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A note on the spectrum of some composition operators on Korenblum type spaces.

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Corresponding Author:	Esther Gómez Orts UPV: Universitat Politecnica de Valencia SPAIN
Corresponding Author Secondary Information:	
Corresponding Author's Institution:	UPV: Universitat Politecnica de Valencia
Corresponding Author's Secondary Institution:	
First Author:	Pablo Galindo
First Author Secondary Information:	
Order of Authors:	Pablo Galindo
	Esther Gómez Orts
Order of Authors Secondary Information:	
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A note on the spectrum of some composition operators on Korenblum type spaces

Pablo Galindo* and Esther Gómez-Orts†

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Abstract

We prove that under some mild conditions on the symbol φ, the spectrum of the corresponding composition operator C_φ on the Korenblum type spaces $A_-^{-\alpha}$ and $A_+^{-\alpha}$ contains a closed ball of positive radius.

1 Introduction

The aim of this note is to study the spectrum of some composition operators on the Korenblum type spaces $A_-^{-\alpha}$ and $A_+^{-\alpha}$ of analytic functions in the open unit disc of the complex plane. The rich Operator Theory on Banach spaces cannot be used here since our spaces are not Banach spaces, but projective and inductive limits of Banach spaces: for instance the basic fact that the spectrum is closed cannot be guaranteed (see [1, Example 4.9] for instance). Nevertheless, we are able to prove that for a family of symbols singled out by H. Kamowitz in [10], the spectrum contains a closed ball of positive radius. A feature that is shared by very many composition operators acting on Banach spaces of analytic functions of which the book [6] is a thorough sample.

A new insight in the approach of Kamowitz study of the spectrum of some composition operators on H^p spaces ([10, Theorem 3.4]) allows us to use his method to prove that for $\varphi : \overline{\mathbb{D}} \to \mathbb{D}$ analytic, with an interior fixed point and a repelling fixed point z_0 in the sphere, the spectrum of the composition operator C_φ on $A_-^{-\alpha}$ and $A_+^{-\alpha}$ contains the closed ball $\overline{B}(0, |\varphi'(z_0)|^{-\alpha})$. At a crucial step Kamowitz approach uses analitycity and that the resolvent set for an operator on a Banach spaces is open; since this fact is not available in our setting, we detour to get a contradiction based on uniform convergence on compact sets of weak* convergent sequences. Our results enlarge the knowledge of the size of $\sigma(C_\varphi)$ that it is known to be a subset of $\overline{B}(0, r_e(C_\varphi, H_\alpha^\infty)) \cup \{\varphi'(0)^n\}_{n=0}^\infty$ according to [7], where $r_e(C_\varphi, H_\alpha^\infty)$ is the essential spectral radius of the composition operator C_φ on H_α^∞.

2 Preliminaries

Let $\varphi : \overline{\mathbb{D}} \to \mathbb{D}$ be an analytic map on the closed unit disc $\overline{\mathbb{D}}$ of the complex plane. The composition operator will be denoted by $C_\varphi f := f \circ \varphi$. The space $H(\mathbb{D})$ of all analytic functions on $\mathbb{D}$ is endowed with the Fréchet topology of uniform convergence on compact sets. When we write a space, we mean a Hausdorff locally convex space. We refer the reader to [13] for results and terminology about functional analysis, and in particular about Fréchet and (LB)-spaces.

For each $\alpha > 0$, the growth Banach spaces of analytic functions are defined as

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$$H_\alpha^\infty := \{f \in H(\mathbb{D})\colon \|f\|_\alpha := \sup_{z\in\mathbb{D}}(1-|z|)^\alpha |f(z)| < \infty\}$$

and

$$H_\alpha^0 := \{f \in H(\mathbb{D})\colon \lim_{|z|\to 1^-}(1-|z|)^\alpha |f(z)| = 0\}\,.$$

These spaces are sometimes defined using the weight $(1-|z|^2)^\alpha$ instead of the weight $(1-|z|)^\alpha$. Since $1-|z| \le 1-|z|^2 \le 2(1-|z|)$, the spaces coincide and the norms are equivalent. Both H_α^∞ and H_α^0 are Banach spaces when endowed with the norm $\|\cdot\|_\alpha$. The space H_α^0, which is a closed subspace of H_α^∞, coincides with the closure of the polynomials on H_α^∞; see e.g. [3]. The space H^∞ of bounded analytic functions on $\mathbb{D}$ is contained in H_α^0 for each $\alpha > 0$. These Banach spaces, as well as their intersections and unions, play a relevant and important role in connection with the interpolation and sampling of analytic functions; see [8, Section 4.3].

For each pair $0 < \beta_1 < \beta_2$ we have $H_{\beta_1}^\infty \subset H_{\beta_2}^0$, with continuous inclusion.

