IDEALS OF MULTILINEAR MAPPINGS VIA ORLICZ SPACES AND TRANSLATION INVARIANT OPERATORS

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Abstract. We study some new summability properties of multilinear operators. We introduce the concepts of φ-summing, φ semi-integral and φ-dominated multilinear maps generated by Orlicz functions. We prove a variant of Pietsch's domination theorem for φ-summing operators, providing also a characterization of φ-dominated operators in terms of factorizations. We analyze vector-valued inequalities associated to these maps, which are applied to obtain general variants of multiple summing operators. We also study translation invariant multilinear operators acting in products of spaces of continuous functions, proving that a factorization theorem can be obtained for them as a consequence of a suitable representation of the corresponding normalized Haar measure.

Multilinear ideals, factorization theorems, Haar measure, translation invariant operators, Orlicz spaces. [MSC 2010]47L20, 46B15

1. Introduction

Banach linear operator ideals play a key role in the theory of operators. One of the most important classes of linear maps between Banach spaces is that of p-absolutely summing operators. These operators are widely recognized as one of the most important developments in modern Banach space theory and found deep applications in many areas of modern analysis (see, e.g., [10, 21, 22]). Motivated by various applications and generalizations of the concept of absolutely p-summing related to L_p-spaces, different classes were defined in a natural way in recent years. We refer to articles [7, 8] and the references therein related to applications of the so-called (E, F)-summing operators to eigenvalues, s-numbers and interpolation of operators. The study of a class of linear operators based on the Orlicz spaces was initiated in [3] and continued later in [12, 13]. In [12] the study of this class of operators was motivated by questions raised in the study of Burkholder–Davis–Gundy inequalities for vector-valued martingales. In his remarkable paper, Geiss [12] used this class of operators generated by Orlicz spaces L_{φ_q} with an exponential function φ_q, given by $\varphi_q(t) = \exp(t^q) - 1$ for all $t \geq 0$ with $q \in [1, \infty)$, for proving the required inequalities. We point out that the notion of absolutely φ-summing was motivated by the consideration of majorizing measures for Gaussian processes.

The ideal structure of multilinear operators acting between Banach spaces has been investigated intensively in recent years. The natural interest is motivated by various applications and generalizations of the rich theory of linear operator ideals. We refer to the article [18], a survey of some classes of multilinear mappings between Banach spaces that generalize the class of absolutely summing linear operators to the multilinear setting. We note that in this article descriptions are provided of the classes of multilinear mappings that are multiple summing, dominated, semi-integral, strongly summing, absolutely summing and strongly multiple summing; the main results concerning these classes of multilinear mappings are presented and also some proofs are included.

We initiate the study of a new class of multilinear operators, whose definition is based on summability properties and integral dominations provided by Orlicz space norms. This is primarily motivated by the natural question that appears in the theory of multilinear operators concerning the existence of multilinear variants of the main results that are known for linear operators in Banach spaces. The main part of these results in the linear setting are related to summability properties and integral dominations, usually involving L^p-type norms. In this paper we will consider the more general Orlicz variant of these notions. The main aim is to study such classes of multilinear operators, for which we are able to prove abstract variants of factorization theorems.

The first progress in the translation of linear-type results to multilinear-type ones was in a work by Alencar and Matos [1], where several classes of multilinear mappings among Banach spaces were investigated. Motivated by that work, the p semi-integral multilinear operators were introduced in [18] for $1 \leq p < \infty$. Following [18], a multilinear operator $T\colon X_1 \times \cdots \times X_n \to Y$ is said to be p semi-integral ($T \in \mathcal{L}_{si,p}(X_1, \dots, X_n; Y)$) if there exist a constant $C > 0$ and a regular probability Borel measure on the σ-algebra $\mathcal{B}(B_{X_1^*} \times \cdots \times B_{X_n^*})$ of $B_{X_1^*} \times \cdots \times B_{X_n^*}$ endowed with the product of the weak* topologies $\sigma(X_j^*, X_j)$ for $1 \leq j \leq n$, such that for all $(x_1, \dots, x_n) \in X_1 \times \cdots \times X_n$,

$$\|T(x_1, \dots, x_n)\|_Y \leq C \left(\int_{B_{X_1^*} \times \cdots \times B_{X_n^*}} |\langle x_1, x_1^* \rangle \cdots \langle x_n, x_n^* \rangle|^p \, d\mu \right)^{1/p}.$$

The infimum of the C defines a norm $\|\cdot\|_{si,p}$ for the space $\mathcal{L}_{si,p}(X_1, \dots, X_n; Y)$ of p semi-integral operators.

The natural counterpart to this definition in terms of summability is given by the p-summing multilinear operators. Following [1], an n-linear operator $T\colon X_1 \times \cdots \times X_n \to Y$ is said to be p-summing if there exists $C > 0$ such that

$$\Big(\sum_{j=1}^m \|T(x_j^1,\dots,x_j^n)\|_Y^p\Big)^{1/p} \le C \sup_{(x_j^*)_{j=1}^n \in B_{X_1^*}\times\cdots\times B_{X_n^*}} \Big(\sum_{j=1}^m |\langle x_j^1, x_1^*\rangle \cdots \langle x_j^n, x_n^*\rangle|^p\Big)^{1/p}$$

for every choice of $m \in \mathbb{N}$ and $(x_j^1,\dots,x_j^n) \in X_1\times\cdots\times X_n$, where $1 \le j \le m$.

These classes of multilinear operators define ideals like the ones that will be introduced in the present paper. Recall that a Banach ideal of multilinear mappings $\mathcal{I}$ is a class of continuous multilinear operators between Banach spaces such that for every $n \in \mathbb{N}$ and Banach spaces $X_1,\dots,X_n,Y$, the associated component, that is denoted by

$$\mathcal{I}(X_1,\dots,X_n;Y) := \mathcal{L}(X_1,\dots,X_n;Y) \cap \mathcal{I}\,,$$

satisfies the following:

(i) $\mathcal{I}(X_1,\dots,X_n;Y)$ is a linear subspace of $\mathcal{L}(X_1,\dots,X_n;Y)$ containing the n-linear mappings of finite type, and

(ii) if $A \in \mathcal{I}(X_1,\dots,X_n;Y)$, $u_j \in \mathcal{L}(G_j;X_j)$ for each $1 \le j \le n$ and $v \in \mathcal{L}(Y;H)$, then $v \circ A \circ (u_1,\dots,u_n)$ belongs to $\mathcal{I}(G_1,\dots,G_n;H)$ and

$$\|v \circ A \circ (u_1,\dots,u_n)\|_{\mathcal{I}} \le \|v\|\,\|A\|_{\mathcal{I}}\,\|u_1\|\cdots\|u_n\|\,,$$

where $\|\cdot\|_{\mathcal{I}}$ is given by a rule for defining the ideal norm.

