(y, x)$ and
(RM4) $(m_r(x, y) - m_{r_{xy}}) \leq (m_r(x, u) - m_{r_{xu}}) + (m_r(u, v) - m_{r_{uv}}) + (m_r(v, y) - m_{r_{vy}})$ for all $u, v \in X/\{x, y\}$.
Then, the pair (X, m_r) is called an rectangular M-metric space.

Example 2.5 ([26]). Let m_r be an m_r-metric. Put
(i) $m_r^w(x, y) = m_r(x, y) - 2m_{r_{xy}} + M_{r_{xy}}$
(ii) $m_r^s(x, y) = m_r(x, y) - m_{r_{xy}}$ when $x \neq y$ and $m_r^s(x, y) = 0$ if $x = y$.
Then, m_r^w and m_r^s are ordinary metrics.

Definition 2.6 ([26]). Let (X, m_r) be an rectangular M- metric space. Then,
(1) A sequence $\{x_n\}$ in X converges to a point x, if and only if

$$(2.1) \qquad \lim_{n \to \infty} (m_r(x_n, x) - m_{r_{x_n, x}}) = 0.$$

(2) A sequence $\{x_n\}$ in X is said to be m_r-Cauchy sequence, if and only if

$$(2.2) \quad \lim_{n,m \to \infty} (m_r(x_n, x_m) - m_{r_{x_n, x_m}}) \text{ and } \lim_{n,m \to \infty} (M_r(x_n, x_m) - m_{r_{x_n, x_m}})$$

exist and finite.
(3) An rectangular M-metric space is said to be m_r-complete, if every m_r-Cauchy sequence $\{x_n\}$ converges to a point x such that

$$(2.3) \quad \lim_{n \to \infty} (m_r(x_n, x) - m_{r_{x_n, x}}) = 0 \text{ and } \lim_{n \to \infty} (M_r(x_n, x) - m_{r_{x_n, x}}) = 0.$$

Lemma 2.7 ([26]). *Let (X, m_r) be a rectangular M-metric space. Then,*
(1) $\{\mathfrak{x}_n\}$ is an m_r-Cauchy sequence in (X, m_r) if and only if it is a Cauchy sequence in the metric space (X, m_r^w).
(2) (X, m_r) is m_r-complete if and only if the metric space (X, m_r^w) is complete. Furthermore,

$$\lim_{n \to \infty} (m_r^w(x_n, x) = 0 \Leftrightarrow \lim_{n \to \infty} (m_r(x_n, x) - m_{r_{x_n, x}}) = 0, \lim_{n \to \infty} (M_r(x_n, x) - m_{r_{x_n, x}}) = 0.$$

Likewise the above definition holds also for m_r^s.

Lemma 2.8 ([26]). *Assume that $x_n \to x$ as $n \to \infty$ in an rectangular M-metric space (X, m_r). Then,*

$$\lim_{n \to \infty} (m_r(x_n, y) - m_{r_{x_n, y}}) = m_r(x, y) - m_{r_{x,y}}, \forall \ y \in X.$$

Lemma 2.9 ([26]). *Assume that $x_n \to x$ and $y_n \to y$ as $n \to \infty$ in an rectangular M-metric space (X, m_r). Then,*

$$\lim_{n \to \infty} (m_r(x_n, y_n) - m_{r_{x_n, y_n}}) = m_r(x, y) - m_{r_{x,y}}.$$

Lemma 2.10. [26] *Assume that $x_n \to x$ and $y_n \to y$ as $n \to \infty$ in an rectangular M-metric space (X, m_r). Then, $m_r(x, y) = m_{r_{x,y}}$. Further if $m_r(x, x) = m_r(y, y)$, then $x = y$.*

Lemma 2.11 ([26]). *Let $\{x_n\}$ be a sequence in an rectangular M-metric space (X, m_r), such that there exists $k \in [0, 1)$ such that*

$$(2.4) \qquad m_r(x_{n+1}, x_n) \leq k m_r(x_n, x_{n-1}) \ \text{ for all } \ n \in \mathbb{N}.$$

