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A novel descriptor redundancy method based on delay partition for exponential stability of time delay systems

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ABSTRACT
This paper investigates the exponential stability of uncertain time delay systems using a novel descriptor redundancy approach based on delay partitioning. First, the original system is casted into an equivalent descriptor singular state-space representation by introducing redundant state variables so that the resulting delay is progressively reduced. From the equivalent model and applying Lyapunov Functional method, two LMI-based conditions for exponential stability with guaranteed decay rate performance are obtained. As a result, the inherent conservatism is proved to be arbitrarily reduced by increasing the number of delay partition intervals in both cases. Decay rate performance and model uncertainties in polytopic form are further considered. Various benchmark examples are provided to validate the effectiveness of the proposed method, showing better trade-off between conservatism and performance in comparison to previous approaches.

KEYWORDS
Exponential stability; Time delay systems; Delay partitioning; Descriptor redundancy; Lyapunov-Krasovskii functional; Linear matrix inequalities

1. Introduction

Over the last decades, the study of linear systems with time delays has received increasing attention (Fridman, 2014; Gu, Chen, & Kharitonov, 2003; Lee & Park, 2018; Niculescu, 2001; Richard, 2003) with different control engineering application, such as network control systems (Liu, Selivanov, & Fridman, 2019; W. Zhang, Branicky, & Phillips, 2001), formation control (González, Aragüés, López-Nicolás, & Sagüés, 2018; González, Aranda, López-Nicolás, & Sagüés, 2019) and consensus of multiagent systems (Y. Zhang & Tian, 2014; Zhou, Sang, Li, Fang, & Wang, 2018) among others. The main difficulty in the stability analysis of delayed systems is their infinite dimensional nature (Gu et al., 2003). A common approach to overcome this problem is the Lyapunov-Krasovskii functional (LKF) method, which is a generalization of the direct method of Lyapunov for ordinary differential equations. Hence, the stability analysis of time delay systems is casted into a finite set of Linear Matrix Inequalities (LMI). Nevertheless, the price to pay is the presence of some conservatism, which strongly depends on the choice of functional and the bounding techniques used

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to *convexify* the stability conditions.

Many efforts have been addressed to find a suitable functional, together with the use of accurate bounding methods, in order to obtain less conservative conditions. Different delay dependent LMI-based approaches have been proposed by designing more complex structures for LKF: augmented Lyapunov functional with some triple integral terms method (Lakshmanan, Senthilkumar, & Balasubramaniam, 2011), multiple integral functionals (Gyurkovics & Takacs, 2016), and free-weighting techniques (Xu & Lam, 2005), among others. Discrete delay decomposition approaches (known also as delay partitioning techniques) (Das, Ghosh, & Subudhi, 2018; Gu, 2001a) have also been proven to be very effective for conservatism reduction in stability analysis of time delay systems by using augmented LKF but at the expense of introducing more delayed states. Delay partitioning method has been widely implemented in different applications, such as neutral systems (Han, 2009), singular systems (Ech-charqy, Ouahi, & Tissir, 2018), nonlinear systems in Takagi-Sugeno fuzzy form (Zhao, Gao, Lam, & Du, 2008), discrete-time systems Meng, Lam, Du, and Gao (2010), multiagent systems Cui, Sun, Zhao, and Zheng (2020), and neural networks (Ali, Gunasekaran, & Aruna, 2017; Jiang, Xia, Feng, Zheng, & Jiang, 2020).

Another well-known source of conservativeness using delay-dependent LKF comes from the bounding method used to deal with some integral terms. In the aim to reduce the inequality gap, the so-called Wirtinger-based inequality (Seuret & Gouaisbaut, 2013) extended Jensen's inequality (Gu, 2001b) by introducing additional quadratic terms. More recently, generalized integral inequalities were developed in Seuret and Gouaisbaut (2017) through Bessel inequality and Legendre polynomials, including Jensen's inequality, Wirtinger-based integral inequality, and other auxiliary function-based integral based inequalities (Park, Lee, & Lee, 2015) as particular cases. Nevertheless, most of these studies are focused on mere stability analysis, and only few works investigates more efficient methods to consider exponential stability with guaranteed decay rate performance. For instance, exponential stability analysis is addressed for time delay systems in (Hien & Trinh, 2016; Seuret & Gouaisbaut, 2013) and other related works applied to switched systems with persistent dwell time (Fan, Wang, Liu, Zhang, & Ma, 2019) and H_∞ control (Fan, Wang, Sun, Yi, & Liu, 2020), among others. Nevertheless, to the best author's knowledge, the research of more efficient methods for exponential stability of uncertain time delay systems, where the conservatism introduced by LKF vanishes when delay partitioning goes thinner, still remains an open issue.

In this paper, we propose a novel LMI-based conditions for exponential stability of uncertain time delay systems obtained by combining descriptor redundancy with delay partitioning, LKF approaches and a state transformation for decay rate analysis. The main contribution is summarized as follows:

- (i) Differently from previous related methods, the inherent conservatism of the exponential stability conditions is theoretically proved and illustrated through simulation results to be asymptotically reduced by refining more and more the delay partition even in the presence of polytopic uncertainties.
- (ii) An improved trade-off between conservatism and complexity is obtained compared to other existing approaches. Concretely, to alleviate the sharp increment of decision variables when the number of partitioning intervals increases, elimination lemma is further applied to reduce the number of decision variables (NoV) to the greatest extent without introducing extra conservatism.

The remainder of the paper is organized as follows: Section 2 describes the problem statement. Section 3 describes the proposed descriptor redundancy method based on delay partitioning. Section 4 presents two LMI-based conditions for exponential stability analysis obtained from the singular descriptor model obtained in Section 3 via LKF approaches. Simulation examples are provided in Section 5, and finally some conclusions are gathered in Section 6.

2. Problem statement and preliminaries

Consider the following delayed system:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + A_d(t)x(t-h) \\ x(\tau) &= \phi(\tau), \quad -h \leq \tau \leq 0. \end{aligned} \tag{1}$$

where $x(t) \in \mathcal{R}^n$ is the system state, $h \geq 0$ is the delay value, $\phi(\tau) : \mathcal{R} \to \mathcal{R}^n$ is a function that represents the initial conditions for the system state, and $A(t), A_d(t) \in \mathcal{R}^n$ are time-varying matrices under polytopic uncertainty description (Fan et al., 2020; He, Wu, She, & Liu, 2004):

$$(A(t),\ A_d(t)) = \sum_{i=1}^{r} \lambda_i(t) \left(\hat{A}_i,\ \hat{A}_{d,i} \right), \tag{2}$$

where $\lambda_i(t),\ 1 \leq i \leq r$ are unknown time-varying functions satisfying $0 \leq \lambda_i(t) \leq 1$, $\sum_{i=1}^{r} \lambda_i(t) = 1$, $\forall t \geq 0$, being $\hat{A}_i, \hat{A}_{d,i}$ known time-constant matrices, and r the number of vertices of the polytope.

The objective is to obtain LMI based conditions to ascertain the robust exponential stability of system (1) with guaranteed decay rate, where conservatism against delays can arbitrarily be reduced by systematically introducing more variables. To this end, a singular descriptor state-space model with delay inversely proportional to the number of redundant variables is first obtained by applying decriptor redundancy and delay partitioning methods.

The preliminary result given below will be useful to eliminate decision variables without introducing extra conservatism:

Lemma 2.1. *(Boyd, El Ghaoui, Feron, & Balakrishnan, 1994) (Elimination Lemma) Let Ω be a symmetric matrix, and U, V, $\mathcal{X}$ matrices of suitable dimensions. Then the inequality $\Omega + He(U^T \mathcal{X} V) < 0$ [1] is true if and only if the following LMI constraints hold:*

- *(i)* $\left(U^{\perp}\right)^T \Omega U^{\perp} < 0$,
- *(ii)* $\left(V^{\perp}\right)^T \Omega V^{\perp} < 0$

where $U^{\perp}$ and $V^{\perp}$ are null basis of U and V respectively.