The spaces of analytic functions we consider are defined in the following way.

$$A_+^{-\alpha} := \bigcap_{\beta>\alpha} H_\beta^\infty = \{f \in H(\mathbb{D})\colon \sup_{z\in\mathbb{D}}(1-|z|)^\beta |f(z)| < \infty\ \forall \beta > \alpha\},$$

in which case also

$$A_+^{-\alpha} = \bigcap_{\beta>\alpha} H_\beta^0$$

for each $\alpha \ge 0$. And

$$A_-^{-\alpha} := \bigcup_{\beta<\alpha} H_\beta^\infty = \{f \in H(\mathbb{D})\colon \sup_{z\in\mathbb{D}}(1-|z|)^\beta |f(z)| < \infty \text{ for some } \beta < \alpha\}$$

in which case also

$$A_-^{-\alpha} = \bigcup_{\beta<\alpha} H_\beta^0,$$

for each $0 < \alpha \le \infty$.

The space $A_+^{-\alpha}$ is a Fréchet space when endowed with the locally convex topology generated by the increasing sequence of norms $\|f\|_k := \sup_{z\in\mathbb{D}}(1-|z|)^{\alpha+\frac{1}{k}}|f(z)|$, for $f \in A_+^{-\alpha}$ and each $k \in \mathbb{N}$. The space $A_+^{-\alpha}$ is also a projective limit and it is dense in H_β^0, $\beta > 0$, since it contains the polynomials. We note, for $0 < \beta_1 < \beta_2$, that the natural inclusion $H_{\beta_1}^\infty \subset H_{\beta_2}^0$ is actually a compact operator. This follows for example as a consequence of [4, Theorem 3.3]. Therefore $A_+^{-\alpha}$ is a Fréchet Schwartz space, [9, §21.1 Example 1(b)]. In particular, bounded subsets of $A_+^{-\alpha}$ are relatively compact, [13, Remark 24.24]. Moreover, for every $\beta > \alpha > 0$ we have $H_\alpha^\infty \subset A_+^{-\alpha} \subset H_\beta^0$ with continuous inclusions.

The space $A_-^{-\alpha}$ is endowed with the finest locally convex topology such that all the natural inclusion maps $H_\beta^\infty \subset A_-^{-\alpha}$ for $\beta < \alpha$ are continuous. In particular, the space $A_-^{-\alpha}$ is the complete (DFS)-space

$$A_-^{-\alpha} := \operatorname*{ind}_k H_{\alpha-\frac{1}{k}}^\infty = \operatorname*{ind}_k H_{\alpha-\frac{1}{k}}^0,$$

(see [13, Proposition 25.20]). The inductive limit is taken over all $k \in \mathbb{N}$ such that $(\alpha - \frac{1}{k}) > 0$.

The classical Korenblum space $A_{-}^{-\infty}$, [11], denoted $A^{-\infty}$, is defined by

$$A^{-\infty} := \bigcup_{0<\alpha<\infty} H_\alpha^\infty = \bigcup_{n\in\mathbb{N}} H_n^\infty$$

and is endowed with the finest locally convex topology such that all the natural inclusion maps $H_n^\infty \subset A^{-\infty}$ are continuous, that is, $A^{-\infty} = \mathrm{ind}_n H_n^\infty$.

For a positive integer m and $\beta > 0$, let $H_{\beta,m}^\infty$ denote the closed subspace of H_β^∞ given by

$$H_{\beta,m}^\infty := \{f \in H_\beta^\infty : f \ \text{ has a zero of at least order m at } 0\}.$$

Lemma 2.1. *Let $0 < p < \infty$. For any positive integer $m \in \mathbb{N}$, the map $F \in H_p^\infty \to z^m F \in H_{p,m}^\infty$ is an isomorphism.*

Proof. First, we will see it is continuous. The norm in $H_{p,m}^\infty$ is the same as in H_p^∞. Let $F \in H_p^\infty$.

$$\|z^m F\|_p = \sup_{z\in\mathbb{D}}(1-|z|)^p |z|^m |F(z)| \le \sup_{z\in\mathbb{D}}(1-|z|)^p |F(z)| = \|F\|_p .$$

On the other hand, any $f \in H_{p,m}^\infty$ can be written as $f = z^m F$, for some holomorphic function F that indeed belongs to H_p^∞. Take $0 < r < 1$ and so,

$$\sup_{z\in\mathbb{D}}(1-|z|)^p|F(z)| = \max\Big\{ \sup_{z\in\overline{B}(0,r)} (1-|z|)^p|F(z)|, \sup_{z\in\mathbb{D}\setminus\overline{B}(0,r)} (1-|z|)^p|F(z)| \Big\}.$$