Then,
(A) $\lim\limits_{n\to\infty} m_r(x_n, x_{n-1}) = 0,$
(B) $\lim\limits_{n\to\infty} m_r(x_n, x_n) = 0,$
(C) $\lim\limits_{n,m\to\infty} m_{r_{x_n,x_m}} = 0$ *and*
(D) $\{x_n\}$ *is an m_r-Cauchy sequence.*

Proof. [26] Using the definition of convergence and inequality (2.4), the proof of the condition (A) follows easily. From the Condition (RM2) and the Condition (A), we get

$$\lim_{n\to\infty} \min\{m_r(x_n, x_n), m_r(x_{n-1}, x_{n-1})\} = \lim_{n\to\infty} m_{r_{x_n x_{n-1}}}$$
$$\leq \lim_{n\to\infty} m_r(x_n, x_{n-1}) = 0.$$

Therefore, the Condition (B) holds. Since $\lim\limits_{n\to\infty} m_r(x_n, x_n) = 0$, the Condition (C) holds. Using the previous conditions and the definition (2.6), we see that the Condition (D) holds. $\square$

In 2012, Wardowski [34] introduced F-contraction and the definition are as follows:

Definition 2.12 ([34]). Let $F : \mathbb{R}_+ \to \mathbb{R}$ be a mapping satisfying:
(F1) F is strictly increasing, i.e. for all $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha < \beta, F(\alpha) < F(\beta)$,
(F2) For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim\limits_{n\to\infty} \alpha_n = 0$ if and only if $\lim\limits_{n\to\infty} F(\alpha_n) = -\infty$,
(F3) There exists $k \in (0, 1)$ such that $\lim\limits_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.
Denote $\triangle_F$ by the collection of all those functions which satisfy the conditions (F1-F3).
A mapping $T : X \to X$ is said to be an F-contraction if there exists $\tau > 0$ such that

$$(2.5) \qquad d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq d(x, y) \quad \forall \ x, y \in X.$$

Some examples related to F-contraction [34] are:

Example 2.13. Let $F : \mathbb{R}_+ \to \mathbb{R}$ be given by the formula $F(\alpha) = \ln(\alpha)$, it is clear that F satisfies (F1)-(F3) ((F3) for any $k \in (0, 1)$).

Example 2.14. Let $F : \mathbb{R}_+ \to \mathbb{R}$ be given by the formula $F(\alpha) = \ln(\alpha) + \alpha, \alpha > 0$. Then, F satisfies (F1)-(F3).

Example 2.15. Let $F : \mathbb{R}_+ \to \mathbb{R}$ be given by the formula $F(\alpha) = -\frac{1}{\sqrt{\alpha}}, \ \alpha > 0$. Then, F satisfies (F1)-(F3) ((F3) for any $k \in (1/2, 1)$).

3. Main results

The following definition is new version of the F-contraction for a rectangular M-metric space.

Definition 3.1. Let (X, m_r) be an rectangular M-metric space. The mapping $T : X \to X$ is said to be an F-contraction on X, if there exists $\tau > 0$ and $F \in \triangle_F$ such that $\forall \ x, y \in X$

$$(3.1) \qquad m_r(Tx, Ty) > 0 \Rightarrow \tau + F(m_r(Tx, Ty)) \leq F(m_r(x, y)).$$

Theorem 3.2. *Let (X, m_r) be a complete rectangular M-metric space and let $T : X \to X$ be a continuous F-contraction. Then, T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ a sequence $\{T^n(x_0)\}_{n \in \mathbb{N}}$ is convergent to x^*.*