Definition 2.2. Given $\alpha > 0$, system (1) is said to be exponentially stable if there exists a positive number $\mathcal{K} > 0$ such that, for any initial condition $x_0 = x(0)$, the

[1] Given a matrix W, we denote $He(W) = W + W^T$.

system state $x(t)$ satisfies the inequality:

$$||x(t)|| = \mathcal{K}||x_0||e^{-\alpha t}, \qquad \forall t \geq 0 \tag{3}$$

3. Descriptor redundancy method

This section describes the proposed descriptor redundancy method based on delay partitioning in order to fulfil the objective given in Section 2. First, let us introduce the following redundant state variables:

$$\begin{cases} z_1(t) = x(t-h+\delta) \\ z_2(t) = x(t-h+2\delta) \\ \cdots \\ z_{N-1}(t) = x\left(t-h+(N-1)\delta\right) = x(t-\delta) \end{cases} \tag{4}$$

where $\delta = h/N$ and $N > 1$. From the above defined $z_i(t),\ i = 1, ..., N-1$, system (1) can be reformulated as:

$$\begin{aligned} &\dot{x}(t) = A(t)x(t) + A_d(t)z_1(t-\delta) \\ &\begin{cases} z_1(t) = z_2(t-\delta), \\ \cdots, \\ z_{N-1}(t) = x(t-\delta) \end{cases} \end{aligned} \tag{5}$$

Now, let us define the following augmented states:

$$\begin{aligned} \bar{x}(t) &= \begin{bmatrix} x^T(t) & z_1^T(t) & \cdots & z_{N-1}^T(t) \end{bmatrix}^T, \\ \bar{\eta}(t) &= \left[x^T(t),\ \int_{t-\delta}^{t} x^T(s)ds,\ z_1^T(t), \cdots, z_{N-1}^T(t) \right]^T, \\ \bar{\xi}(t) &= \left[x^T(t),\ \frac{1}{\delta}\int_{t-\delta}^{t} x^T(s)ds,\ z_1^T(t), \cdots, z_{N-1}^T(t) \right]^T \end{aligned} \tag{6}$$

and the following equivalences:

$$\bar{\eta}(t) = \Pi_1\bar{\xi}(t), \qquad \bar{x}(t) = \Pi_2\bar{\xi}(t) \tag{7}$$

where $\bar{x}(t)$ is defined in (32) and

$$\Pi_1 = \begin{bmatrix} I_n & 0 & 0 \\ 0 & \delta I_n & 0 \\ 0 & 0 & I_{(N-1)n} \end{bmatrix}, \qquad \Pi_2 = \begin{bmatrix} I_n & 0_n & 0 \\ 0 & 0 & I_{(N-1)n} \end{bmatrix}, \tag{8}$$

To include exponential decay rate in the stability analysis, let us introduce the new state variables: $\bar{\eta}_\alpha(t) = e^{\alpha t}\bar{\eta}(t)$, $\bar{\xi}_\alpha(t) = e^{\alpha t}\bar{\xi}(t)$ and $\bar{x}_\alpha(t) = e^{\alpha t}\bar{x}(t)$. Time-derivative of $\bar{\eta}_\alpha(t)$ yields:

$$\begin{aligned}\dot{\bar{\eta}}_\alpha(t) &= \alpha\bar{\eta}_\alpha(t) + e^{\alpha t}\dot{\bar{\eta}}(t) \\ &= \alpha\Pi_1\bar{\xi}_\alpha(t) + e^{\alpha t}\dot{\bar{\eta}}(t)\end{aligned} \tag{9}$$

where Π_1 is defined in (8). Pre-multiplying both sides of (9) by

$$\bar{\mathcal{E}} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0_{(N-1)n} \end{bmatrix}$$

and taking into account (31), we obtain:

$$\bar{\mathcal{E}}\dot{\bar{\eta}}_\alpha(t) = \bar{\mathcal{A}}(t)\bar{\xi}_\alpha(t) + \bar{\mathcal{A}}_d(t)\bar{x}_\alpha(t-\delta). \tag{10}$$

where

$$\bar{\mathcal{A}}(t) = \begin{bmatrix} A(t)+\alpha I_n & 0 & 0 \\ I_n & \alpha\delta I_n & 0 \\ 0 & 0 & (\alpha-1)I_{(N-1)n} \end{bmatrix}, \tag{11}$$
$$\bar{\mathcal{A}}_d(t) = \begin{bmatrix} 0_n & e^{\alpha\delta}\mathcal{A}_d(t) \\ -e^{\alpha\delta}I_n & 0 \\ e^{\alpha\delta}U_1 & e^{\alpha\delta}U_2 \end{bmatrix}$$

and

$$\mathcal{A}_d(t) = u_0 \otimes A_d(t), \tag{12}$$
$$U_1 = u_1^T \otimes I_n, \qquad U_2 = \begin{cases} 0_n & \text{if } N=2 \\ \begin{bmatrix} 0 & I_{(N-2)n} \\ 0_n & 0 \end{bmatrix} & \text{if } N>2, \end{cases}$$
$$u_0 = \begin{cases} 1 & \text{if } N=2 \\ \begin{bmatrix} 1 & 0_{1\times(N-2)} \end{bmatrix} & \text{if } N>2, \end{cases}$$
$$u_1 = \begin{cases} 1 & \text{if } N=2 \\ \begin{bmatrix} 0_{1\times(N-2)} & 1 \end{bmatrix} & \text{if } N>2, \end{cases}$$

where the symbol $\otimes$ stands for the Kronecker product.

Hence, the exponential stability analysis of system (1) can be casted into an admissibility analyisis of the singular descriptor system (10). The advantage of using the new model (10) is that the delay value $\delta = h/N$ goes to 0 as long as N increases by introducing more redundant state variables. This fact allows an asymptotic reduction of the conservatism of LKF when $N \to \infty$, as later discussed in Remark 1.

4. Exponential stability analysis

This section addresses the robust exponential stability with decay rate performance of (1) by means of the proposed descriptor redundancy based on delay partition described in Section 3 and the LKF method using Wirtinger's inequality (Theorem 4.1) and Jensen's inequality (Theorem 4.2). At the end of this section, Corollary 4.3 shows that the proposed methods renders non-conservative as long as N increases.

Theorem 4.1. *System (1) is robustly exponentially stable with decay rate α for a given delay partition (4) with $N > 1$ and $\delta = h/N$ if there exist symmetric matrices $P_1, P_3, Q_i, Z, S_{f,i} \in \mathcal{R}^n > 0$ with $f = 1, ..., N-1$, $i = 1, ..., r$, and matrices $P_2, R_{f,i} \in \mathcal{R}^n$ $f = 1, ..., N-1$ such that the following LMIs hold $\forall \{i, j\} \in \{1, ..., r\} \times \{1, ..., r\}$:*

$$\begin{bmatrix} \hat{\Omega}_{1,i} + \hat{\Omega}_{2,ij} & \delta^2 \hat{\Omega}_{3,i}^T Z \\ (*) & -\delta^2 Z \end{bmatrix} < 0, \quad \bar{P} > 0, \quad \bar{Q}_i > 0 \tag{13}$$

where

$$\bar{P} = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}, \quad \bar{Q}_i = \begin{bmatrix} Q_i & \bar{R}_i \\ \bar{R}_i^T & \bar{S}_i \end{bmatrix}, \tag{14}$$
$$\hat{\Omega}_{1,i} = \begin{bmatrix} \hat{\Omega}_i & \delta(\hat{A}_i^T P_2 + 2\alpha P_2 + P_3) & -e^{\alpha\delta} P_2 & e^{\alpha\delta} P_1 \hat{\mathcal{A}}_{d,i} \\ (*) & 2\delta^2 \alpha P_3 & -\delta e^{\alpha\delta} P_3 & \delta e^{\alpha\delta} P_2^T \hat{\mathcal{A}}_{d,i} \\ (*) & (*) & 0 & 0 \\ (*) & (*) & (*) & 0_{(N-1)n} \end{bmatrix},$$
$$\hat{\Omega}_{2,ij} = \begin{bmatrix} Q_i - 4Z & 6Z & -2Z + e^{\alpha\delta} \bar{R}_i U_1 & e^{\alpha\delta} \bar{R}_i U_2 \\ (*) & -12Z & 6Z & 0 \\ (*) & (*) & e^{2\alpha\delta} U_1^T \bar{S}_i U_1 - Q_j - 4Z & e^{2\alpha\delta} U_1^T \bar{S}_i U_2 - \bar{R}_j \\ (*) & (*) & (*) & e^{2\alpha\delta} U_2^T \bar{S}_i U_2 - \bar{S}_j \end{bmatrix},$$
$$\hat{\Omega}_{3,i} = \begin{bmatrix} \hat{A}_i + \alpha I_n & 0_n & 0_n & e^{\alpha\delta} \hat{\mathcal{A}}_{d,i} \end{bmatrix}$$

and

$$\begin{aligned} \hat{\Omega}_i &= \hat{A}_i^T P_1 + P_1 \hat{A}_i + P_2 + P_2^T + 2\alpha I_n, \\ \bar{R}_i &= \begin{bmatrix} R_{1,i} & R_{2,i} & \cdots & R_{N-1,i} \end{bmatrix}, \\ \bar{S}_i &= diag\left(S_{1,i},\ S_{2,i}, \cdots\ S_{N-1,i}\right), \end{aligned} \tag{15}$$

where $\hat{\mathcal{A}}_{d,i} = u_0 \otimes \hat{A}_{d,i}$, being $u_0, U_1,\ U_2$ defined in (12).