In the present paper we investigate some classes of multilinear mappings generated by Orlicz spaces. We define the class of φ-summing multilinear operators that are defined by Orlicz sequence spaces. The core of this concept lies in a variant of Pietsch's domination theorem proved in Section 3. This result motivates us to introduce the notion of φ semi-integral multilinear operators, which involve Orlicz spaces $L_\varphi(\mu)$ generated by regular probability Borel measure measures on $B_{X_1^*}\times\cdots\times B_{X_n^*}$ endowed with the product of the weak* topologies. We prove that for a wide class of Orlicz functions, the φ-summing and φ semi-integral multilinear operators coincide. In Section 3 we present also general vector-valued inequalities for semi-integral φ-summing operators.

In Section 4, we introduce the φ-dominated multilinear operators, a subclass of the φ semi-integral mappings. We characterize this family in terms of factorizations, under the restriction on the Orlicz function φ to be submultiplicative. We analyze the relationship of this class with the general class of φ semi-integral operators. General examples of φ-dominated multilinear operators are also presented in this section.

In Section 5, we study translation invariant φ semi-integral operators on the products of a particular type of subspaces of $C(K)$-spaces, and investigate the Pietsch's measures that appear in the Domination Theorem. As a by-product of our result, we deduce in particular that for a translation invariant φ-summing multilinear operator defined on the product $X_1 \times \cdots \times X_n$ of closed invariant subspaces X_j of $C(G_j)$-spaces on compact topological groups G_j for each $1 \leq j \leq n$, the Pietsch's measure is the normalized Haar measure on $G_1 \times \cdots \times G_n$. This, together with the previous results on φ-summing and φ semi-integral operators, will be used for showing that, in this case, a complete factorization scheme can be obtained. We obtain in this way our main application: under some mild conditions on the Orlicz function φ, φ-summability, integral domination and factorization through products of Orlicz spaces are equivalent properties for translation invariant multilinear operators. In the last part of this Section, we prove some vector-valued inequalities for a universal class of function-lattice semi-integral operators. We show that these operators are multiple summing operators in a more general sense, underlying the fact that the results that are obtained for φ semi-integral mappings are actually true in a broader setting. Finally, let us point out that multiple summing operators generated by scales of ℓ_p-sequence spaces is an active current area of research that has found several interesting applications (see, e.g., [4, 9, 20]). Our study generalizes some domination properties of multilinear operators associated to ℓ_p-spaces to the case of Orlicz sequence spaces.

2. Notation and background

Throughout the paper we use standard notation from Banach space theory and operator theory. Given a Banach space X, X^* will be the dual space of X, B_X its closed unit ball and S_X its unit sphere. If (Ω, Σ, μ) is a σ-finite measure space and X is a Banach space, $L^0(\mu, X)$ denotes the space of (equivalence classes of) strongly μ-measurable X-valued functions. As usual $L^0(\mu) := L^0(\mu, \mathbb{R})$ is equipped with the topology of convergence in measure on μ-finite sets. A linear subspace X of $L^0(\mu)$ is called an (order) ideal whenever it follows from $f \in X$, $g \in L^0(\mu)$ and $|g| \leq |f|$, that $g \in X$. An order ideal $X \subset L^0(\mu)$ provided with a monotone norm $\|\cdot\|$ is called a normed function lattice in $L^0(\mu)$. If the normed (function) lattice X is norm complete and there exists $u \in X$ such that $u > 0$ on Ω, then X is called a Banach function lattice in $L^0(\mu)$. A Banach lattice X is said to have the Fatou property if its unit ball B_X is closed in $L^0(\mu)$.

Let E be a (Banach) ideal on a measure space (Ω, Σ, μ) and let X be a Banach space. The Köthe–Bochner space $E(X)$ is defined to consist of all strongly measurable functions $x\colon \Omega \to X$ with $\|x(\cdot)\|_X \in E$, and is equipped with the norm $\|x\|_{E(X)} := \| \|x(\cdot)\|_X \|_E$. The Köthe–Bochner spaces are connected with the mixed norm spaces which will be used in our paper. Let $(\Omega_1, \Sigma_1, \nu)$ and $(\Omega_2, \Sigma_2, \mu)$ be measure spaces and let E and F be

Banach lattices in $L^0(\nu)$ and $L^0(\mu)$, respectively. Assume (for measurability reasons) that either the measure ν is discrete or the norm $\|\cdot\|_F$ is semi-continuous, i.e., if $0 \le f_n \uparrow f$ ν-a.e., with $f \in F$, then $\|f_n\|_F \to \|f\|_F$.

In what follows, for every $f \in L^0(\Omega_1 \times \Omega_2, \nu \times \mu)$ and $(s,t) \in \Omega_1 \times \Omega_2$, we define $f_s \in L^0(\mu)$ and $f^t \in L^0(\nu)$ by $f_s(\cdot) = f(s,\cdot)$ and $f^t(\cdot) = f(\cdot,t)$. The mixed Banach lattice $E[F]$ in $L^0(\nu \times \mu)$ is defined to be the space of all $f \in L^0(\nu \times \mu)$ such that $f_s \in F$ with $s \mapsto \|f_s\|_F \in E$ equipped with the norm

$$\|f\|_{E[F]} := \|\|f_s\|_F\|_E \,.$$

Similarly, we define $[E]F$ to be the Banach lattice of all $f \in L^0(\nu \times \mu)$ equipped with the norm

$$\|f\|_{[E]F} := \|\|f^t\|_E\|_F \,.$$

We will use Orlicz spaces. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be an *Orlicz function* (that is, a convex increasing and continuous positive function with $\varphi(0) = 0$). The Orlicz space $L_\varphi(\mu)$ (L_φ for short) on a measure space $(\Omega, \mathcal{A}, \mu)$ is defined to be the space of $f \in L^0(\mu)$ such that $\int_\Omega \varphi(\lambda|f|)\,d\mu < \infty$ for some $\lambda > 0$ and is equipped with the Luxemburg norm given by

$$\|x\|_{L_\varphi} := \inf\left\{\varepsilon > 0;\ \int_\Omega \varphi\Big(\frac{|f|}{\varepsilon}\Big)\,d\mu \le 1\right\}.$$

If Ω is a finite or countable set and $\mathcal{A} = 2^\Omega$, we will often write $\ell_\varphi(\mu)$ instead of $L_\varphi(\mu)$. Let L_φ be an Orlicz space on $(\Omega, \mathcal{A}, \mu)$. In what follows we will need two simple observations:

- For every $0 \ne f \in L_\varphi$, $\quad \int_\Omega \varphi(|f|/\|f\|_{L_\varphi})\,d\mu \le 1\,;$
- If $g \in L_\varphi(\mu)$ satisfies $\int_\Omega \varphi\big(|g|/\lambda\big)\,d\mu \ge 1$, then $\lambda \le \|g\|_{L_\varphi}$.