Proof. Let $x_0 \in X$ be arbitrary and fixed. We define a sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$, $x_{n+1} = Tx_n$, $n = 0, 1, \cdots$. Denote $\xi_n = m_r(x_n, x_{n+1}) - m_{r_{x_n, x_{n+1}}}$, $n = 0, 1, \cdots$,
if there exist $n_0 \in \mathbb{N}$ for which $x_{n_0+1} = x_{n_0}$. Then, $Tx_{n_0} = x_{n_0}$ and the proof is finished.
Suppose that $x_{n+1} \neq x_n$ for every $n \in \mathbb{N}$. Then $\xi_n > 0$, for all $n \in \mathbb{N}$.
Using(3.1), then following holds for every $n \in \mathbb{N}$.

$$(3.2) \qquad F(\xi_n) \leq F(\xi_{n-1}) - \tau \leq F(\xi_{n-2}) - 2\tau \leq ... \leq F(\xi_0) - n\tau$$

From (3.2) we obtain $\lim_{n \to \infty} F(\xi_n) = -\infty$ that together with condition (F2) gives

$$(3.3) \qquad \lim_{n \to \infty} m_r(x_n, x_{n+1}) - m_{r_{x_{n+1}, x_n}} = 0.$$

We shall prove that

$$\lim_{n \to \infty} m_r(x_n, x_{n+2}) = 0.$$

We assume that $x_n \neq x_m$ for every $n, m \in \mathbb{N}, n \neq m$.
Indeed, suppose that $x_n = x_m$ for some $n = m + k$ with $k > 0$.

$$
\begin{aligned}
F(m_r(x_m, x_{m+1}) - m_{r_{x_m, x_{m+1}}}) &= F(m_r(x_n, x_{n+1}) - m_{r_{x_n, x_{n+1}}}) \\
&= F(m_r(x_{m+k}, x_{m+k+1}) - m_{r_{x_{m+k}, x_{m+k+1}}}) \\
&\leq F(m_r(x_m, x_{m+1}) - m_{r_{x_m, x_{m+1}}})) - k\tau \\
&< F(m_r(x_m, x_{m+1}) - m_{r_{x_m, x_{m+1}}}))
\end{aligned}
$$

a contradiction. Therefore, $m_r(x_n, x_m) - m_{r_{x_n, x_m}} > 0$ for every $n, m \in \mathbb{N}$ with $n \neq m$.

$$(3.4) \qquad \tau + F(m_r(x_n, x_{n+2})) \leq F(m_r(x_{n-1}, x_{n+1})), \quad \forall \ n \in \mathbb{N}.$$

Hence

$$(3.5) \qquad F(m_r(x_n, x_{n+2})) \leq F(m_r(x_{n-1}, x_{n+1})) - \tau$$

$$\begin{aligned}
F(m_r(x_n, x_{n+2})) &\leq F(m_r(x_{n-2}, x_n)) - 2\tau \\
&\leq \cdots \leq F(m_r(x_0, x_2)) - n\tau.
\end{aligned}$$

Taking limit as $n \to \infty$ in above inequality, we get

$$\lim_{n \to \infty} F(m_r(x_n, x_{n+2})) = -\infty.$$

Then, from the Condition (F2) of Definition (2.12), we conclude that

(3.6)
$$\lim_{n \to \infty} (m_r(x_n, x_{n+2})) = 0.$$

Next, we shall show that $\{x_n\}_{n \in \mathbb{N}}$ is a m_r-Cauchy sequence, that is

$$\lim_{n,m \to \infty} (m_r(x_n, x_m)) - m_{r_{x_n,x_m}} = 0, \qquad n, m \in \mathbb{N}.$$

Now, from Definition (2.12) there exist $k \in (0,1)$ such that

$$\lim_{n \to \infty} m_r(x_n, x_{n+1})^k F(m_r(x_n, x_{n+1})) = 0.$$
$$F(m_r(x_n, x_{n+1})) \leq F(m_r(x_0, x_1)) - n\tau.$$

We have

$$m_r(x_n, x_{n+1})^k (F(m_r(x_n, x_{n+1})) - F(m_r(x_0, x_1))) \leq m_r(x_n, x_{n+1})^k (F(m_r(x_0, x_1)) - n\tau)$$