Proof. To prove the admissibility of the equivalent singular descriptor system (10), the following Lyapunov-Krasovskii functional is proposed:

$$V(t) = V_1(t) + V_2(t) + V_3(t) > 0 \tag{16}$$

where

$$\begin{aligned} V_1(t) &= \bar{\eta}_\alpha^T(t) \bar{\mathcal{E}} \bar{\mathcal{P}} \bar{\eta}_\alpha(t), \\ V_2(t) &= \int_{t-\delta}^{t} \bar{x}_\alpha^T(s) \bar{Q}(s) \bar{x}_\alpha(s) ds, \\ V_3(t) &= \delta \int_{-\delta}^{0} \int_{t+s}^{t} \dot{\bar{\eta}}_\alpha^T(\theta) \bar{\mathcal{E}} \bar{\mathcal{Z}} \bar{\mathcal{E}} \dot{\bar{\eta}}_\alpha(\theta) d\theta ds \end{aligned} \tag{17}$$

being $\bar{Q}(s) = \sum_{i=1}^{r} \lambda_i(s)\bar{Q}_i$ with $\bar{Q}_i$ defined in (14), and

$$\bar{\mathcal{P}} = \begin{bmatrix} P_1 & P_2 & 0 \\ P_2^T & P_3 & 0 \\ X_1 & X_2 & X_3 \end{bmatrix}, \ \bar{\mathcal{Z}} = \begin{bmatrix} Z & 0 & Y_1 \\ 0 & 0_n & Y_2 \\ Y_1^T & Y_2^T & Y_3 + Y_3^T \end{bmatrix}, \tag{18}$$

where $X_1, X_2 \in \mathcal{R}^{(N-1)n\times n}$, $X_3 \in \mathcal{R}^{(N-1)n\times(N-1)n}$, $Y_1, Y_2 \in \mathcal{R}^{n\times(N-1)n}$ and $Y_3 \in \mathcal{R}^{(N-1)n\times(N-1)n}$. Note that $\bar{\mathcal{P}}$ is defined to deal with singular descriptor matrix through the equivalence $\bar{\mathcal{E}}\bar{\mathcal{P}} = \bar{\mathcal{P}}^T\bar{\mathcal{E}}$. Hence, time-derivative of the above terms $V_i(t), i = 1, 2, 3$ yields:

$$\begin{aligned} \dot{V}_1(t) &= \dot{\bar{\eta}}_\alpha^T(t)\bar{\mathcal{E}}\bar{\mathcal{P}}\bar{\eta}_\alpha(t) + \bar{\eta}_\alpha^T(t)\bar{\mathcal{E}}\bar{\mathcal{P}}\dot{\bar{\eta}}_\alpha(t), \\ &= \dot{\bar{\eta}}_\alpha^T(t)\bar{\mathcal{E}}\bar{\mathcal{P}}\bar{\eta}_\alpha(t) + \bar{\eta}_\alpha^T(t)\bar{\mathcal{P}}^T\bar{\mathcal{E}}\dot{\bar{\eta}}_\alpha(t), \\ &= \bar{\xi}_\alpha^T(t)\left(\bar{\mathcal{A}}^T(t)\bar{\mathcal{P}}\Pi_1 + \Pi_1^T\bar{\mathcal{P}}^T\bar{\mathcal{A}}(t)\right)\bar{\xi}_\alpha(t) + He\left(\bar{x}_\alpha^T(t-\delta)\bar{\mathcal{A}}_d^T(t)\bar{\mathcal{P}}\Pi_1\bar{\xi}_\alpha(t)\right) \end{aligned} \tag{19}$$

$$\begin{aligned} \dot{V}_2(t) &= \bar{x}_\alpha^T(t)\bar{Q}(t)\bar{x}_\alpha(t) - \bar{x}_\alpha^T(t-\delta)\bar{Q}(t-\delta)\bar{x}_\alpha(t-\delta) \\ &= \bar{\xi}_\alpha^T(t)\Pi_2^T\bar{Q}(t)\Pi_2\bar{\xi}_\alpha(t) - \bar{x}_\alpha^T(t-\delta)\bar{Q}(t-\delta)\bar{x}_\alpha(t-\delta), \end{aligned}$$

$$\begin{aligned} \dot{V}_3(t) &= \delta^2\left(\dot{\bar{\eta}}_\alpha^T(t)\bar{\mathcal{E}}\bar{\mathcal{Z}}\bar{\mathcal{E}}\dot{\bar{\eta}}_\alpha(t)\right) - \delta\int_{t-\delta}^{t}\dot{\bar{\eta}}_\alpha^T(\theta)\bar{\mathcal{E}}\bar{\mathcal{Z}}\bar{\mathcal{E}}\dot{\bar{\eta}}_\alpha(\theta)d\theta \\ &= \delta^2\bar{\xi}_\alpha^T(t)\left(\bar{\mathcal{A}}^T(t)\bar{\mathcal{Z}}\bar{\mathcal{A}}(t)\right)\bar{\xi}_\alpha(t) + \delta^2 He\left(\bar{x}_\alpha^T(t-\delta)\bar{\mathcal{A}}_d^T(t)\bar{\mathcal{Z}}\bar{\mathcal{A}}(t)\bar{\xi}_\alpha(t)\right) \\ &+ \delta^2\bar{x}_\alpha^T(t-\delta)\left(\bar{\mathcal{A}}_d^T(t)\bar{\mathcal{Z}}\bar{\mathcal{A}}_d(t)\right)\bar{x}_\alpha(t-\delta) - \delta\int_{t-\delta}^{t}\dot{x}_\alpha^T(\theta)Z\dot{x}_\alpha(\theta)d\theta, \end{aligned}$$

where Π_1, Π_2 are defined in (8).

Let $x_\alpha(t) = e^{\alpha t}x(t)$, $\xi_\alpha^*(t) = \begin{bmatrix} x_\alpha^T(t) & x_\alpha^T(t-\delta) & \int_{t-\delta}^{t} x_\alpha^T(s)ds \end{bmatrix}^T$ and

$$\Xi_3^* = \begin{bmatrix} -4 & -2 & 6 \\ -2 & -4 & 6 \\ 6 & 6 & -12 \end{bmatrix} \otimes Z. \tag{20}$$

Applying Wirtinger's inequality, we have that

$$-\delta\int_{t-\delta}^{t}\dot{x}_\alpha^T(\theta)Z\dot{x}_\alpha(\theta)d\theta \le -\xi_\alpha^{*T}(t)\Xi_3^*\xi_\alpha^*(t). \tag{21}$$

or equivalently

$$\begin{aligned} &-\delta\int_{t-\delta}^{t}\dot{x}_\alpha^T(\theta)Z\dot{x}_\alpha(\theta)d\theta \\ &\le -\left(\Pi_3\bar{\xi}_\alpha(t) + \Pi_4\bar{x}_\alpha(t-\delta)\right)^T\Xi_3^*\left(\Pi_3\bar{\xi}_\alpha(t) + \Pi_4\bar{x}_\alpha(t-\delta)\right). \end{aligned} \tag{22}$$

where

$$\Pi_3 = \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I_n & 0_{n\times(N-1)n} \end{bmatrix}, \; \Pi_4 = \begin{bmatrix} 0 & 0 \\ I_n & 0 \\ 0 & 0_{n\times(N-1)n} \end{bmatrix}, \tag{23}$$

From (19) and (22), together with the exponential stability condition given by $\dot{V}(t)$, the following inequality can be obtained:

$$\begin{aligned} &\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) \\ &\leq \chi_\alpha^T(t)\left(\Xi_0(t) + He\left(\mathcal{N}_1^T \mathcal{X} \mathcal{N}_2(t)\right)\right)\chi_\alpha(t) < 0 \end{aligned} \tag{24}$$

where $\Xi_0(t) = \Xi_1(t) + \Xi_2(t) + \Xi_3 + \Xi_4(t)$, $\chi_\alpha^T(t) = \begin{bmatrix} \bar{\xi}_\alpha^T(t) & \bar{x}_\alpha^T(t-\delta) \end{bmatrix}$, and

$$\begin{aligned} &\Xi_1(t) = He\left(\begin{bmatrix} \Pi_1^T \\ 0 \end{bmatrix} \begin{bmatrix} \bar{P} & 0 \\ 0 & 0_{(N-1)n} \end{bmatrix} \begin{bmatrix} \bar{\mathcal{A}}(t) & \bar{\mathcal{A}}_d(t) \end{bmatrix}\right), \\ &\Xi_2(t) = \begin{bmatrix} \Pi_2^T \bar{Q}(t)\Pi_2 & 0 \\ 0 & -\bar{Q}(t-\delta) \end{bmatrix}, \\ &\Xi_3 = \begin{bmatrix} \Pi_3^T \\ \Pi_4^T \end{bmatrix} \Xi_3^* \begin{bmatrix} \Pi_3 & \Pi_4 \end{bmatrix} = \begin{bmatrix} -4 & 6 & 0 & -2 & 0 \\ 6 & -12 & 0 & 6 & 0 \\ 0 & 0 & 0_{(N-1)} & 0 & 0 \\ -2 & 6 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0_{(N-1)} \end{bmatrix} \otimes Z, \\ &\Xi_4(t) = \delta^2 \begin{bmatrix} \bar{\mathcal{A}}^T(t) \\ \bar{\mathcal{A}}_d^T(t) \end{bmatrix} \begin{bmatrix} \bar{Z} & 0 \\ 0 & 0_{(N-1)n} \end{bmatrix} \begin{bmatrix} \bar{\mathcal{A}}(t) & \bar{\mathcal{A}}_d(t) \end{bmatrix}, \\ &\mathcal{X} = \begin{bmatrix} X_1 & X_2 & X_3 & \delta^2 Y_1^T & \delta^2 Y_2^T & \delta^2 Y_3^T \end{bmatrix}, \\ &\mathcal{N}_1 = \begin{bmatrix} 0_{(N-1)n\times n} & 0_{(N-1)n\times n} & -I_{(N-1)n} & e^{\alpha\delta}U_1 & e^{\alpha\delta}U_2 \end{bmatrix}, \\ &\mathcal{N}_2(t) = \begin{bmatrix} \Pi_1 & 0_{(N+1)n\times Nn} \\ \bar{\mathcal{A}}(t) & \bar{\mathcal{A}}_d(t) \end{bmatrix}, \; \bar{Z} = \begin{bmatrix} Z & 0 \\ 0 & 0_n \end{bmatrix} \end{aligned} \tag{25}$$