Since F is holomorphic and $\overline{B}(0,r)$ compact, there is $M > 0$ such that

$$\sup_{z\in\overline{B}(0,r)} (1-|z|)^p|F(z)| = M < \infty.$$

Moreover, since $f \in H_p^\infty$,

$$\sup_{z\in\mathbb{D}\setminus\overline{B}(0,r)} (1-|z|)^p|F(z)| = \sup_{z\in\mathbb{D}\setminus\overline{B}(0,r)} (1-|z|)^p \frac{|z^m F(z)|}{|z|^m} \le \sup_{z\in\mathbb{D}\setminus\overline{B}(0,r)} (1-|z|)^p \frac{|z^m F(z)|}{r^m} = \frac{\|f\|_p}{r^m} < \infty.$$

Thus, the map is surjective and in addition it is clearly one-to-one, hence an isomorphism by the Open Mapping Theorem. □

This section includes also a number of lemmas needed in the sequel.

Recall that a sequence $(z_k) \subset \mathbb{D}$ is an *iteration sequence* for φ if $\varphi(z_k) = z_{k+1}$ for all k. We need the following crucial lemmas due to Cowen and MacCluer.

Lemma 2.2 ([6], Lemma 7.34)**.** *If φ is an analytic map, not an automorphism, of the unit disk into itself and $\varphi(0) = 0$. For a given $0 < r < 1$, there exists $1 \le M < \infty$ such that if $(z_k)_{k=-K}^\infty$ is an iteration sequence with $|z_n| \ge r$ for some non-negative integer n and if $(w_k)_{k=-K}^n$ are arbitrary numbers, then there is $f \in H^\infty$ such that*

$$f(z_k) = w_k, \quad -K \le k \le n$$

and

$$||f||_\infty \le M \sup\{|w_k| : -K \le k \le n\}.$$

Lemma 2.3 ([6], Lemma 7.35)**.** *Let φ be as in the previous lemma. For any iteration sequence $(z_k)_k$ there exists $c < 1$ such that*

$$\frac{|z_{k+1}|}{|z_k|} \le c$$

whenever $|z_k| \le 1/2$.

Lemma 2.4. *Let $1 \leq p < \infty$, $m \in \mathbb{N}$. Suppose $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic on $\mathbb{D}$ with $\varphi(0) = 0$. Consider $\lambda \neq \varphi'(0)^n$ for all non-negative integers n and $g \in H^\infty_{p,m}$. If there is an analytic function $f \in H^\infty_p$ with $g = \lambda f - f \circ \varphi$, then f also belongs to $H^\infty_{p,m}$.*

Proof. Observe that $f(0) = 0$ since $0 = g(0) = \lambda f(0) - f(0)$. If we differentiate the expression $g(z) = \lambda f(z) - f(\varphi(z))$ we obtain:

$$\begin{aligned} g'(z) &= \lambda f'(z) - f'(\varphi(z))\varphi'(z) \\ g''(z) &= \lambda f''(z) - f''(\varphi(z))\varphi'(z)^2 - f'(\varphi(z))\varphi''(z) \\ g'''(z) &= \lambda f'''(z) - f'''(\varphi(z))\varphi'(z)^3 - f''(\varphi(z))2\varphi''(z) - f''(\varphi(z))\varphi'(z)\varphi''(z) - f'(\varphi(z))\varphi'''(z) \\ &\vdots \end{aligned}$$

In particular, for $z = 0$,

$$\begin{aligned} 0 = g'(0) &= \lambda f'(0) - f'(0)\varphi'(0) \\ 0 = g''(0) &= \lambda f''(0) - f''(0)\varphi'(0)^2 - f'(0)\varphi''(0) \\ 0 = g'''(0) &= \lambda f'''(0) - f'''(0)\varphi'(0)^3 - f''(0)2\varphi''(0) - f''(0)\varphi'(0)\varphi''(0) - f'(0)\varphi'''(0) \\ &\vdots \end{aligned}$$

Since $\lambda \neq \varphi'(0)^n$ for all $n \in \mathbb{N}$, from the first equation we get that $f'(0) = 0$, from the second one, that $f''(0) = 0$, and so on, we get that

$$f(0) = f'(0) = f''(0) = \ldots = f^{\eta)}(0) = 0.$$

Thus, $f(z) = z^\eta F(z)$ for some holomorphic function F and from Lemma 2.1, $F \in H^\infty_p$. Therefore $f \in H^\infty_{p,\eta}$. □

For $f, g \in A^{-\alpha}_{-}$ and a complex number $\lambda \neq 0$, satisfying $\lambda f - f \circ \varphi = g$, one can check inductively that

$$\lambda^k f(x_{-k}) = f(x_k)\lambda^{-k} + \lambda^{-1} \sum_{i=-k}^{k-1} g(x_i)\lambda^{-i} \quad \text{for each positive integer } k. \tag{2.1}$$