(3.7)
$$m_r(x_n, x_{n+1})^k (F(m_r(x_n, x_{n+1})) - F(m_r(x_0, x_1))) \leq -m_r(x_n, x_{n+1})^k n\tau \leq 0.$$

Taking limit as $n \to \infty$ in above inequality, we conclude that

$$\lim_{n \to \infty} m_r(x_n, x_{n+1})^k n\tau = 0.$$

Then, there exist $n_1 \in \mathbb{N}$ such that $n m_r(x_n, x_{n+1})^k \leq 1, \quad \forall \ n \geq n_1$

$$m_r(x_n, x_{n+1}) \leq \frac{1}{n^{1/k}}, \quad \forall \, n \geq n_1.$$

Now, from Definition (2.12) there exists $k \in (0,1)$ such that

(3.8)
$$\lim_{n \to \infty} (m_r(x_n, x_{n+2}))^k F(m_r(x_n, x_{n+2})) = 0.$$

Since

$$F(m_r(x_n, x_{n+2})) \leq F(m_r(x_0, x_2)) - n\tau.$$

We have

$$(m_r(x_n, x_{n+2}))^k F(m_r(x_n, x_{n+2})) \leq (m_r(x_n, x_{n+2}))^k (F(m_r(x_0, x_2)) - n\tau)$$

$$(m_r(x_n, x_{n+2}))^k (F(m_r(x_n, x_{n+2})) - F(m_r(x_0, x_2))) \leq -n\tau (m_r(x_n, x_{n+2}))^k$$

$(m_r(x_n, x_{n+2})) \leq \frac{1}{n^{1/k}}$. Next, we show that $\{x_n\}$ is m_r-Cauchy sequence, that is

$$\lim_{n \to \infty} (m_r(x_n, x_{n+p}) - m_{r_{x_n,x_{n+p}}}) = 0, \quad \forall \ p \in \mathbb{N}.$$

The cases $p = 1$ and $p = 2$ are proved respectively by (3.3) and (3.4). Now, we take $p \geq 3$. It is sufficient to examine two cases.

Case 1. Firstly, let p is odd that is $p = 2m + 1$ for any $m \geq 1, n \in \mathbb{N}$. From the Condition (RM4) of definition of the m_r-metric, we get

$$
\begin{aligned}
m_r(x_n, x_{n+p}) &= m_r(x_n, x_{n+2m+1}) \\
(m_r(x_n, x_{n+p}) - m_{r_{x_n,x_{n+p}}}) &\leq (m_r(x_n, x_{n+1}) - m_{r_{x_n,x_{n+1}}}) \\
&\quad + (m_r(x_{n+1}, x_{n+2}) - m_{r_{x_{n+1},x_{n+2}}}) \\
&\quad + (m_r(x_{n+2}, x_{n+p}) - m_{r_{x_{n+2},x_{n+p}}}) \\
(m_r(x_n, x_{n+2m+1}) - m_{r_{x_n,x_{n+2m+1}}}) &\leq (m_r(x_n, x_{n+1}) - m_{r_{x_n,x_{n+1}}}) \\
&\quad + (m_r(x_{n+1}, x_{n+2}) - m_{r_{x_{n+1},x_{n+2}}}) \\
&\quad + (m_r(x_{n+2}, x_{n+2m+1}) - m_{r_{x_{n+2},x_{n+2m+1}}}) \\
&\leq (m_r(x_n, x_{n+1}) - m_{r_{x_n,x_{n+1}}}) \\
&\quad + (m_r(x_{n+1}, x_{n+2}) - m_{r_{x_{n+1},x_{n+2}}}) \\
&\quad + (m_r(x_{n+2}, x_{n+3}) - m_{r_{x_{n+2},x_{n+3}}}) + \\
&\quad \vdots \\
&\quad + (m_r(x_{n+2m}, x_{n+2m+1}) - m_{r_{x_{n+2m},x_{n+2m+1}}}) \\
&\leq [(m_r(x_n, x_{n+1}) - m_{r_{x_n,x_{n+1}}}) \\
&\quad + (m_r(x_{n+2}, x_{n+3}) - m_{r_{x_{n+2},x_{n+3}}}) + \\
&\quad \vdots \\
&\quad + (m_r(x_{n+2m-2}, x_{n+2m-1}) - m_{r_{x_{n+2m-2},x_{n+2m-1}}})] \\
&\quad + [(m_r(x_{n+1}, x_{n+2}) - m_{r_{x_{n+1},x_{n+2}}}) + \\
&\quad \vdots \\
&\quad + (m_r(x_{n+2m-1}, x_{n+2m}) - m_{r_{x_{n+2m-1},x_{n+2m}}}) \\
&\quad + (m_r(x_{n+2m}, x_{n+2m+1}) - m_{r_{x_{n+2m},x_{n+2m+1}}})] \\
(m_r(x_n, x_{n+p}) - m_{r_{x_n,x_{n+p}}}) &\leq \sum_{i=n}^{n+p-1} (m_r(x_i, x_{i+1}) - m_{r_{x_i,x_{i+1}}}) \\
&\leq \sum_{i=n}^{\infty} (m_r(x_i, x_{i+1}) - m_{r_{x_i,x_{i+1}}}) \\
&\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}.
\end{aligned}
$$