where Ξ_3^* and $\bar{P}$ are defined in (20) and (14) respectively. Note from (24) that $\dot{V}(t) < 0$ is true $\forall \chi_\alpha(t) \neq 0$ if and only if $\Xi_0(t) + He\left(\mathcal{N}_1^T \mathcal{X} \mathcal{N}_2(t)\right) < 0$. Taking into account that the null-subspace of $\mathcal{N}_2(t)$ is the empty set and applying Lemma 2.1, the inequality $\Xi_0(t) + He\left(\mathcal{N}_1^T \mathcal{X} \mathcal{N}_2(t)\right) < 0$ is equivalent to:

$$\left(\mathcal{N}_1^\perp\right)^T \Xi_0(t)\, \mathcal{N}_1^\perp < 0 \tag{26}$$

where $\mathcal{N}_1^\perp$ is a null-subspace of $\mathcal{N}_1$, which renders:

$$\mathcal{N}_1^\perp = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & e^{\alpha\delta}U_1 & e^{\alpha\delta}U_2 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_{(N-1)n} \end{bmatrix} \tag{27}$$

Taking into account (2), the inequality (26) is equivalent to:

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\lambda_i(t)\lambda_j(t-\delta)\begin{bmatrix}\hat{\Omega}_{1,i}+\hat{\Omega}_{2,ij} & \delta^2\hat{\Omega}_{3,i}^T Z\\ (*) & -\delta^2 Z\end{bmatrix}<0 \tag{28}$$

with $\hat{\Omega}_{1,i}$, $\hat{\Omega}_{2,ij}$ and $\hat{\Omega}_{3,i}$ defined in (14). Finally, applying convex sum properties, the inequality (28) is true $\forall t \geq 0$ if LMIs (13) hold, concluding the proof. □

Also, a LMI condition based on Jensen's inequality is obtained through the following theorem:

Theorem 4.2. *System (1) is robustly exponentially stable with decay rate α for a given delay partition (4) with $N > 1$ and $\delta = h/N$ if there exist symmetric matrices $P, Q_i, Z, S_{f,i} \in \mathcal{R}^n > 0$ with $f = 1, ..., N-1$, $i = 1, ..., r$, and matrices $R_{f,i} \in \mathcal{R}^n$ $f = 1, ..., N-1$ such that the following LMIs hold $\forall\{i, j\} \in \{1, ..., r\} \times \{1, ..., r\}$:*

$$\begin{bmatrix}\hat{\Omega}_{4,ij} & \delta^2\hat{\Omega}_{5,i}^T Z\\ (*) & -\delta^2 Z\end{bmatrix}<0,\quad \begin{bmatrix}Q_i & \bar{R}_i\\ \bar{R}_i^T & \bar{S}_i\end{bmatrix}>0 \tag{29}$$

where

$$\hat{\Omega}_{4,ij}=\begin{bmatrix}\hat{\Omega}_i+Q_i-Z & e^{\alpha\delta}\bar{R}_iU_1+Z & e^{\alpha\delta}P\hat{\mathcal{A}}_{d,i}+e^{\alpha\delta}\bar{R}_iU_2\\ (*) & e^{2\alpha\delta}U_1^T\bar{S}_iU_1-Q_j-Z & e^{2\alpha\delta}U_1^T\bar{S}_iU_2-\bar{R}_j\\ (*) & (*) & e^{2\alpha\delta}U_2^T\bar{S}_iU_2-\bar{S}_j\end{bmatrix}, \tag{30}$$

$$\hat{\Omega}_{5,i}=\begin{bmatrix}\hat{A}_i+\alpha I_n & 0_n & e^{\alpha\delta}\hat{\mathcal{A}}_{d,i}\end{bmatrix},$$
$$\bar{R}_i=\begin{bmatrix}R_{1,i} & R_{2,i} & \cdots & R_{N-1,i}\end{bmatrix},$$
$$\bar{S}_i=diag\left(S_{1,i},\ S_{2,i},\cdots\ S_{N-1,i}\right),$$
$$\hat{\Omega}_i=\hat{A}_i^TP+P\hat{A}_i+2\alpha P$$

where $\hat{\mathcal{A}}_{d,i} = u_0 \otimes \hat{A}_{d,i}$, being u_0, U_1, U_2 defined in (12).

Proof. Let $\bar{x}_\alpha(t) = e^{\alpha t}\bar{x}(t)$, where $\bar{x}(t)$ is defined in (6). Note that (5) can be written in descriptor state-space representation with $\bar{x}_\alpha(t)$ as:

$$\bar{E}\dot{\bar{x}}_\alpha(t)=\bar{A}(t)\bar{x}_\alpha(t)+\bar{A}_d(t)\bar{x}_\alpha(t-\delta), \tag{31}$$

where

$$\bar{E}=\begin{bmatrix}I_n & 0\\ 0 & 0_{(N-1)n}\end{bmatrix}, \tag{32}$$
$$\bar{A}(t)=\begin{bmatrix}A(t)+\alpha I_n & 0\\ 0 & (\alpha-1)I_{(N-1)n}\end{bmatrix},\quad \bar{A}_d(t)=\begin{bmatrix}0_n & e^{\alpha\delta}\mathcal{A}_d(t)\\ e^{\alpha\delta}U_1 & e^{\alpha\delta}U_2\end{bmatrix}$$

Consider the following Lyapunov-Krasovskii functional:

$$V(t)=V_1(t)+V_2(t)+V_3(t)>0 \tag{33}$$

where

$$V_1(t) = \bar{x}_\alpha^T(t)\bar{E}\bar{P}_0\bar{x}_\alpha(t), \quad (34)$$
$$V_2(t) = \int_{t-\delta}^{t} \bar{x}_\alpha^T(s)\bar{Q}(s)\bar{x}_\alpha(s)ds,$$
$$V_3(t) = \delta \int_{-\delta}^{0}\int_{t+s}^{t} \dot{x}_\alpha^T(\theta)\bar{E}\bar{Z}_0\bar{E}\dot{x}_\alpha(\theta)d\theta ds$$

with $\bar{P}_0 = \begin{bmatrix} P & 0 \\ X_1 & X_2 \end{bmatrix}$, and $\bar{Z}_0 = \begin{bmatrix} P & Y_1 \\ Y_1^T & Y_2 \end{bmatrix}$

Now, consider Jensen's inequality instead of Wirtinger's inequality to find an upper bound for the term $-\delta\int_{t-\delta}^{t} \dot{\bar{x}}_\alpha^T(s)\bar{E}\bar{Z}_0\bar{E}\dot{\bar{x}}_\alpha(s)ds$ coming from the time-derivative of $V_3(t)$:

$$\begin{aligned} & -\delta\int_{t-\delta}^{t} \dot{\bar{x}}_\alpha^T(s)\bar{E}\bar{Z}_0\bar{E}\dot{\bar{x}}_\alpha(s)ds \quad (35)\\ & \le -\left(\int_{t-\delta}^{t} \dot{\bar{x}}_\alpha^T(s)\right)\bar{E}\bar{Z}_0\bar{E}\left(\int_{t-\delta}^{t} \dot{\bar{x}}_\alpha^T(s)\right) \\ & = -\left(\bar{x}_\alpha(t) - \bar{x}_\alpha(t-\delta)\right)\bar{E}\bar{Z}_0\bar{E}\left(\bar{x}_\alpha(t) - \bar{x}_\alpha(t-\delta)\right) \\ & = -\left(x_\alpha(t) - x_\alpha(t-\delta)\right)Z\left(x_\alpha(t) - x_\alpha(t-\delta)\right) \end{aligned}$$

Following similar procedure as in Theorem 4.1 and applying (35), the sufficient condition for robust exponential stability is obtained:

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\lambda_i(t)\lambda_j(t-\delta)\begin{bmatrix} \hat{\Omega}_{4,ij} & \delta^2\hat{\Omega}_{5,i}^T Z \\ (*) & -\delta^2 Z \end{bmatrix} < 0 \quad (36)$$

with $\hat{\Omega}_{4,ij}$, $\hat{\Omega}_{5,i}$ defined in (30). Finally, applying convex sum properties, the inequality (29) is true $\forall t \ge 0$ if LMIs (13) hold, concluding the proof. □

The following corollary shows that, for a sufficiently high value for delay partition N given in (4), the LMI-based conditions given in Theorems 4.1 and 4.2 become non-conservative for the estimation of the maximum allowable delay h.