Lemma 2.5 ([10], Theorem 2.5)**.** *Suppose φ is analytic in a neighbourhood of a fixed point z_1 and $c = \varphi'(z_1)$, $|c| < 1$. Then there is a function A, analytic at z_1 such that $((\varphi_n(z) - z_1)/c^n) \to A(z)$ uniformly near z_1. In fact, $\varphi_n(z) = z_1 + c^n A(z) + \mathcal{O}(|c^n A(z)|^2)$. Further, if $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic on $\overline{\mathbb{D}}$ and z_1 is a fixed point of φ with $|z_1| = 1$ and $\varphi'(z_1) = c < 1$, then for each $z \in \mathbb{D}$, z near z_1, we have $A_0(z) = \mathrm{Re}A(z) > 0$ and $|\varphi_n(z)| = 1 + c^n A_0(z) + \mathcal{O}(|c^n A(z)|^2)$.*

3 Results

Let $T : X \to X$ be a continuous operator on a space X. We write $T \in \mathcal{L}(X)$. The *resolvent set* $\rho(T)$ of T consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ is a continuous linear operator, that is $T - \lambda I : X \to X$ is bijective and has a continuous inverse. Here I stands for the identity operator on X. The set $\sigma(T) := \mathbb{C} \setminus \rho(T)$ is called the *spectrum* of T. The *point spectrum* $\sigma_p(T)$ of T consists of all $\lambda \in \mathbb{C}$ such that $(\lambda I - T)$ is not injective. If we need to stress the space X, then we write $\sigma(T; X)$, $\sigma_p(T; X)$ and $\rho(T; X)$. Unlike for Banach spaces X, it may happen that $\rho(T) = \emptyset$ or that $\rho(T)$ is not open.

The following result is the paper's main one and provides the same conclusion as [10, Theorem 3.4].

Theorem 3.1. *Consider $A_-^{-\alpha}$ with $\alpha > 0$. Suppose that $\varphi : \overline{\mathbb{D}} \to \mathbb{D}$ is analytic and that it has a fixed point $a \in \mathbb{D}$. Suppose that there is a positive integer N such that φ_N has, at least, a fixed point z_0 in the unit sphere and that $\varphi'(z_0) > 1$. Then $\sigma(C_\varphi; A_-^{-\alpha}) \supseteq \{\lambda : |\lambda| \le \varphi_N'(z_0)^{-\alpha/N}\}$.*

Proof. Without loss of generality we assume that $a = 0$. Notice that φ cannot be an automorphism of $\mathbb{D}$ since then φ_N would be as well an automorphism and its fixed point structure prevents it. Thus $0 \in \sigma(C_\varphi)$.

Lemma 2.5 can be applied to φ_N^{-1} because it exists locally near its fixed point z_0 and $(\varphi_N^{-1})'(z_0) = \frac{1}{(\varphi_N)'(z_0)} < 1$. Thus we may choose $x_0 \in \mathbb{D}$ with $\lim_n \varphi_N^{-n}(x_0) = z_0$. Relying on it we construct an iterated sequence $(x_k)_{k=-\infty}^{+\infty}$ as follows. Define

$$x_k := \begin{cases} \varphi_k(x_0) & \text{if } k > 0, \\ \varphi_N^{-n}(x_0) & \text{if } k = -nN \text{ with } n > 0, \\ \varphi_p(x_{-nN}) & \text{if } k = -nN + p \text{ with } p = 0, \dots, N-1 \text{ and } n > 0. \end{cases}$$

Then, for all integers k, $\varphi(x_k) = x_{k+1}$.

Again by Lemma 2.5, if $n > 0$, we have

$$z_0 - x_{-nN} = z_0 - \varphi_N^{-n}(x_0) \sim \varphi_N'(z_0)^{-n} A(x_0) \tag{3.1}$$

and

$$1 - |x_{-nN}| \sim \varphi_N'(z_0)^{-n} A_0(x_0), \tag{3.2}$$

where $A(x_0)$ and $A_0(x_0) = \mathrm{Re} A(x_0)$ are not zero.

The choice of x_0 allows us to assume that $|x_0| > \frac{1}{2}$. Let $m_0 := \max\{k : |x_k| \ge 1/4\}$. Then $m_0 \ge 0$ and $|x_k| < 1/4$ for $k > m_0$. By Lemma 2.3 there is b with $1/2 \le b < 1$ for which $|x_{k+1}/x_k| \le b$ for all $k \ge m_0$. This implies that

$$|x_k| \le b^{k-m_0} |x_{m_0}|, \text{ for } k \ge m_0 . \tag{3.3}$$

Denote $c := \varphi_N'(z_0) = (\varphi'(z_0))^N$. Thus $c > 1$.