Let $X_1, \dots, X_n$ and Y be Banach spaces. We equip the product $X_1 \times \dots \times X_n$ with the norm $\|(x_1, \dots, x_n)\| = \max_{1\le j\le n}\|x_j\|_{X_j}$. We denote by $\mathcal{L}(X_1, \dots, X_n; Y)$ the Banach space of multilinear and continuous operators defined on $X_1 \times \dots \times X_n$ with values in Y equipped with the norm

$$\|T\| := \sup\big\{\|T(x_1, \dots, x_n)\|_Y;\ (x_1, \dots, x_n) \in B_{X_1} \times \dots \times B_{X_n}\big\}\,.$$

In the case Y is the scalar field ($\mathbb{R}$ or $\mathbb{C}$), we denote the space of multilinear forms by $\mathcal{L}(X_1, \dots, X_n)$.

As usual $C(K)$ stands for the Banach space of real-valued continuous functions on a compact Hausdorff space K and is endowed with the supremum norm. In what follows, we will consider also topological groups that are assumed to be T_1-spaces and so Hausdorff spaces, as is well-known. For the operation on a topological group we use multiplicative notation. We recall that the Haar measure on a compact topological group G is a regular Borel measure μ on the Borel sets which is left and right invariant, that is,

$\mu(gB) = \mu(B)$ and $\mu(Bg) = \mu(B)$ for every Borel set B and every $g \in G$. It is well-known that there is only one normalized Haar measure on a compact topological group G (see, e.g., [25]). The uniqueness of the normalized Haar measure on a compact topological group will be used in this paper.

3. φ-summing and φ semi-integral multilinear operators

Let φ be an Orlicz function. Throughout the paper, for a fixed $m \in \mathbb{N}$ and a positive sequence $\{\nu_j\}_{j=1}^m \in S_{\ell_1^m}$, we denote by $\ell_\varphi^m(\nu)$ the m-dimensional Orlicz sequence space over the probability measure space $\big([m], 2^{[m]}, \mu\big)$, where $[m] := \{1, \dots, m\}$ and $\mu(\{j\}) = \nu_j$ for each $j \in [m]$, endowed with the norm given by

$$\|\xi\|_{\ell_\varphi^m(\nu)} := \inf\Big\{\lambda > 0;\, \sum_{j=1}^m \varphi(|\xi_j|/\lambda)\nu_j \le 1\Big\}, \quad \xi = \{\xi_j\}_{j=1}^m .$$

Let $X_1, \dots, X_n, Y$ be Banach spaces. We introduce the following definition: an n-linear operator $T\colon X_1 \times \cdots \times X_n \to Y$ is said to be *φ-summing* if there exists a constant $C > 0$ such that

$$\big\|\big\{\|T(x_j^1, \dots, x_j^n)\|_Y\big\}_{j=1}^m\big\|_{\ell_\varphi^m(\nu)} \le C \sup_{(x_j^*)_{j=1}^n \in B_{X_1^*}\times\cdots\times B_{X_n^*}} \big\|\big\{\langle x_j^1, x_1^*\rangle \cdots \langle x_j^n, x_n^*\rangle\big\}_{j=1}^m\big\|_{\ell_\varphi^m(\nu)}$$

for every m-dimensional Orlicz space $\ell_\varphi^m(\nu)$ and for all $x_j^i \in X_i$, $1 \le i \le n$ and $1 \le j \le m$. We denote by $\mathcal{L}_\varphi(X_1, \dots, X_n; Y)$ the space of all φ-summing n-linear operators from $X_1 \times \cdots \times X_n$ into Y. It is a Banach space endowed with the norm $\pi_\varphi(T)$, that is defined to be the least constant C satisfying the above requirements. It can be easily checked that $(\mathcal{L}_\varphi, \pi_\varphi)$ is an ideal of multilinear operators.

If S_j are nonempty sets and $f_j\colon S_i \to \mathbb{R}$ are functions for each $1 \le j \le n$, then $\odot$ denotes the pointwise product map defined by

$$\odot\,(f_1, \dots, f_n)(s_1, \dots, s_n) := f_1(s_1)\cdots f_n(s_n), \quad (s_1, \dots, s_n) \in S_1 \times \cdots \times S_n .$$

As usual, for a Banach space X, we denote by κ_X the canonical embedding $\kappa_X\colon X \to C(B_{X^*})$, where B_{X^*} is equipped with the weak* topology. Using the pointwise product $\odot$, we define the multiplication operator $\circledast\colon X_1 \times \cdots \times X_n \to C(B_{X_1^*} \times \cdots \times B_{X_n^*})$ by

$$\circledast\,(x_1, \dots, x_n) := \odot\big(\kappa_{X_1}(x_1), \dots, \kappa_{X_n}(x_n)\big), \quad \big(x_1, \dots, x_n\big) \in X_1 \times \cdots \times X_n .$$

Next we prove a variant of the domination theorem for the specific case of φ-summing multilinear operators.