From the above from the convergence of the series

$$\sum_{i=n}^{n+p-1} \frac{1}{i^{1/k}} \Rightarrow \lim_{n\to\infty} (m_r(x_n, x_{n+p}) - m_{r_{x_n, x_{n+p}}}) = 0.$$

Case 2. Secondly, assume p is even that is $p = 2m$ for any $m \geq 1, n \in \mathbb{N}$.
From the Condition (RM4) of definition of the m_r-metric, we get

$$
\begin{aligned}
(m_r(x_n, x_{n+p}) - m_{r_{x_n, x_{n+p}}}) \;\leq\; & (m_r(x_n, x_{n+2}) - m_{r_{x_n, x_{n+2}}}) \\
& + (m_r(x_{n+2}, x_{n+3}) - m_{r_{x_{n+2}, x_{n+3}}}) \\
& + (m_r(x_{n+3}, x_{n+2m}) - m_{r_{x_{n+3}, x_{n+2m}}}) \\
(m_r(x_n, x_{n+2m}) - m_{r_{x_n, x_{n+2m}}}) \;\leq\; & (m_r(x_n, x_{n+2}) - m_{r_{x_n, x_{n+2}}}) \\
& + (m_r(x_{n+2}, x_{n+3}) - m_{r_{x_{n+2}, x_{n+3}}}) \\
& + (m_r(x_{n+3}, x_{n+4}) - m_{r_{x_{n+3}, x_{n+4}}}) + \\
& \quad\vdots \\
& + (m_r(x_{n+2m-3}, x_{n+2m-2}) - m_{r_{x_{n+2m-3}, x_{n+2m-2}}}) \\
& + (m_r(x_{n+2m-2}, x_{n+2m-1}) - m_{r_{x_{n+2m-2}, x_{n+2m-1}}}) \\
& + (m_r(x_{n+2m-1}, x_{n+2m}) - m_{r_{x_{n+2m-1}, x_{n+2m}}}) \\
\;\leq\; & (m_r(x_n, x_{n+2}) - m_{r_{x_n, x_{n+2}}}) \\
& + (m_r(x_{n+2}, x_{n+3}) - m_{r_{x_{n+2}, x_{n+3}}}) \\
& + (m_r(x_{n+3}, x_{n+4}) - m_{r_{x_{n+3}, x_{n+4}}}) \\
& + (m_r(x_{n+4}, x_{n+5}) - m_{r_{x_{n+4}, x_{n+5}}}) + \\
& \quad\vdots \\
& + (m_r(x_{n+2m-2}, x_{n+2m-1}) - m_{r_{x_{n+2m-2}, x_{n+2m-1}}}) \\
& + (m_r(x_{n+2m-1}, x_{n+2m}) - m_{r_{x_{n+2m-1}, x_{n+2m}}}) \\
\;\leq\; & (m_r(x_n, x_{n+2}) - m_{r_{x_n, x_{n+2}}}) + \sum_{i=n+2}^{n+2m-1} (m_r(x_i, x_{i+1}) - m_{r_{x_i, x_{i+1}}}) \\
\;\leq\; & (m_r(x_n, x_{n+2}) - m_{r_{x_n, x_{n+2}}}) + \sum_{i=n+2}^{n+p-1} (m_r(x_i, x_{i+1}) - m_{r_{x_i, x_{i+1}}}) \\
\;\leq\; & (m_r(x_n, x_{n+2}) - m_{r_{x_n, x_{n+2}}}) + \sum_{i=n+2}^{\infty} (m_r(x_i, x_{i+1}) - m_{r_{x_i, x_{i+1}}}) \\
\;\leq\; & (m_r(x_n, x_{n+2}) - m_{r_{x_n, x_{n+2}}}) + \sum_{i=n+2}^{\infty} \frac{1}{i^{1/k}} \\
\;\leq\; & \frac{1}{n^{1/k}} + \sum_{i=n+2}^{\infty} \frac{1}{i^{1/k}}.
\end{aligned}
$$