Fix λ so that $0 < |\lambda| \le c^{-\alpha/N}$. Suppose for a contradiction that $\lambda \notin \sigma(C_\varphi)$.

Choose n_0 so large that

$$\frac{b^n}{|\lambda|} < 1 \ \forall n \ge n_0. \tag{3.4}$$

Fix $m \in \mathbb{N}$, $m > n_0$ such that $|\varphi'(0)|^m < |\lambda|$. Given $f \in A_-^{-\alpha}$, there exists $0 < \beta < \alpha$ such that $f \in H_\beta^\infty$, so $|f(x_{-nN})| \le \|f\|_\beta (1 - |x_{-nN}|)^{-\beta}$. Therefore bearing in mind (3.2),

$$|\lambda^{nN} f(x_{-nN})| \lesssim \|f\|_\beta \frac{1}{A_0(x_0)^\beta} \left(\frac{|\lambda|^N}{c^{-\beta}} \right)^n . \tag{3.5}$$

Consequently, for all $f \in A_-^{-\alpha}$ we have

$$\lim_n |\lambda^{nN} f(x_{-nN})| = 0. \tag{3.6}$$

Let us denote by $A_{-,m}^{-\alpha}$ the inductive limit

$$A_{-,m}^{-\alpha} := \bigcup_{0<\beta<\alpha} H_{\beta,m}^\infty .$$

We claim that $\big(C_\varphi - \lambda I\big)(A^{-\alpha}_{-,m}) = A^{-\alpha}_{-,m}$. Let $g \in A^{-\alpha}_{-,m}$. Since $C_\varphi - \lambda I$ is onto, there is $f \in A^{-\alpha}_{-}$ such that $g = (C_\varphi - \lambda I)(f)$. Thus there is $0 < \beta < \alpha$ such that $f \in H^\infty_\beta$, and so also $g \in H^\infty_\beta$, hence $g \in H^\infty_{\beta,m}$. According to Lemma 2.4, f belongs as well to $H^\infty_{\beta,m}$, as claimed. In addition, $f(z) = z^m F(z)$ for some $F \in H^\infty_\beta$, as pointed out by Lemma 2.1, therefore $|f(x_{nN})| = |\varphi_{nN}(x_0)^m F(\varphi_{nN}(x_0))|$. Now applying [10, Lemma 2.6], we obtain for such an f,

$$\begin{aligned} \overline{\lim_n}\, |\lambda|^{-nN} |f(x_{nN})| &= \overline{\lim_n}\, |\varphi_{nN}(x_0)|^m |\lambda|^{-nN} |F(\varphi_{nN}(x_0))| \\ &\le \overline{\lim_n}\, |\varphi'(0)|^{mnN} |\lambda|^{-nN} |F(\varphi_{nN}(x_0))| = 0. \end{aligned} \tag{3.7}$$

From (2.1), we have

$$\lambda^{nN} f(x_{-nN}) = f(x_{nN})\lambda^{-nN} + \lambda^{-1} \sum_{i=-nN}^{Nn-1} g(x_i)\lambda^{-i}, \tag{3.8}$$

which together with limits (3.6) and (3.7) above show that if $g \in (C_\varphi - \lambda I)(A^{-\alpha}_{-,m})$, then

$$\sum_{k=-\infty}^{+\infty} g(x_k)\lambda^{-k} = 0.$$

Next, given the iteration sequence $(x_k)_{k=-K}^{+\infty}$, define the linear functionals L_K on $A^{-\alpha}_{-,m}$ by

$$L_K(f) := \sum_{k=-K}^{\infty} \frac{f(x_k)}{\lambda^k}\,. \tag{3.9}$$

If we denote the topological dual of $A^{-\alpha}_{-,m}$ with the inductive limit topology by $(A^{-\alpha}_{-,m})'$, then the functionals $L_K \in (A^{-\alpha}_{-,m})'$. In order to prove this, recall that $L_K : A^{-\alpha}_{-,m} \to \mathbb{C}$ is continuous if $L_K : H^\infty_{\beta,m} \to \mathbb{C}$ is continuous for all $\beta < \alpha$, that is, if for all $\beta < \alpha$ there exists a positive constant C such that $|L_K(f)| \le C\|f\|_\beta$ for all $f \in H^\infty_{\beta,m}$. Fix $\beta < \alpha$. For each $f \in H^\infty_{\beta,m}$ there is $F \in H^\infty_\beta$ such that $f(z) = z^m F(z)$ for all $z \in \mathbb{D}$. Then using estimate (3.3) and according to the choice of m_0, we obtain:

$$|L_K(f)| \le \sum_{k=-K}^{\infty} \frac{(1-|x_k|)^\beta |x_k|^m |F(x_k)|}{(1-|x_k|)^\beta |\lambda|^k} \le \|F\|_\beta \sum_{k=-K}^{m_0} \frac{|x_k|^m}{(1-|x_k|)^\beta |\lambda|^k} + \|F\|_\beta \left(\frac{4}{3}\right)^\beta \frac{|x_{m_0}|^m}{|\lambda|^{m_0}} \sum_{i=1}^{\infty} \frac{b^{im}}{|\lambda|^i}\,.$$