Theorem 3.1. *Let φ be a normalized Orlicz function (i.e., $\varphi(1) = 1$) and let $T\colon X_1\times\cdots\times X_n \to Y$ be a φ-summing multilinear operator with $\pi_\varphi(T) \le M$. If, for each $1 \le j \le n$, $J_j\colon X_j \to C(K_j)$ is an isometric embedding of* *a Banach space X_j into $C(K_j)$, then there exists a regular Borel probability measure μ on the product $K_1 \times \cdots \times K_n$ such that, for all $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$,*

$$\|T(x_1, \ldots, x_n)\|_Y \leq M \| \odot (J_1(x_1), \ldots, J_n(x_n))\|_{L_\varphi(\mu)} .$$

Proof. Let $K_1 \times \cdots \times K_n$ be equipped with the product topology. Consider the set

$$F_1 := \Big\{ f \in C(K_1 \times \cdots \times K_n); \sup_{(k_1,\ldots,k_n) \in K_1 \times \cdots \times K_n} f(k_1, \ldots, k_n) < 1 \Big\} .$$

For every $x = (x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$, we define the function $f_x \in C(K_1 \times \cdots \times K_n)$ by

$$f_x := \varphi(M|J_1(x_1)| \cdots |J_n(x_n)|) .$$

Let

$$F_2 := \mathrm{conv}\big\{ f_x;\, x = (x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n,\, \|Tx\|_Y = 1 \big\} .$$

Clearly, F_1 and F_2 are convex sets in $C(K_1 \times \cdots \times K_n)$. It is clear that F_1 is an open set. We claim that $F_1 \cap F_2 = \emptyset$. To see this, let $f \in F_2$. We will show that $f \notin F_1$. By the definition of F_2 there exist $\nu_1 > 0, \ldots, \nu_m > 0$ and $\widehat{x}_j := (x_j^1, \ldots, x_j^n) \in X_1 \times \cdots \times X_n$, with $\sum_{j=1}^m \nu_j = 1$ and $\|T\widehat{x}_j\|_Y = 1$ for each $1 \leq j \leq m$, such that

$$f = \sum_{j=1}^m \nu_j f_{\widehat{x}_j} .$$

Thus by $\pi_\varphi(T) \leq M$, we have that for $(\nu_1, \ldots, \nu_m)$,

$$\begin{aligned} \|\{1\}\|_{\ell_\varphi^m(\nu)} &= \big\| \big\{ \|T(x_j^1, \ldots, x_j^n)\|_Y \big\}_{j=1}^m \big\|_{\ell_\varphi^m(\nu)} \qquad (*) \\ &\leq M \sup_{(x_1^*,\ldots,x_n^*) \in B_{X_1^*} \times \cdots \times B_{X_n^*}} \big\| \big\{ \langle x_j^1, x_1^* \rangle \cdots \langle x_j^n, x_n^* \rangle \big\}_{j=1}^m \big\|_{\ell_\varphi^m(\nu)} \\ &\leq M \sup_{(\mu_1,\ldots,\mu_n) \in B_{C(K_1)^*} \times \cdots \times B_{C(K_n)^*}} \big\| \big\{ \langle J_1(x_j^1), \mu_1 \rangle \cdots \langle J_n(x_j^n), \mu_n \rangle \big\}_{j=1}^m \big\|_{\ell_\varphi^m(\nu)} \\ &= M \sup_{(k_1,\ldots,k_n) \in K_1 \times \cdots \times K_n} \big\| \big\{ J_1(x_j^1)(k_1) \cdots \langle J_n(x_j^n)(k_n) \big\}_{j=1}^m \big\|_{\ell_\varphi^m(\nu)} . \end{aligned}$$

We observe that the last equality follows by the fact that

$$(\mu_1, \ldots, \mu_n) \mapsto \big\| \big\{ \langle J_1(x_j^1), \mu_1 \rangle \cdots \langle J_n(x_j^n), \mu_n \rangle \big\}_{j=1}^m \big\|_{\ell_\varphi^m(\nu)}$$

is a $\sigma(C(K_j)^*, C(K_j))$-continuous, convex function in each variable μ_j on $B_{C(K_j)^*}$ and so attains its supremum on the set of extreme points $\{\delta_{k_j};\, k_j \in K_j\}$ for each $1 \leq j \leq n$.

We claim now that $f \notin F_1$. In fact, otherwise we would have

$$\sup_{(k_1,\dots,k_n)\in K_1\times\cdots\times K_n} \sum_{j=1}^{m} \varphi\big(M|J_1(x_j^1)(k_1)|\cdots|J_n(x_j^n)(k_n)|\big)\nu_j < 1\,.$$

Hence, by the definition of the norm in $\ell_\varphi^m(\nu)$, we would get that

$$\sup_{(k_1,\dots,k_n)\in K_1\times\cdots\times K_n} \big\|\big\{J_1(x_j^1)(k_1)\cdots\langle J_n(x_j^n)(k_n)\big\}_{j=1}^m\big\|_{\ell_\varphi^m(\nu)} < 1/M\,.$$

Clearly, $\varphi(1) = 1$ yields $\|\{1\}_{j=1}^m\|_{\ell_\varphi^m(\nu)} = 1$. Combining this with the inequality $(*)$, we get a contradiction. This proves the claim.

Now we can apply both the Hahn–Banach and Riesz representation theorems to get the existence of a constant λ and a regular Borel measure μ on $K_1\times\cdots\times K_n$ such that

$$\int_{K_1\times\cdots\times K_n} f\,d\mu \le \lambda \quad \text{for all} \quad f\in F_1$$

and

$$\int_{K_1\times\cdots\times K_n} f\,d\mu \ge \lambda \quad \text{for all} \quad f\in F_2\,.$$

Since F_1 contains all negative functions, μ has to be a positive measure. Thus, taking a normalization of μ, we get that $\lambda \ge 1$. Hence, if $x = (x_1,\dots,x_n)\in X_1\times\cdots\times X_n$ with $\|T(x_1,\dots,x_n)\|_Y = 1$, then

$$\int_{K_1\times\cdots\times K_n} \varphi(M|J_1(x_1)|\cdots|J_n(x_n)|)\,d\mu \ge \lambda \ge 1.$$

Thus, for all $x = (x_1,\dots,x_n)\in X_1\times\cdots\times X_n$ with $T(x_1,\dots,x_n)\ne 0$, we get

$$\int_{K_1\times\cdots\times K_n} \varphi\bigg(\frac{M|J_1(x_1)|\cdots|J_n(x_n)|}{\|T(x_1,\dots,x_n)\|_Y}\bigg)\,d\mu \ge 1\,.$$

This yields the required estimate

$$\|T(x_1,\dots,x_n)\|_Y \le M\|J_1(x_1)\cdots J_n(x_n)\|_{L_\varphi(\mu)}$$

and completes the proof. $\square$

This theorem yields the following variant of the Pietsch's domination theorem.