From the above from the convergence of the series

$$\sum_{i=n+2}^{n+2m-1} \frac{1}{i^{1/k}} \Rightarrow \lim_{n\to\infty} (m_r(x_n, x_{n+p}) - m_{r_{x_n,x_{n+p}}}) = 0.$$

By Lemma (2.9), we obtain that for any $n, m \in \mathbb{N}$,

$$m_r^s(x_n, x_m) = m_r(x_n, x_m) - m_{r_{x_n,x_m}} \to 0 \text{ as } n \to \infty.$$

This implies that $\{x_n\}_{n\in\mathbb{N}}$ is a m_r-Cauchy sequence with respect to m_r^s and converges by Lemma (2.10). Thus,

$$\lim_{n,m\to\infty} m_r^s(x_n, x_{n+2m+1}) = 0$$

and

$$\lim_{n,m\to\infty} m_r^s(x_n, x_{n+2m}) = 0.$$

We received by Lemma (2.7) is that $\{x_n\}$ is an m_r-Cauchy sequence.
From the completeness of X, there exist $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*.$$

Finally, the continuity of T yields

$$
\begin{aligned}
(m_r(Tx^*, x^*) - m_{r_{Tx^*,x^*}}) &= \lim_{n\to\infty} (m_r(Tx_n, x_n) - m_{r_{Tx_n,x_n}}) \\
&= \lim_{n\to\infty} (m_r(x_{n+1}, x_n) - m_{r_{x_{n+1},x_n}}) = 0.
\end{aligned}
$$

Now, we show that the uniqueness of a fixed point of T. Assume that T has two distinct fixed points $x, y \in X$, such that $x = Tx, y = Ty$.
From the Condition (3.1), we have

$$F(m_r(x,y)) = F(m_r(Tx, Ty)) < \tau + F(m_r(Tx, Ty)) \le F(m_r(x,y)),$$

which is contradiction. Hence, T has unique fixed point. $\qquad\square$

Example: Let $X = [0,1]$ and $m_r(x,y) = \frac{|x|+|y|}{2}$, for all $x, y \in X$. Then, (X, m_r) is complete rectangular M-metric space. Define a mapping $T : X \to X$ such that $T(x) = \frac{x}{2}$, for all $x \in X$.
Define the function $F : \mathbb{R}_+ \to \mathbb{R}$ by $F(r) = \ln(r)$, for all $x, y \in X$ such that $m_r(Tx, Ty) > 0$ this implies that $\tau + F(m_r(Tx, Ty)) = \tau + \ln(\frac{|x|+|y|}{4})$.
Let $\tau \le \ln 2$. Then

$$\tau + \ln\left(\frac{|x|+|y|}{4}\right) \le \ln 2 + \ln\left(\frac{|x|+|y|}{4}\right) = F(m_r(x,y)).$$