In addition, since $m > n_0$, estimate (3.4) guarantees that the series above is convergent, so

$$|L_K(f)| \le C\|F\|_\beta\,,$$

for some positive constant C. Moreover, by Lemma 2.1 the map $F \in H^\infty_\beta \to z^m F \in H^\infty_{\beta,m}$ is an isomorphism, hence there is $M > 0$ such that $\|F\|_\beta \le M\|z^m F\|_\beta$. Therefore, $|L_K(f)| \le CM\|f\|_\beta$, that is, $L_K : H^\infty_{\beta,m} \to \mathbb{C}$ is continuous, as wanted.

Moreover, the surjectivity of $C_\varphi - \lambda I|_{A^{-\alpha}_{-,m}}$ shows that $\lim_K L_K(f) = 0$ for all $f \in A^{-\alpha}_{-,m}$. That is, the sequence (L_K) converges to 0 in the weak* topology $\sigma\big((A^{-\alpha}_{-,m})', A^{-\alpha}_{-,m}\big)$.

Fix β such that $0 < 2\beta < \alpha$, then there exists $f_{x_0} \in H^\infty_\beta \subset A^{-\alpha}_{-}$ such that $||f_{x_0}||_\beta \le 1$ and $|f_{x_0}(x_0)| = 1/(1-|x_0|)^\beta$. Let $1 \le M < \infty$ be the constant in Lemma 2.2 for $r = 1/4$. Then there is $f_K \in H^\infty$ with $\|f_K\|_\infty \le M$, $|f_K(x_0)| = 1$ and satisfying

$$x_0^m f_K(x_0) f_{x_0}(x_0) > 0 \;\text{ and }\; f_K(x_k) = 0 \text{ for } -K \le k \le m_0,\, k \ne 0.$$

Now, the function $g_K(x) := x^m f_K(x) f_{x_0}(x)$ belongs to $H^\infty_{\beta,m}$ and $\|g_K\|_\beta \le M$. Further,

$$L_K(g_K) = x_0^m f_K(x_0) f_{x_0}(x_0) + \sum_{k=m_0+1}^{\infty} \lambda^{-k} x_k^m f_K(x_k) f_{x_0}(x_k).$$

If, in addition, we choose m so that

$$M \frac{1}{|\lambda|^{m_0}} \frac{1}{v_\beta(x_{m_0})} \frac{b^m}{|\lambda| - b^m} < \frac{1}{2(1-|x_0|)^\beta} \ ,$$

and use again (3.3) and (3.4), we obtain

$$|L_K(g_K)| \ge \frac{|x_0|^m}{2(1-|x_0|)^\beta}. \tag{3.10}$$

To conclude, recall that the embedding $H^\infty \hookrightarrow H^\infty_\beta$ is a compact operator and the multiplication operator $M_{z^m f_{x_0}}$ is a continuous linear map from H^∞_β into $H^\infty_{2\beta}$. Hence $(g_K)_K$ is a relatively compact subset of $H^\infty_{2\beta,m}$ as lying in the image of a ball under the composition of those two mappings. This way we are led into a contradiction with (3.10) since we must have $\lim_K L_K(g_K) = 0$ because the pointwise bounded set $(L_K)_K$ is an equicontinuous set in the dual of the barreled space $A^{-\alpha}_{-,m}$.

Finally, since m can be big enough, we conclude that $\sigma(C_\varphi; A^{-\alpha}_-) \supseteq \{\lambda : |\lambda| \le \varphi_N'(z_j)^{-\alpha/N}\}$.

□

Remark 3.2. If φ satisfies the assumptions of Theorem 3.1 with $N = 1$ and $\varphi'(0) \neq 0$, then we can complete the information about the spectrum of C_φ by using [7, Corollary 6] to obtain

$$\{\varphi'(0)^n\}_{n=0}^\infty \cup \{\lambda : |\lambda| \le \varphi'(z_0)^{-\alpha}\} \subseteq \sigma(C_\varphi; A^{-\alpha}_-) \subseteq \overline{B}(0, r_e(C_\varphi, H^\infty_\alpha)) \cup \{\varphi'(0)^n\}_{n=0}^\infty.$$

Example 3.3. Consider the symbol $\varphi(z) = \frac{z}{2-z}$, $z \in \mathbb{D}$. It is analytic on $\overline{\mathbb{D}}$ and 0 is an interior fixed point, while $z_0 = 1$ is a boundary fixed point with $\varphi'(1) = 2 > 1$. Then, the hypotheses of Theorem 3.1 hold and, for a given $\alpha > 0$, the composition operator $C_\varphi : A^{-\alpha}_- \to A^{-\alpha}_-$ verifies $\sigma(C_\varphi; A^{-\alpha}_-) \supseteq \{\lambda : |\lambda| \le 2^{-\alpha}\}$. In [5, Example 1] it is noticed that $\varphi_n(z) = \frac{z}{2^n - (2^n-1)z}$.