Corollary 4.3. *There exists a sufficiently high value for N such that LMI conditions given in Theorem 4.2 are always feasible if functions $\lambda_i(t)$ are differentiable functions, and there exists $P > 0$ such that $(\hat{A}_i + \hat{A}_{d,i})^T P + P(\hat{A}_i + \hat{A}_{d,i}) + 2\alpha P < 0,\ i = 1, ..., r$, that is to say, the equivalent delay-free system (1) is proven to be exponentially stable with decay rate α.*

Proof. Considering that $\lambda_i(t)$ are differentiable functions, by the Mean Value Theorem, we can write $\lambda_i(t-\delta) = \lambda_i(t) - \delta\dot{\lambda}_i(t^*)$ where $t \le t^* \le t-\delta$. Hence, recalling that $\bar{Q}(t) = \sum_{i=1}^{r}\lambda_i(t)\bar{Q}_i$ and denoting $\bar{\lambda}_d = max_i(|\dot{\lambda}_i(t)|) < \infty$, time-derivative of $V_2(t)$

can be reformulated as:

$$\dot{V}_2(t) = \bar{x}_\alpha^T(t)\bar{Q}(t)\bar{x}_\alpha(t) - \bar{x}_\alpha^T(t-\delta)\bar{Q}(t-\delta)\bar{x}_\alpha(t-\delta) \tag{37}$$
$$= \bar{\xi}_\alpha^T(t)\Pi_2^T\bar{Q}(t)\Pi_2\bar{\xi}_\alpha(t) - \bar{x}_\alpha^T(t-\delta)\bar{Q}(t)\bar{x}_\alpha(t-\delta)$$
$$- \bar{x}_\alpha^T(t-\delta)\left(\delta\sum_{i=1}^{r}\dot{\lambda}_i(t^*)Q_i\right)\bar{x}_\alpha(t-\delta)$$
$$\leq \bar{\xi}_\alpha^T(t)\Pi_2^T\bar{Q}(t)\Pi_2\bar{\xi}_\alpha(t) - \bar{x}_\alpha^T(t-\delta)\bar{Q}(t)\bar{x}_\alpha(t-\delta)$$
$$+ \bar{x}_\alpha^T(t-\delta)\left(\delta\bar{\lambda}_d\sum_{i=1}^{r}\bar{Q}_i\right)\bar{x}_\alpha(t-\delta)$$

Hence, the double convex sum (36) can be expressed as

$$\sum_{i=1}^{r}\lambda_i(t)\begin{bmatrix}\hat{\Omega}_{4,i}^* & \delta^2\hat{\Omega}_{5,i}^T Z \\ (*) & -\delta^2 Z\end{bmatrix} < 0 \tag{38}$$

where

$$\hat{\Omega}_{4,i}^* = \begin{bmatrix}\hat{\Omega}_i + Q_i - Z & \bar{R}_i U_1 + Z & P\hat{\mathcal{A}}_{d,i} + \bar{R}_i U_2 \\ (*) & U_1^T\bar{S}_i U_1 - (1-\delta\bar{\lambda}_d)Q_i - Z & U_1^T\bar{S}_i U_2 - (1-\delta\bar{\lambda}_d)\bar{R}_i \\ (*) & (*) & U_2^T\bar{S}_i U_2 - (1-\delta\bar{\lambda}_d)\bar{S}_i\end{bmatrix} \tag{39}$$

First, pre-and post-multiplying (38) by $diag\left(I,\ Z^{-1}\right)$, and further applying Schur Complement, we obtain the equivalent matrix inequality:

$$\sum_{i=1}^{r}\lambda_i(t)\left(\hat{\Omega}_{4,i}^* + \delta^2\hat{\Omega}_{5,i}^T\bar{Z}\hat{\Omega}_{5,i}\right) < 0 \tag{40}$$

Assume a sufficiently large value N such that $\delta = h/N$ becomes neglectable. Then, the above inequality renders:

$$\sum_{i=1}^{r}\lambda_i(t)\hat{\Omega}_{4,i} < 0 \tag{41}$$

with

$$\hat{\Omega}_{4,i} = \begin{bmatrix}\hat{\Omega}_i + Q_i - Z & \bar{R}_i U_1 + Z & P\hat{\mathcal{A}}_{d,i} + \bar{R}_i U_2 \\ (*) & U_1^T\bar{S}_i U_1 - Q_i - Z & U_1^T\bar{S}_i U_2 - \bar{R}_i \\ (*) & (*) & U_2^T\bar{S}_i U_2 - \bar{S}_i\end{bmatrix} \tag{42}$$

From the structure of $\bar{R}$ and $\bar{S}$ given in Theorem 4.1, and taking into account that

$\hat{\mathcal{A}}_{d,i} = u_0 \otimes \hat{A}_{d,i}$, the following equivalences are obtained:

$$\begin{aligned} &P\hat{\mathcal{A}}_{d,i} + \bar{R}_i U_2 = \begin{bmatrix} P\hat{A}_{d,i} & R_{1,i} & \cdots & R_{N-2,i} \end{bmatrix}, \\ &\bar{R}_i U_1 = R_{N-1,i}, \qquad U_1^T \bar{S}_i U_1 = S_{N-1,i}, \\ &U_1^T \bar{S}_i U_2 - \bar{R}_i = \begin{bmatrix} -R_{1,i} & -R_{2,i} & \cdots & -R_{N-1,i} \end{bmatrix}, \\ &U_2^T \bar{S}_i U_2 - \bar{S}_i \\ &= diag\left(-S_{1,i},\ -S_{2,i} + S_{1,i},\ \cdots,\ -S_{N-1,i} + S_{N-2,i}\right). \end{aligned} \tag{43}$$

Define $\tilde{S}_{1,i} = S_{1,i}$ and $\tilde{S}_{f,i} = S_{f,i} - S_{f-1,i},\ f = 2, ..., N$. Hence, noting that $S_{N-1,i} = \sum_{f=1}^{N-1} \tilde{S}_{f,i}$ we obtain

$$\begin{aligned} &U_1^T \bar{S}_i U_1 = \sum_{f=1}^{N-1} \tilde{S}_{f,i}, \\ &U_2^T \bar{S}_i U_2 - \bar{S}_i = diag\left(-\tilde{S}_{1,i},\ -\tilde{S}_{2,i},\ \cdots,\ -\tilde{S}_{N-1,i}\right). \end{aligned} \tag{44}$$

Taking into account the above equivalences, the inequality (41) is true if $\begin{bmatrix} \hat{\Phi}_{1,i} & \hat{\Phi}_{2,i} \\ \hat{\Phi}_{2,i}^T & \Phi_{3,i} \end{bmatrix} < 0,\ \forall i = 1, ..., r$, where

$$\begin{aligned} &\hat{\Phi}_{1,i} = \begin{bmatrix} \hat{\Omega}_i + Q_i - Z & R_{N-1,i} + Z \\ (*) & \sum_{f=1}^{N-1} \tilde{S}_{f,i} - Q_i - Z \end{bmatrix}, \\ &\hat{\Phi}_{2,i} = \begin{bmatrix} P\hat{\mathcal{A}}_{d,i} + \bar{R}_i U_2 \\ U_1^T \bar{S}_i U_2 - \bar{R}_i \end{bmatrix} = \begin{bmatrix} P\hat{A}_{d,i} & R_{1,i} & \cdots & R_{N-2,i} \\ -R_{1,i} & -R_{2,i} & \cdots & -R_{N-1,i} \end{bmatrix}, \\ &\Phi_{3,i} = diag\left(-\tilde{S}_{1,i},\ -\tilde{S}_{2,i},\ \cdots,\ -\tilde{S}_{N-1,i}\right) \end{aligned} \tag{45}$$

Hereinafter, let us consider $S_{f,i} \equiv \epsilon f I_n,\ f = 1, 2, ..., N-1$ for any scalar $\epsilon > 0$. Then, it can be deduced that $\tilde{S}_{1,i} \equiv \epsilon I_n$ and $\Phi_{3,i} \equiv \Phi_3 = -\epsilon I_{(N-1)n} < 0$. Then, by Schur complement and applying the definition of Ξ_1 in Theorem 4.2, the above inequality is equivalent to:

$$\begin{aligned} &\hat{\Phi}_{1,i} - \hat{\Phi}_{2,i}\Phi_3^{-1}\hat{\Phi}_{2,i}^T \\ &= \begin{bmatrix} \hat{A}_i^T P + P\hat{A}_i + 2\alpha P \\ +Q_i - Z + \Gamma_{1,i} & Z + \Gamma_{2,i} \\ (*) & -Q_i - Z + \Gamma_{3,i} \end{bmatrix} < 0 \end{aligned} \tag{46}$$

where

$$\Gamma_{1,i} = \epsilon^{-1}(P\hat{A}_{d,i})(P\hat{A}_{d,i})^T + \epsilon^{-1}\sum_{f=1}^{N-2}\left(R_{f,i}R_{f,i}^T\right),$$

$$\Gamma_{2,i} = R_{N-1,i} - \epsilon^{-1}(P\hat{A}_{d,i})R_{1,i}^T - \epsilon^{-1}\sum_{f=1}^{N-2}\left(R_{f,i}R_{f+1,i}^T\right),$$

$$\Gamma_{3,i} = \sum_{f=1}^{N-1}\epsilon I_n + \epsilon^{-1}\sum_{f=1}^{N-1}\left(R_{f,i}R_{f,i}^T\right) \tag{47}$$