Theorem 3.2. *Let φ be a normalized Orlicz function. Suppose that $T\colon X_1\times\cdots\times X_n\to Y$ is a φ-summing multilinear operator with $\pi_\varphi(T)\le C$. Then there exists a regular Borel probability measure μ on $B_{X_1^*}\times\cdots\times B_{X_n^*}$ equipped with the product of the weak* topologies so that for every $(x_1,\dots,x_n)\in X_1\times\cdots\times X_n$,*

$$\|T(x_1,\dots,x_n)\|_Y \le C\|\circledast(x_1,\dots,x_n)\|_{L_\varphi(\mu)}\,.$$

Proof. Since $J_j := \kappa_{X_j}$ is an isometric embedding of X_j into $C(K_j)$, where $K_j := B_{X_j^*}$ is equipped with the weak* topology for each $1 \le j \le n$, Theorem 3.1 applies. $\square$

The following definition is motivated by Theorem 3.2: let $X_1, \ldots, X_n$ be Banach spaces and let $\mathcal{M}(B_{X_1^*} \times \cdots \times B_{X_n^*})$ be the space of all Borel probability measures on $B_{X_1^*} \times \cdots \times B_{X_n^*}$, endowed with the product of the weak* topologies. A multilinear operator $T \colon X_1 \times \cdots \times X_n \to Y$ is said to be $F(\mu)$ *semi-integral* whenever there exist a constant $C > 0$, a measure $\mu \in \mathcal{M}(B_{X_1^*} \times \cdots \times B_{X_n^*})$ and a Banach lattice E in $L^0(B_{X_1^*} \times \cdots \times B_{X_n^*})$ such that, for every $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$, we have

$$\|T(x_1, \ldots, x_n)\|_Y \le C \| \circledast (x_1, \ldots, x_n)\|_{F(\mu)} .$$

The infimum of the constant C for which the inequality holds is denoted by $\pi_{si,F}(T)$. In what follows μ is called a Pietsch's measure (or a representing measure) for T.

In the case that T is $F(\mu)$ semi-integral, where $F(\mu) = L_\varphi(\mu)$ is an Orlicz space, then T is called φ semi-integral for short, and we write $\pi_{si,\varphi}$ instead of $\pi_{si,L_\varphi}(T)$. In this case we denote by $\mathcal{L}_{si,\varphi}(X_1, \ldots, X_n; Y)$ the space of all φ semi-integral multilinear operators from $X_1 \times \cdots \times X_n$ into Y.

In order to motivate the developments that follow, let us write here an immediate —but relevant— consequence of Theorem 3.2.

Corollary 3.3. *Let φ be a normalized Orlicz function. If $T \colon X_1 \times \cdots \times X_n \to Y$ is a φ-summing multilinear operator, then T is φ semi-integral with $\pi_{si,\varphi}(T) \le \pi_\varphi(T)$.*

We show some properties of φ-summing multilinear operators. We recall that weakly sequentially continuous operators between Banach spaces are those that carry weakly convergent sequences to norm convergent sequences. Motivated by the importance of this class of operators in the linear setting multilinear variants were investigated. We recall, following for instance [2, 6] that an n-linear operator $T \colon X_1 \times \cdots \times X_n \to Y$ is said to be a *weakly sequentially continuous* multilinear operator if $x_k^j \to x^j$ weakly in X_j as $k \to \infty$ for each $1 \le j \le n$ implies that $\|T(x_k^1, \ldots, x_k^n) - T(x^1, \ldots, x^n)\|_Y \to 0$ as $k \to \infty$. This class of multilinear operator is well known in literature with also a different denomination: completely continuous multilinear operators (see, for instance, [5, 14]).

Clearly, every multilinear operator $T \colon X_1 \times \cdots \times X_n \to Y$ is a weakly sequentially continuous operator if X_j has the Schur property for each $1 \le j \le n$ (i.e., such that weakly compact sets in X_j are compact).

The following characterization of the weakly sequentially continuous multilinear operators, in terms of weak Cauchy sequences, follows from [2, Corollary 2.5]. We note that it is pointed out in this article that only trivial modifications in Lemma 2.4 and in Proposition 2.2 are needed to prove this Corollary. We also mention that the proof of Proposition 2.2 is presented for polynomials between Banach spaces, and it is given by induction. For the sake of completeness, we include a direct proof to the multilinear case, without using induction.

Proposition 3.4. *Suppose that $T\colon X_1 \times \cdots \times X_n \to Y$ is a weakly sequentially continuous multilinear operator. If for each j with $1 \leq j \leq n$, $\{x_k^j\}_k$ is a weakly Cauchy sequence in X_j, then $\{T(x_k^1, \dots, x_k^n)\}_k$ is a norm convergent sequence in Y.*

Proof. Suppose that T is a weakly sequentially continuous operator, and let $\{x_k\}_k = \{(x_k^1, \dots, x_k^n)\}$ be a sequence in $X_1 \times \cdots \times X_n$ such that $\{x_k^j\}_k$ is a weakly Cauchy sequence in X_j for each $1 \leq j \leq n$. Now assume for the sake of contradiction that $\{Tx_k\}$ is not a norm Cauchy sequence of Y. Then, we can find some $\varepsilon > 0$ and a subsequence $\{z_k\}$ of $\{x_k\}$ such that,

$$\|Tz_{2k} - Tz_{2k-1}\|_Y > \varepsilon \quad \text{for all} \quad k \in \mathbb{N}\,.$$

We will use the following obvious algebraic identity:

$$\begin{aligned} Tz_{2k} - Tz_{2k-1} &= T\big(z_{2k}^1, \dots, z_{2k}^n\big) - T\big(z_{2k-1}^1, \dots, z_{2k-1}^n\big) \\ &= T\big(z_{2k}^1 - z_{2k-1}^1, z_{2k}^2, \dots, z_{2k}^n\big) + T\big(z_{2k-1}^1, z_{2k}^2 - z_{2k-1}^2, z_{2k}^3, \dots, z_{2k}^n\big) \\ &+ \cdots + T\big(z_{2k-1}^1, \dots, z_{2k-1}^{n-1}, z_{2k}^n - z_{2k-1}^n\big)\,. \end{aligned}$$

Since T is weakly sequentially continuous and $z_{2k}^j - z_{2k-1}^j \to 0$ weakly in X_j for each $1 \leq j \leq n$, it follows that each term on the right side of the equality is norm convergent to 0 in Y, which contradicts the above inequality. Thus, $\{Tx_k\}$ is a norm Cauchy sequence, and so norm convergent in Y. This finishes the proof. □

Similarly as in the linear case compact mappings form an important class in the multilinear setting. A multilinear operator is said to be *compact* if it carries each product of bounded sets to a relatively compact set. This class of operators forms a multi-ideal that is closed in the operator topology. The properties of compact bilinear operators are studied in [23].