Recall that for any symbol φ the essential norm of $C_\varphi : H^\infty_\alpha \to H^\infty_\alpha$ can be computed according to $\|C_\varphi\|_e = \limsup_{|z|\to 1} \frac{(1-|z|)^\alpha}{(1-|\varphi(z)|)^\alpha}$ (see [5] for instance). Moreover, from [6, Proposition 2.46] we obtain that $\|C_\varphi\|_e = \max_{|\xi|=1} |\varphi'(\xi)|^{-\alpha}$. For the iterates φ^n, this maximum is achieved at $\xi = 1$, with value $(\varphi^n)'(1) = 2^n$. So $\|C_\varphi^n\|_e = (2^n)^{-\alpha}$, from where it follows that $r_e(C_\varphi, H^\infty_\alpha) = (\frac{1}{2})^\alpha$. And $\sigma(C_\varphi, H^\infty_\alpha) = \{0\} \cup \{\varphi'(0)^n : n \in \mathbb{N}\} \cup \overline{B}(0, r_e(C_\varphi, H^\infty_\alpha))$ as proved in [2, Theorem 8]. Realize that in this example $\sigma(C_\varphi; H^\infty_\alpha) = \sigma(C_\varphi; A^{-\alpha}_-)$.

The following theorem is the analogous to Theorem 3.1 for the Fréchet space $A^{-\alpha}_+$. Since the proof is quite similar we will simply mention the claims that have been proved in Theorem 3.1 and detail the differences. The main difference is the choice of the function f_{x_0}. For a weight v in $\mathbb{D}$, the space H^∞_v is defined in an absolutely analogous way to H^∞_α by simply replacing the weight $(1-|z|)^\alpha$ by $v(z)$.

Theorem 3.4. *Consider $A^{-\alpha}_+$, with $\alpha \ge 0$. Suppose that $\varphi : \overline{\mathbb{D}} \to \mathbb{D}$ is analytic and that it has a fixed point $a \in \mathbb{D}$. Suppose that there is a positive integer N such that φ_N has, at least, a fixed point z_0 in the unit sphere and that $\varphi'(z_0) > 1$. Then $\sigma(C_\varphi; A^{-\alpha}_+) \supseteq \{\lambda : |\lambda| \le \varphi_N'(z_0)^{-\alpha/N}\}$.*

Proof. We may assume that $a = 0$. Again since φ cannot be an automorphism then $0 \in \sigma(C_\varphi)$. Construct the sequence $(x_k)_k$ as in Theorem 3.1, which verifies the statements (3.1), (3.2) and (3.3). Denote $c := \varphi_N'(z_0)$ then $c > 1$.

Let $\lambda \notin \sigma(C_\varphi)$ with $|\lambda| \le c^{-\alpha/N}$. We now choose n_0 so large that (3.4) holds.

Fix $m \in \mathbb{N}$, $m > n_0$ such that $|\varphi'(0)|^m < |\lambda|$. Each $f \in A_+^{-\alpha}$ satisfies $|f(z)| \le \|f\|_{\alpha+\varepsilon}(1-|z|)^{-(\alpha+\varepsilon)}$ for all $z \in \mathbb{D}$, $\varepsilon > 0$ and so,

$$|\lambda^{nN} f(x_{-nN})| \le \|f\|_{\alpha+\varepsilon} \frac{1}{A_0(x_0)^{\alpha+\varepsilon}} \left(\frac{|\lambda|^N}{c^{-(\alpha+\varepsilon)}} \right)^n .$$

This implies that (3.6) holds for all $f \in A_+^{-\alpha}$.

Consider $f, g \in A_+^{-\alpha}$ with $\lambda f - f \circ \varphi = g$. Then they satisfy equation (3.8).