Let $W_i = Q_i + Z - \Gamma_{3,i}$. Therefore, replacing $Z = W_i - Q_i + \Gamma_{3,i}$ into (46) we obtain:

$$\begin{bmatrix} \hat{A}_i^T P + P\hat{A}_i + 2\alpha P \\ +2Q_i - W_i + \Gamma_{1,i} - \Gamma_{3,i} & W_i - Q_i + \Gamma_{2,i} + \Gamma_{3,i} \\ (*) & -W_i \end{bmatrix} < 0 \tag{48}$$

Choosing $Q_i > \Gamma_{3,i} - Z$ implies that $W_i > 0$. Hence, applying again Schur Complement in the above LMI, we obtain

$$\begin{aligned} &\hat{A}_i^T P + P\hat{A}_i + 2\alpha P + 2Q_i - W_i + \Gamma_{1,i} - \Gamma_{3,i} \\ &+ \left(W_i - Q_i + \Gamma_{2,i} + \Gamma_{3,i}\right)^T W_i^{-1}\left(W_i - Q_i + \Gamma_{2,i} + \Gamma_{3,i}\right) < 0 \end{aligned} \tag{49}$$

Taking into account that

$$\begin{aligned} &\left(W_i - Q_i + \Gamma_{2,i} + \Gamma_{3,i}\right)^T W_i^{-1}\left(W_i - Q_i + \Gamma_{2,i} + \Gamma_{3,i}\right) \\ &= W_i + \Gamma_{2,i} + \Gamma_{2,i}^T + 2\left(\Gamma_{3,i} - Q_i\right) + \left(\Gamma_{2,i} + \Gamma_{3,i} - Q_i\right)^T W_i^{-1}\left(\Gamma_{2,i} + \Gamma_{3,i} - Q_i\right) \end{aligned} \tag{50}$$

we have that (49) renders:

$$\begin{aligned} &\hat{A}_i^T P + P\hat{A}_i + 2\alpha P + \Gamma_{1,i} + \Gamma_{2,i} + \Gamma_{2,i}^T + \Gamma_{3,i} \\ &+ \left(\Gamma_{2,i} + \Gamma_{3,i} - Q_i\right)^T W_i^{-1}\left(\Gamma_{2,i} + \Gamma_{3,i} - Q_i\right) < 0 \end{aligned} \tag{51}$$

By choosing $R_{f,i} = P\hat{A}_{d,i} - \epsilon f I_n,\ f = 1, ..., N-1$, we have that the following equivalence is true (see Appendix for details):

$$\Gamma_{1,i} + \Gamma_{2,i} + \Gamma_{2,i}^T + \Gamma_{3,i} = \hat{A}_{d,i}^T P + P\hat{A}_{d,i} \tag{52}$$

Replacing (52) into (51), we obtain:

$$\begin{aligned} &\hat{A}_i^T P + P\hat{A}_i + \hat{A}_{d,i}^T P + P\hat{A}_{d,i} + 2\alpha P \\ &+ \left(\Gamma_{2,i} + \Gamma_{3,i} - Q_i\right)^T W_i^{-1}\left(\Gamma_{2,i} + \Gamma_{3,i} - Q_i\right) < 0 \end{aligned} \tag{53}$$

A sufficiently high choice for matrix W_i implies that LMIs (53) are feasible if

$$(\hat{A}_i + \hat{A}_{d,i})^T P + P(\hat{A}_i + \hat{A}_{d,i}) + 2\alpha P < 0 \tag{54}$$

Note that the fulfilment of the above conditions (54) implies that

$$\sum_{i=1}^{r} \lambda_i(t) \Big((\hat{A}_i + \hat{A}_{d,i})^T P + P(\hat{A}_i + \hat{A}_{d,i}) + 2\alpha P \Big) < 0,$$

which demonstrates the robust stability of the delay-free system (1). To satisfy the rightmost inequality of (29), one can freely choose Q_i sufficiently high to satisfy $Q_i > \sum_{f=1}^{N-1} (\epsilon f)^{-1} R_{f,i}^T R_{f,i}$.

By continuity, there exists a sufficiently small $\delta = h/N$ such that, if (54) are true $\forall i = 1, ..., r$, then LMIs (29) hold.

Finally, taking into account that Theorem 4.1 is a generalization of Theorem 4.2 due to the use of Wirtinger's inequality instead of Jensen's inequality, it is evident that the same property holds for Theorem 4.1. □

Remark 1. In view of Corollary 4.3 and the equivalence between (1) and the descriptor singular models (10) and (31), Theorems 4.1 and 4.2 render non-conservative for exponential stability with decay rate performance for maximum allowable delay estimation when $N \to \infty$.

Remark 2. A novelty introduced in the proposed LKF is that the positive semidefinite matrix $\bar{Q}(t)$ in $V_2(t)$ (see (17) and (34)) has been defined time-dependent through the unknown functions $\lambda_i(t)$ given in (2). This definition has also been crucial for the asymptotic conservatism reduction stated in Corollary 4.3 in the presence of time-varying model mismatches under polytopic uncertainties.

Remark 3. The use of Elimination Lemma 2.1 has been helpful to reduce the NoV without introducing extra conservatism by removing X_1, X_2, X_3, Y_1, Y_2 and Y_3 in (17) and (18), whose size is proportional to N^2. As a result, the NoV has a linear dependence on the number of delay partition intervals N: $NoV = 0.5Nr(3n^2+n)+2.5n^2-rn^2+1.5n$ for LMIs in Theorem 4.1 and $NoV = 0.5Nr(3n^2 + n) + n^2(1 - r) + n$ for LMIs in Theorem 4.2 respectively.

5. Simulation results

In order to illustrate the effectiveness of the proposed stability analysis method discussed in Remark 1 and Remark 3, the following three benchmark examples are provided. The first one illustrates that better trade-off between conservatism and complexity can be achieved in comparison to previous approaches. The second and third examples show that the conservatism from Jensen's inequality in (35) decreases as long as N increases considering model uncertainties and decay rate performance.

5.1. Example 1

Consider (1) with $r = 1$ and system matrices:

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}, \quad A_{d,1} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{55}$$

This example is already known to be stable for delays $h \in [0.10017, 1.7178]$ if only stability without decay rate performance analysis is considered (i.e., $\alpha = 0$) (Liu et al., 2019).

N	2	5	10	20	30
h_{min}	0.10026	0.10020	0.10019	0.10019	0.10018
h_{max}	1.6803	1.7106	1.7159	1.7173	1.7176

Table 1. Estimation of delay bounds h_{min} and h_{max} obtained by Theorem 4.2 as a function of N in Example 1.

h	0.3	1.5	1.6	NoV
Seuret and Gouaisbaut (2013)	0.0971	0.0175	-	16
Hien and Trinh (2016)	0.0971	0.1039	0.045	30
Th. 4.2 (N=2)	0.0986	0.1022	0.0419	23
Th. 4.2 (N=3)	0.1002	0.1140	0.0522	30
Th. 4.2 (N=4)	0.1006	0.1175	0.0557	37

Table 2. Maximum exponential decay rate α for different values of delay h in Example 1.

Table 1 depicts the lower and upper delays obtained from Theorem 4.2 ($\alpha = 0$) for different numbers of delay partition N. It can be seen that delays converge to the analytical bounds as long as N increases.

For exponential stability with decay rate, Table 2 compares the maximum exponential decay rate α and the NoV obtained from Theorem 4.2 in comparison to previous approaches considering $h = 0.3$, $h = 1.5$ and $h = 1.6$ respectively. Note that a better estimation of decay rate is obtained with respect to Seuret and Gouaisbaut (2013) and Hien and Trinh (2016) by choosing $N \geq 2$ and $N \geq 3$ respectively. In particular, from the second and fourth rows in Table 2, it can be appreciated that the decay rate estimation is better than (Hien & Trinh, 2016) for $N = 3$ using the same number of decision variables (NoV=30), leading comparatively to a better trade-off between complexity and conservatism.