Combining Proposition 3.4 with Rosenthal's theorem [24], which states that a Banach space X does not contain an isomorphic copy of ℓ_1 if and only if any norm bounded sequence in X contains a weak Cauchy subsequence, yields immediately the following well known fact. The polynomial version of this result is proved in [2, Proposition 2.12].

Corollary 3.5. *Suppose that each Banach space $X_1, \dots, X_n$ does not contain isomorphic copy of ℓ_1. Then every weakly sequentially continuous multilinear operator $T\colon X_1 \times \cdots \times X_n \to Y$ is compact.*

We note that the assumption on isomorphic copies of ℓ_1 in Corollary 3.5 is essential in general. To see this observe that, it follows by the Schur property of ℓ_1 that the bilinear operator $T\colon \ell_1 \times \ell_2 \to \ell_2$ given, for all $x = \{x_k\} \in \ell_1$, $y = \{y_k\} \in \ell_2$ by the formula:

$$T(x, y) := \{x_k y_k\}$$

is a weakly sequentially continuous bilinear operator. Since $T(e_k, e_k) = e_k$ for each $k \in \mathbb{N}$ where e_k is the standard unit vector basis in c_0, T is not compact.

We conclude our discussion about weakly sequentially continuous multilinear operators with the following application:

Theorem 3.6. *The following statements are true for φ-summing multilinear operator $T\colon X_1 \times \cdots \times X_n \to Y$.*

(i) *T is a weakly sequentially continuous operator;*
(ii) *If each Banach space $X_1, \ldots, X_n$ does not contain isomorphic copy of ℓ_1, then T is compact.*

Proof. (i). Let $x_k^j \to x^j$ weakly in X_j as $k \to \infty$ for each $1 \le j \le n$. By algebraic identity, we have

$$\begin{aligned} T\big(x_k^1, \ldots, x_k^n\big) - T\big(x^1, \ldots, x^n\big) &= T\big(x_k^1 - x^1, x_k^2, \ldots, x_k^n\big) \\ &+ T\big(x^1, x_k^2 - x^2, x_k^3, \ldots, x_k^n\big) + T\big(x^1, \ldots, x^{n-1}, x_k^n - x^n\big)\,. \end{aligned}$$

From Theorem 3.2, it follows that T is φ semi-integral operator. Combining this with Lebesgue's Dominated Convergence Theorem reveals that each term on the right side of the above equality is norm convergent to 0 in Y. In consequence, we deduce that $T(x_k^1, \ldots, x_k^n) \to T(x^1, \ldots, x^n)$ in Y as required. Combining (i) with Lemma 3.5 yields the statement (ii). This completes the proof. ☐

Now let us give a fundamental example of a φ semi-integral multilinear operators. As in the case of the classical integral operators, the canonical embedding from products of $C(K)$-spaces into the corresponding Orlicz space is φ semi-integral.

Lemma 3.7. *Let φ be an Orlicz function and let $K_1, \ldots, K_n$ be compact Hausdorff spaces. Then for every Borel probability measure on $K_1 \times \cdots \times K_n$ the multilinear operator*

$$\odot\colon C(K_1) \times \cdots \times C(K_n) \to L_\varphi(K_1 \times \cdots \times K_n, \mu)$$

is φ semi-integral with $\pi_{si,\varphi}(\odot) \le 1$.

Proof. Fix $f_j \in C(K_j)$ for each $1 \le j \le n$ such that $\lambda := \|\odot(f_1, \ldots, f_n)\|_{L_\varphi(\mu)} > 0$. Since $\odot(f_1, \ldots, f_n)$ is a continuous function on $K_1 \times \cdots \times K_n$, it follows that

$$\int_{K_1\times\cdots\times K_n} \varphi(|\odot(f_1,\ldots,f_n)|/\lambda)\,d\mu = 1.$$

Let $B_{C(K_1)^*}\times\cdots\times B_{C(K_n)^*}$ be equipped with the product of the weak* topologies. Consider the mapping $\delta\colon K_1\times\cdots\times K_n \to B_{C(K_1)^*}\times\cdots\times B_{C(K_n)^*}$ given by

$$\delta(s) := (\delta_{s_1},\ldots,\delta_{s_n}), \quad s=(s_1,\ldots,s_n)\in K_1\times\cdots\times K_n\,.$$

Clearly, δ is continuous and one-to-one (and so δ is a homeomorphism onto $\delta(K_1\times\cdots\times K_n)$). Thus, δ is a Borel mapping. Let $\nu := (\delta)\mu$ be the image measure of μ (via δ) on Borel sets of $B_{C(K_1)^*}\times\cdots\times B_{C(K_n)^*}$. Then we have

$$\begin{aligned} 1 &= \int_{K_1\times\cdots\times K_n} \varphi(|\odot(f_1,\ldots,f_n)|/\lambda)\,d\mu \\ &= \int_{K_1\times\cdots\times K_n} \varphi\big(|\odot\big(\kappa_{C(K_1)}(f_1),\ldots,\kappa_{C(K_n)}(f_n)\big)(\delta(s))|/\lambda\big)\,d\mu \\ &\le \int_{B_{C(K_1)^*}\times\cdots\times B_{C(K_n)^*}} \varphi(|\langle f_1,\cdot\rangle\cdots\langle f_n,\cdot\rangle|/\lambda)\,d\nu\,, \end{aligned}$$

whence

$$\|\odot(f_1,\ldots,f_n)\|_{L_\varphi(\mu)} \le \|\langle f_1,\cdot\rangle\cdots\langle f_n,\cdot\rangle\|_{L_\varphi(\nu)}\,.$$

This shows that $\odot$ is φ semi-integral with $\pi_{si,\varphi}(\odot)\le 1$. □