Let us denote by $A_{+,m}^{-\alpha}$ the projective limit

$$A_{+,m}^{-\alpha} := \bigcap_{n=1}^{+\infty} H_{\alpha+\frac{1}{n},m}^{\infty} = \bigcap_{\varepsilon>0} H_{\alpha+\varepsilon,m}^{\infty}.$$

We claim that $\big(C_\varphi - \lambda I\big)(A_{+,m}^{-\alpha}) = A_{+,m}^{-\alpha}$. Let $g \in A_{+,m}^{-\alpha}$. Since $C_\varphi - \lambda I$ is onto, there is $f \in A_+^{-\alpha}$ such that $g = (C_\varphi - \lambda I)(f)$. Thus, for all $\varepsilon > 0$, $f \in H_{\alpha+\varepsilon}^{\infty}$, and so also $g \in H_{\alpha+\varepsilon}^{\infty}$, hence $g \in H_{\alpha+\varepsilon,m}^{\infty}$. According to Lemma 2.4, f belongs as well to $H_{\alpha+\varepsilon,m}^{\infty}$, as claimed and (3.7) holds.

Therefore, we deduce that if $g \in (C_\varphi - \lambda I)(A_{+,m}^{-\alpha})$ and $|\varphi'(0)|^m < |\lambda|$ then,

$$\sum_{k=-\infty}^{+\infty} g(x_k)\lambda^{-k} = 0.$$

The functionals L_K are defined on $A_{+,m}^{-\alpha}$ according to the same expression as in (3.9). That they are continuous and that the sequence (L_K) is null for the weak*-topology $\sigma\big((A_{+,m}^{-\alpha})', A_{+,m}^{-\alpha}\big)$ follows as in Theorem 3.1.

Consider the function

$$\rho(r) := \begin{cases} 1, & r \in [0, 1 - \frac{1}{e}[\\ -\log(1-r), & r \in [1 - \frac{1}{e}, 1[\end{cases} .$$

It is non-decreasing, continuous and $\lim_{r\to 1^-} \rho(r) = +\infty$. Define $v(z) := \frac{1}{\rho(|z|)}$, $z \in \mathbb{D}$. For each $0 < \varepsilon < 1$, it can be seen that there exists C_ε such that

$$(1-|z|)^\varepsilon \le C_\varepsilon v(z), \quad \text{for all } z \in \mathbb{D}.$$

In other words, the space H_v^∞ is contained in $A_+^{-\alpha}$. We can take $f_{x_0} \in H_v^\infty$ such that $\|f_{x_0}\|_v \le 1$ and $|f_{x_0}(x_0)| = 1/v(x_0)$. Let $1 \le M < \infty$ be the constant in Lemma 2.2 for $r = 1/4$. Then there is $f_K \in H^\infty$ with $\|f_K\|_\infty \le M$ and $|f_K(x_0)| = 1$, satisfying

$$x_0^m f_K(x_0) f_{x_0}(x_0) > 0 \ \text{ and } \ f_K(x_k) = 0 \text{ for } -K \le k \le m_0,\ k \ne 0.$$

Now, the function $g_K(x) := x^m f_K(x) f_{x_0}(x)$ belongs to $H_{v,m}^\infty$ and $\|g_K\|_v \le M$. Further,

$$L_K(g_K) = x_0^m f_K(x_0) f_{x_0}(x_0) + \sum_{k=m_0+1}^{+\infty} \lambda^{-k} x_k^m f_K(x_k) f_{x_0}(x_k).$$

If, in addition, we choose m so that

$$M \frac{1}{|\lambda|^{m_0}} \frac{1}{v(x_{m_0})} \frac{b^m}{|\lambda| - b^m} < \frac{1}{2v(x_0)} \quad ,$$

and use again (3.3) and (3.4), we obtain

$$|L_K(g_K)| \geq \frac{|x_0|^m}{2v(x_0)}. \tag{3.11}$$

However, since $(g_K)_K$ is a relatively compact subset of $A^{-\alpha}_{+,m}$ and the sequence $(L_K)_K$ is weak*-null and equicontinuous, we would have that $\lim_K L_K(g_K) = 0$, which contradicts (3.11).

In conclusion, since m can be large enough, we deduce that $\sigma(C_\varphi; A^{-\alpha}_+) \supseteq \{\lambda : |\lambda| \leq \varphi'_N(z_j)^{-\alpha/N}\}$.
□

4 Declarations

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Conflicts of interest/Competing interests

Not applicable

Availability of data and material

Not applicable

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Authors' address

Pablo Galindo. Departamento de Análisis Matemático. Universidad de Valencia. 46100 Burjasot, Valencia, Spain. *email:*galindo@uv.es

Esther Gómez-Orts. Instituto Universitario de Matemática Pura y Aplicada IUMPA. Universidad Politécnica de Valencia. 46022 Valencia, Spain. *email:*esgoor@upvnet.upv.es

Dear editor,

Please find attached the revision of the submision RIMA-D-20-01226, with title “A note on the spectrum of some composition operators on Korenblum type spaces”.

We have considered the referee’s criticisms and suggestions and modified the file accordingly. Forward them our appreciation.

Yours sincerely,
Pablo Galindo and Esther Gómez-Orts