5.2. Example 2

Consider the uncertain system (1) with $r = 2$ and system matrices (Gu, 2001a):

$$A_1 = \begin{bmatrix} -2-a & -a \\ -a & -0.9-a \end{bmatrix}, \quad A_{d,1} = \begin{bmatrix} -1-a & 0 \\ -1 & -1+a \end{bmatrix}, \tag{56}$$

$$A_2 = \begin{bmatrix} -2+a & a \\ a & -0.9+a \end{bmatrix}, \quad A_{d,2} = \begin{bmatrix} -1+a & 0 \\ -1 & -1-a \end{bmatrix}, \tag{57}$$

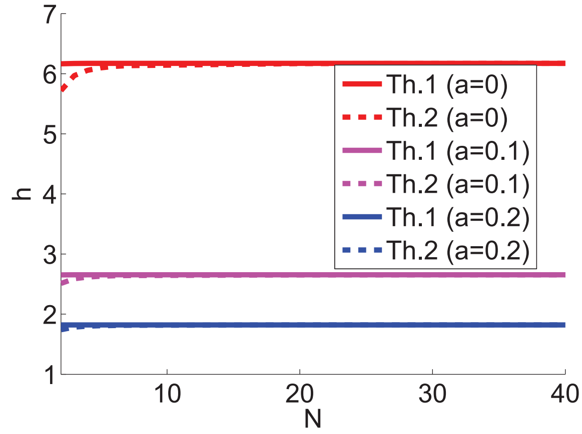

Figure 1. Maximum allowable delay h for stability as a function of the number of delay partition intervals N (Example 2) using Theorem 4.1 (Wirtinger's inequality) and Theorem 4.2 (Jensen's inequality) for different values of uncertainties a.

The maximum allowable delay obtained by Theorem 4.1 (solid-blue line) and Theorem 4.2 (dashed-red line) are depicted in Fig. 1 considering $a = 0$ (uncertain-free case), $a = 0.1$ and $a = 0.2$ respectively. It can be appreciated that the gap between Jensen's and Wirtinger's inequalities reduces when $N \to \infty$. Note also that the estimation of the maximum allowable delay h asymptotically converges to the analytical bound $h_{max} = 6.1725$ (Gu et al., 2003) considering the nominal model (i.e., $a = 0$).

5.3. *Example 3*

Consider (1) with $r = 1$ and system matrices:

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad A_{d,1} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \tag{58}$$

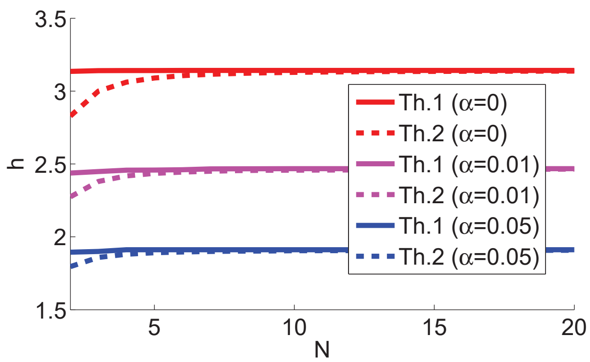

Figure 2. Maximum allowable delay h for exponential stability as a function of the number of delay partitions N (Example 3) using Theorem 4.1 (applying Wirtinger's inequality) and Theorem 4.2 (Jensen's inequality) for different decay rates α.

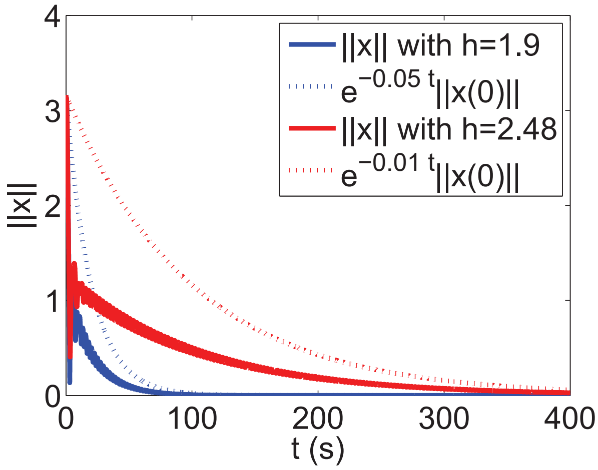

Figure 3. State evolution $||x(t)||$ for $h = 1.9$ (blue line) and $h = 2.48$ (red line) in Example 3.

Fig. 2 shows the maximum allowable delay h that guarantees the stability with exponential decay rate α obtained by Theorem 4.1 (solid-blue line) and Theorem 4.2 (dashed-red line) respectively for $\alpha = 0$, $\alpha = 0.01$ and $\alpha = 0.05$, where different numbers of delay partition intervals N has been considered. As in the previous examples, it can be appreciated that the obtained delay converges to the analytical bound ($h_{max} = \pi$ for $\alpha = 0$) Gu et al. (2003) when $N \to \infty$. In light of the given results, it is remarkable that Wirtinger's inequality combined with delay partitioning method (Theorem 4.1) offers a sensible trade-off between complexity and conservatism since the analytical bound of the maximum allowable delay is reached in a reduced number of delay partitions N, whereas using Jensen's inequality (Theorem 4.2) requires larger values for N to reach the same delay bound with only a difference of 7 NoV between them $\forall N \geq 2$ (see Remark 3). Therefore, an extension of Theorem 4.1 to other generalized integral inequalities (Seuret & Gouaisbaut, 2017) could not be expected to gain much more conservatism reduction when combined with delay partitioning approaches.

Simulation results of the state evolution are presented in Fig. 3 considering the maximum allowable delay h for exponential stability with decay rates: $\alpha = 0.01$ (with $h = 1.9$) and $\alpha = 0.01$ (with $h = 2.48$). The value of h in both cases has been obtained by Theorem 4.1 and Theorem 4.2 choosing a sufficiently large N (see Fig. 2). Initial condition for system state has been chosen $x(0) = [3\ 1]$. Note that state evolution $||x(t)||$ (solid lines) is tightly bounded by $||x(0)||e^{-\alpha t}$ (dashed lines) in both cases, which confirms the effectiveness of the proposed method.

5.4. Example 4 (inverted pendulum)

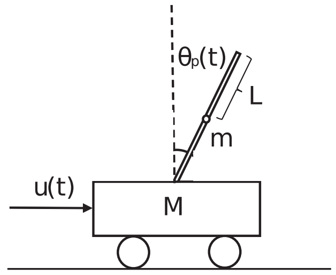

Figure 4. Inverted pendulum system

This example studies the control system of an inverted pendulum system with delayed control input. The system consists in a pendulum attached to the side of a cart by means of a pivot which allows the pendulum to swing in a 2D-plane (see Fig. 4). Consider the linearized state-space model $\dot{x}(t) = A(t)x(t)+B(t)u(t-h)$, where h is the input delay, $x(t) = \begin{bmatrix}\theta_p(t) & \dot{\theta}_p(t)\end{bmatrix}^T$ is the state variable containing the angular position and velocity of the pendulum from the top vertical ($\theta_p(t)$ and $\dot{\theta}_p(t)$ respectively), $u(t)$ is the control input which consists in a force applied to the cart with the purpose of keeping the pendulum balanced upright, and $A(t), B(t)$ are the state-space matrices:

$$A(t) = \begin{bmatrix} 0 & 1 \\ (1+\rho_a(t))\left(\frac{3(M+m)}{L(4M+m)}\right) & 0 \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ -(1+\rho_b(t))\left(\frac{3}{L(4M+m)}\right) \end{bmatrix} \tag{59}$$

where M and m are the masses of the cart and the pendulum, respectively; L is the half length of the pendulum (i.e., the distance from the pivot to the center of mass of the pendulum, and $\rho_a(t) \leq \bar{\rho}$, $\rho_b(t) \leq \bar{\rho}$ are functions introduced to include time-varying model mismatches, where $\bar{\rho} \geq 0$ determines the size of model uncertanties. Let $M = 8.00Kg$, $m = 2.00Kg$, $l = 0.50m$ and $g = 9.81ms^{-2}$. Hence, we have that the closed-loop system renders as (1) with $\big(A(t),\ B(t)\big) = \sum_{i=1}^{4} \lambda_i(t)\left(\hat{A}_i,\ \hat{B}_i\right)$ with

$$\begin{aligned}
\hat{A}_1 &= \begin{bmatrix} 0 & 1 \\ 17.31-\bar{\rho} & 0 \end{bmatrix}, & \hat{B}_1 &= \begin{bmatrix} 0 \\ -0.1765-\bar{\rho} \end{bmatrix}, \\
\hat{A}_2 &= \begin{bmatrix} 0 & 1 \\ 17.31+\bar{\rho} & 0 \end{bmatrix}, & \hat{B}_2 &= \begin{bmatrix} 0 \\ -0.1765-\bar{\rho} \end{bmatrix}, \\
\hat{A}_3 &= \begin{bmatrix} 0 & 1 \\ 17.31-\bar{\rho} & 0 \end{bmatrix}, & \hat{B}_3 &= \begin{bmatrix} 0 \\ -0.1765+\bar{\rho} \end{bmatrix}, \\
\hat{A}_4 &= \begin{bmatrix} 0 & 1 \\ 17.31+\bar{\rho} & 0 \end{bmatrix}, & \hat{B}_4 &= \begin{bmatrix} 0 \\ -0.1765+\bar{\rho} \end{bmatrix}
\end{aligned} \tag{60}$$

and scalar functions $\lambda_i(t)$, $i = 1, ..., 4$ satisfying the conditions given in (2). Note that the system is open-loop unstable with poles at $\{4.1598,\ -4.1598\}$. Consider the stabilizing control law $u(t) = Kx(t)$ with $K = [102.9100 \quad 80.7916]$ (Gao & Chen, 2007) with closed-loop poles at $\{-0.0598,\ -14.1976\}$. The closed-loop control system can therefore be described by (1) with $\hat{A}_i$ given in (60), and $\hat{A}_{d,i} = \hat{B}_i K, i = 1, ..., 4$.