Further, we will prove vector-valued inequalities for φ semi-integral multilinear operators generated by Orlicz functions satisfying some minor conditions. As a by-product, we deduce that these operators are φ-summing. In the proof we will use the following characterization of the embeddings between mixed Orlicz spaces and Orlicz spaces defined on a product of measure spaces: Let $(\mathcal{X}_1,\Sigma_1,\nu)$ and $(\mathcal{X}_2,\Sigma_2,\mu)$ be σ-finite measure spaces. Then the inclusion map

$$\mathrm{id}\colon L_\varphi(\nu\times\mu)\hookrightarrow L_\varphi(\nu)[L_\varphi(\mu)]$$

is bounded with norm $\|\mathrm{id}\|\le C$ if and only if φ is C-supermultiplicative (i.e., $\varphi(Cuv)\ge\varphi(u)\varphi(v)$ for all $u,v>0$ and some $C>0$). If additionally μ and ν are finite measures, then the above statement holds if and only φ is C-supermultiplicative at ∞ (i.e., there exists $a>0$ such that $\varphi(Cuv)\ge\varphi(u)\varphi(v)$ for all $u,v\ge a$).

Let us say that the above result was proven for the case of non-atomic finite measures in [26]. A minor modification of the proof gives the result for the general case.

We need the following lemma.

Lemma 3.8. *Let $(\mathcal{X}_1, \Sigma_1, \nu)$ and $(\mathcal{X}_2, \Sigma_2, \mu)$ be σ-finite measure spaces with μ finite. Suppose that φ is a C-supermultiplicative Orlicz function. Then the inclusion map*

$$\mathrm{id}\colon [L_\varphi(\nu)]L_\infty(\mu) \hookrightarrow L_\varphi(\nu)[L_\varphi(\mu)]$$

is bounded with $\|\mathrm{id}\| \le C\max\{1, \mu(\mathcal{X}_2)\}$. Moreover, if μ and ν are finite measures, then the above statement holds whenever φ is C-supermultiplicative at ∞.

Proof. For any $f \in [L_\varphi(\nu)]L_\infty(\mu)$ with norm 1, we have $\|f^t\|_{L_\varphi(\nu)} \le 1$ for μ-almost all $t \in \mathcal{X}_2$. Hence, for μ-almost all $t \in \mathcal{X}_2$,

$$\int_{\mathcal{X}_1} \varphi(|f^t(s)|)\, d\nu(s) \le \int_{\mathcal{X}_1} \varphi\bigg(\frac{|f^t(s)|}{\|f^t\|_{L_\varphi(\nu)}}\bigg)\, d\nu(s) \le 1\,.$$

Combining with Fubini's Theorem, it follows that

$$\int_{\mathcal{X}_1\times\mathcal{X}_2} \varphi(|f|)\, d(\nu\times\mu) = \int_{\mathcal{X}_2}\bigg(\int_{\mathcal{X}_1} \varphi(|f(s,t)|)\, d\nu(s)\bigg)\, d\mu(t) \le \max\{1, \mu(\mathcal{X}_2)\}\,.$$

This shows that $f \in L_\varphi(\nu\times\mu)$ with norm less than or equal to $\max\{1, \mu(\mathcal{X}_2)\}$. To finish, it is enough to apply the result mentioned above. ∎

We are ready to prove a vector-valued estimate for φ semi-integral multilinear operators which seems to be of independent interest.

Theorem 3.9. *Let $T\colon X_1 \times \cdots \times X_n \to Y$ be a φ semi-integral multilinear operator with $\pi_{si,\varphi}(T) \le M$, where φ is a C-supermultiplicative Orlicz function, and let $(\Omega_j, \Sigma_j, \nu_j)$ be σ-finite measure spaces for each $1 \le j \le n$. Suppose that $L_\varphi(\nu)$ is an Orlicz space in $L^0(\Omega_1 \times \cdots \times \Omega_n, \nu)$ with $\nu = \nu_1 \times \cdots \times \nu_n$. Then the following vector-valued estimate holds for $A = CM$:*

$$\big\|\|T(f_1(\cdot), \ldots, f_n(\cdot))\|_Y\big\|_{L_\varphi(\nu)} \le A \sup_{(x_1^*,\ldots,x_n^*)\in B_{X_1^*}\times\cdots\times B_{X_n^*}} \|\langle f_1(\cdot), x_1^*\rangle \cdots \langle f_n(\cdot), x_n^*\rangle\|_{L_\varphi(\nu)}$$

for all $(f_1, \ldots, f_n) \in L^0(\nu_1, X_1) \times \cdots \times L^0(\nu_n, X_n)$ such that the value of the expression on the right hand side of the inequality is finite. If the measures ν_j are finite for each $1 \le j \le n$, then the above statement is true if φ is supermultiplicative at ∞.

Proof. We assume that φ is C-supermultiplicative for a given $C > 0$; that is, there is $C > 0$ such that $\varphi(Cuv) \ge \varphi(u)\varphi(v)$ for all $u, v > 0$. Our hypothesis implies that there exists a probability Borel measure μ on $B_{X_1^*} \times \cdots \times B_{X_n^*}$ equipped with the product of the weak* topologies such that for every $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$,

$$\|T(x_1, \ldots, x_n)\|_Y \le M\|\langle x_1, \cdot\rangle \cdots \langle x_n \cdot\rangle\|_{L_\varphi(\mu)}\,.$$

An application of Lemma 3.8 gives the required estimate. Indeed,

$$\begin{aligned}\big\|\|T(f_1(\cdot),\dots,f_n(\cdot))\|_Y\big\|_{L_\varphi(\nu)} &\le M\big\|\langle f_1(\cdot),\cdot\rangle\cdots\langle f_n(\cdot),\cdot\rangle\big\|_{L_\varphi(\nu)[L_\varphi(\mu)]}\\ &\le M\big\|\langle f_1(\cdot),\cdot\rangle\cdots\langle f_n(\cdot),\cdot\rangle\big\|_{[L_\varphi(\nu)]L_\infty(\mu)}\\ &= CM\sup_{(x_1^*,\dots,x_n^*)\in B_{X_1^*}\times\cdots\times B_{X_n^*}}\big\|\langle f_1(\cdot),x_1^*\rangle\cdots\langle f_n(\cdot),x_n^*\rangle\big\|_{L_\varphi(\nu)}.\end{aligned}$$

These computations together, adapted by means of a standard argument, give the proof also for the case of finite measures. □

The following result is an immediate consequence of Theorems 3.1 and 3.9.