Thus, the contractive condition is satisfied for all $x, y \in X$. Hence, all hypotheses of the Theorem (3.2) are satisfied and T has a unique fixed point $x = 0$.

4. Applications

In this section, we apply Theorem (3.2) to investigate the existence and uniqueness of solution of the Fredholm integral equation [7].
Let $X = C([a, b], \mathbb{R})$ be the set of continuous real valued functions defined on $[a, b]$.
Now, we consider the following Fredholm type integral equation:

$$(4.1) \qquad x(p) = \int_a^b G(p, q, x(p))dq + h(p), \quad \text{for } p, q \in [a, b]$$

where $G, h \in C([a, b], \mathbb{R})$. Define $m_r : X \times X \to \mathbb{R}_+$ by

$$(4.2) \qquad m_r(x(p), y(p)) = \sup_{p \in [a,b]} \frac{(|x(p)| + |y(p)|)}{2}, \quad \forall \ x, y \in X.$$

Then, (X, m_r) is an m_r-complete in rectangular M-metric space.

Theorem 4.1. *Suppose that there exist $\tau > 0$ and for all $x, y \in C([a, b], \mathbb{R})$*

$$|G(p, q, x(p)) + G(p, q, y(p)) + 2h(p)| \quad \leq \quad \frac{e^{-\tau}}{(b - a)}|x(p) + y(p)|, \quad \forall \ p, q \in [a, b].$$

Then, the integral equation (4.1) has a unique solution.

Proof. Define $T : X \to X$ by,

$$T(x(p)) = \int_a^b G(p, q, x(p))dq + h(p), \quad \forall \ p, q \in [a, b].$$

Observe that existence of a fixed point of the operator T is equivalent to the existence of a solution of the integral equation (4.1). Now, for all $x, y \in X$. We

have

$$m_r(Tx, Ty) = \left| \frac{T(x(p)) + T(y(p))}{2} \right|$$

$$= \left| \int_a^b \left(\frac{G(p, q, x(p)) + G(p, q, y(p)) + 2h(p)}{2} \right) dq \right|$$

$$\leq \int_a^b \left| \left(\frac{G(p, q, x(p)) + G(p, q, y(p)) + 2h(p)}{2} \right) dq \right|$$

$$\leq \frac{e^{-\tau}}{(b-a)} \int_a^b \frac{|x(p) + y(p)|}{2} dq$$

$$\leq \frac{e^{-\tau}}{(b-a)} \int_a^b \frac{|x(p)| + |y(p)|}{2} dq$$

$$\leq \frac{e^{-\tau}}{(b-a)} \sup_{p \in [a,b]} \frac{(|x(p)| + |y(p)|)}{2} \left(\int_a^b dq \right)$$

$$\leq \frac{e^{-\tau}}{(b-a)} m_r(x, y)(b-a)$$

$$\leq e^{-\tau} m_r(x, y).$$

Thus, the Condition (3.1) is satisfied with $F(\alpha) = \ln(\alpha)$. Therefore, all the conditions of Theorem (3.2) are satisfied. Hence the operator T has a unique fixed point, which means that the Fredholm integral equation (4.1) has a unique solution. This completes the proof. □

5. CONCLUSION

As the rectangular M-metric is new generalization of m-metric and rectangular metric. In this article, we introduced F-contraction in rectangular M-metric space and utilized a fixed point for it. We give a suitable example which supported to the fixed point theorem. We give an application in Fredholm integral equation in rectangular M-metric space.

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