N	2	3	4	5	6	7
h ($\alpha = 0.00$)	0.1091	0.1103	0.1108	0.1110	0.1112	0.1112
h ($\alpha = 0.03$)	0.1088	0.1100	0.1105	0.1107	0.1107	0.1107

Table 3. Maximum allowable delay h obtained by Theorem 4.1 for robust stability (first row, $\alpha = 0$) and guaranteed exponential decay rate (second row, $\alpha = 0.03$) in Example 4 (inverted pendulum) with $\bar{\rho} = 0.003$.

The first row in Table 3 depicts the maximum allowable input delay h obtained with Theorem 4.1 with different number of delay partitions N that guarantees the robust closed-loop stability of the inverted pendulum considering the size of uncertainties $\bar{\rho} = 0.003$ (first row). The second row in Table 3 shows the maximum allowable input delay that ensures the exponenial stability with guaranteed decay rate $\alpha = 0.03$ (second row). From the results given in Table 3, one can see that the maximum allowable delay converges to $h = 0.1112$ for robust closed-loop stability, and $h = 0.1107$ for robust stability with decay rate $\alpha = 0.03$ as long as N increases.

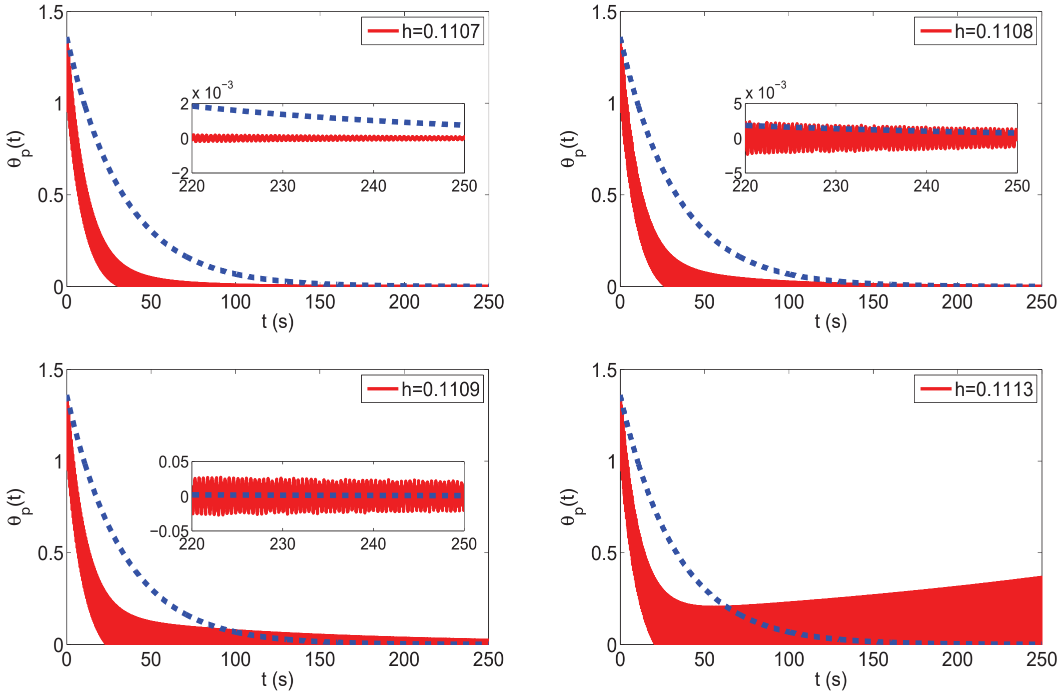

Figure 5. Red solid lines: angular position of the inverted pendulum $||\theta_p(t)||$ for different input delays with $\bar{\rho} = 0.003$ after 100 simulations with different time-varying patterns for uncertain functions $\lambda_i(t)$. Blue dashed line: envelope function for an exponential decay rate $\alpha = 0.03$

To corroborate the effectiveness of the maximum input delay estimation, Fig. 5 depict the time evolution of the angular position of the inverted pendulum $\theta_p(t)$ (solid red lines) considering four cases with input delays of $h = 0.1107$, $h = 0.1108$, $h = 0.1109$ and 0.1113 respectively. Each case has been obtained after running 100 simulations using different time-varying uncertain functions $\lambda_i(t)$, $i = 1, ..., 4$ (2) randomly generated. It can be appreciated that exponential decay rate is accomplished in the first case with $h = 0.1107$, such as expected from Theorem 4.1. Note that slight increments in the input delay keeps the system stable, but the decay rate is not always fulfilled (See cases with $h = 0.1108$ and $h = 0.1109$ in Fig. 5). Note also from Fig. 5 that the system becomes unstable for an input delay slightly greater than the estimated, i.e., $h = 0.1113$. Therefore, it is illustrated that the proposed stability analysis method renders non-conservative for sufficiently high values of N.

6. Conclusions

This paper has addressed the exponential stability analysis of uncertain time delay systems with guaranteed decay rate performance under delay partitioning, descriptor redundancy and LKF approach. The key aspect is that conservatism of the two proposed LMI-based conditions (based on Jensen's and Wirtinger's inequalities respectively) has been proved to arbitrarily be reduced by systematically introducing more variables. Various benchmark examples have been provided to illustrate this fact and show that the proposed delay partitioning descriptor redundancy method combined with Wirtinger's inequality offers a reasonable compromise between complexity and conservatism, outperforming with respect to previous approaches.

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Appendix: Proof of the equivalence (52) for Corollary 4.3

Let $X_i = P\hat{A}_{d,i}$ and $\mathcal{G} = \epsilon^{-1}\sum_{f=1}^{N-2}\left(R_{f,i}R_{f,i}^T\right)$. Then, we have that $R_{f,i} = X_i - \epsilon f I_n,\ f = 1, ..., N-1$ and the following three equivalences:

$$(i)\ :\ R_{f+1,n} = R_{f,n} - \epsilon I_n, \tag{61}$$

$$\begin{aligned}(ii)\ :\ \epsilon^{-1}\sum_{f=1}^{N-2}\left(R_{f,i}R_{f+1,i}^T\right) &= \epsilon^{-1}\sum_{f=1}^{N-2}\left(R_{f,i}R_{f,i}^T\right) - \sum_{f=1}^{N-2}R_{f,i} \\ &= \mathcal{G} - (N-2)X_i + \epsilon\left(\sum_{f=1}^{N-2}f\right)I_n \\ (iii)\ :\ \epsilon^{-1}\sum_{f=1}^{N-1}\left(R_{f,i}R_{f,i}^T\right) &= \mathcal{G} + \epsilon^{-1}R_{N-1,i}R_{N-1,i}^T \\ &= \mathcal{G} + \epsilon^{-1}X_iX_i^T - (N-1)X_i - (N-1)X_i^T + \epsilon(N-1)^2 I_n\end{aligned}$$

Using the above definitions of terms X_i and $\mathcal{G}$, and taking into account the equivalences (61), the expressions $\Gamma_{1,i}, \Gamma_{2,i}, \Gamma_{3,i}$ given in (47) can be rewritten as:

$$\begin{aligned}\Gamma_{1,i} &= \epsilon^{-1}X_iX_i^T + \mathcal{G}, \\ \Gamma_{2,i} &= X_i - \epsilon(N-1)I_n - \epsilon^{-1}X_iX_i^T + X_i - \mathcal{G} + (N-2)X_i - \epsilon\left(\sum_{f=1}^{N-2}f\right)I_n, \\ \Gamma_{3,i} &= \epsilon(N-1)I_n + \mathcal{G} + \epsilon^{-1}X_iX_i^T - (N-1)X_i - (N-1)X_i^T + \epsilon(N-1)^2 I_n\end{aligned} \tag{62}$$

Noting that $\mathcal{G} = \mathcal{G}^T$, it can easily be checked that

$$\Gamma_{1,i} + \Gamma_{2,i} + \Gamma_{2,i}^T + \Gamma_{3,i} = X_i + X_i^T + \epsilon\kappa I_n \tag{63}$$

where $\kappa = (N-1)^2 - (N-1) - 2\left(\sum_{f=1}^{N-2}f\right) = (N-1)^2 - (N-1) - \left((N-2)^2 + (N-2)\right) = 0$. Finaly, replacing $X_i = P\hat{A}_{d,i}$ and $\kappa = 0$, the equivalence (52) is obtained.