Corollary 3.10. *Let φ be a C-supermultiplicative Orlicz function at ∞ for some $C>0$. Then a multilinear operator $T\colon X_1\times\cdots\times X_n\to Y$ is φ summing if and only if T is φ semi-integral.*

Combining Theorems 3.1, 3.2 and 3.9, we obtain the following variant of Pietsch's characterization of φ-summing multilinear operators.

Theorem 3.11. *Let φ be a normalized Orlicz function which is 1-supermultiplicative at ∞. The following statements are equivalent for a multilinear operator $T\colon X_1\times\cdots\times X_n\to Y$.*

(i) *T is φ-summing with $\pi_\varphi(T)\le C$;*
(ii) *T is φ semi-integral with $\pi_{si,\varphi}(T)\le C$;*
(iii) *For every (equivalently, for some) isometric embeddings $J_j\colon X_j\to C(K_j)$ for each $1\le j\le n$ there exist a Borel probability measure μ on the product $K_1\times\cdots\times K_n$ of compact Hausdorff spaces and a constant $C>0$ such that, for all $(x_1,\dots,x_n)\in X_1\times\cdots\times X_n$,*

$$\|T(x_1,\dots,x_n)\|_Y\le C\|\odot(J_1(x_1),\dots,J_n(x_n))\|_{L_\varphi(\mu)}.$$

We conclude this section with a result related to strongly φ-summing multilinear operators. The next definition is motivated by Dimant's paper [11], where the concept of strongly p-summing multilinear operator was introduced.

Let $X_1,\dots,X_n$, Y be Banach spaces. Following the definition that was given in [11] for the case of ℓ_p-spaces, we say that an n-linear operator $T\colon X_1\times\cdots\times X_n\to Y$ is *strongly φ-summing* if there exists a constant $C>0$ such that

$$\big\|\big\{\|T(x_j^1,\dots,x_j^n)\|_Y\big\}_{j=1}^m\big\|_{\ell_\varphi^m(\mu)}\le C\sup_{\phi\in B_{\mathcal{L}(X_1,\dots,X_n)}}\big\|\big\{\|\phi(x_j^1,\dots,x_j^n)\|\big\}_{j=1}^m\big\|_{\ell_\varphi^m(\mu)}$$

for every m-dimensional Orlicz space $\ell_\varphi^m(\mu)$ with a probability measure μ on $[m]$ and all $x_j^i\in X_i$ for each $1\le i\le n$ and $1\le j\le m$. The least constant satisfying the above requirements is denoted by $\pi_{S,\varphi}(T)$.

A multilinear operator $T\colon X_1\times\cdots\times X_n \to Y$ is said to be *strongly φ semi-integral* if there exists a constant $C>0$ such that for every $(x_1,\dots,x_n)\in X_1\times\cdots\times X_n$, there is a Borel probability measure μ on $B_{\mathcal{L}(X_1,\dots,X_n)}$ endowed with the weak* topology such that

$$\|T(x_1,\dots,x_n)\|_Y \le C\|\phi(x_1,\dots,x_n)\|_{L_\varphi(\mu)}\,.$$

We can state the following Domination Theorem for strongly φ-summing multilinear operators. The proof is similar to the proofs of Theorems 3.1 and 3.9, and so we skip it.

Theorem 3.12. *Let φ be a normalized Orlicz function. Suppose that $T\colon X_1\times\cdots\times X_n \to Y$ is a strongly φ-summing multilinear operator. Then T is strongly φ semi-integral, that is, there exists a regular Borel probability measure μ on $B_{\mathcal{L}(X_1,\dots,X_n)}$ equipped with the weak* topology so that for every $(x_1,\dots,x_n)\in X_1\times\cdots\times X_n$,*

$$\|T(x_1,\dots,x_n)\|_Y \le C\pi_{S,\varphi}(T)\|\phi(x_1,\dots,x_n)\|_{L_\varphi(\mu)}\,.$$

Moreover, the converse also holds if φ is supermultiplicative at ∞.

4. φ-dominated multilinear operators

In this section we introduce a new class of operators that are directly related to the ones presented in the previous section, in order to analyze suitable factorization schemes for these classes. For a given Orlicz function φ, a multilinear operator $T\colon X_1\times\cdots\times X_n \to Y$ is said to be φ-dominated if there is a constant C and Borel probability measures μ_j on $B_{X_j^*}$ endowed with the weak* topology for each $1\le j\le n$ such that for all $x_1\in X_1,\dots,x_n\in X_n$, we have

$$\|T(x_1,\dots,x_n)\|_Y \le C\|\langle x_1,\cdot\rangle\|_{L_\varphi(\mu_1)}\cdots\|\langle x_n,\cdot\rangle\|_{L_\varphi(\mu_n)}\,.$$

We write $\mathcal{L}^d_\varphi(X_1,\dots,X_n,Y)$ for the space of all φ-dominated multilinear operators from $X_1\times\cdots\times X_n$ into Y, and we denote by $\pi^d_\varphi(T)$ the least constant C satisfying the above requirements. We note that in the case when $\varphi(t)=t^p$ for all $t\ge 0$ with $1\le p<\infty$, we recover the existing notions: p-dominated linear operators (see [10, p. 188]) and p-dominated multilinear mappings (see [17, 18]).

Similarly to what happens in the case of p-dominated operators, φ-dominated multilinear maps admit factorization.

Theorem 4.1. *Let $T\colon X_1\times\cdots\times X_n \to Y$ be a multilinear map. Then the following statements are equivalent.*

(i) *T is φ-dominated;*

(ii) *T admits a factorization*:

$$\begin{tikzcd} X_1 \times \cdots \times X_n \arrow[r, "T"] \arrow[d, dotted, "i_1\times\cdots\times i_n"'] & Y \\ S_1 \times \cdots \times S_n \arrow[ur, dotted, "\widehat{T}"] & \end{tikzcd}$$

where $\widehat{T}$ is a multilinear operator, $S_1, \ldots, S_n$ are the subspaces of $C(B_{X_j^*})$ for each $1 \le j \le n$ defined by the functions $\{\langle x_j, \cdot\rangle;\, x_j \in X_j\}$ with the Orlicz norms $\|\cdot\|_{L_\varphi(\mu_j)}$ and i_j are the maps $X_j \ni x_j \mapsto \langle x_j, \cdot\rangle$;

(iii) *T admits a factorization*:

$$\begin{tikzcd} X_1 \times \cdots \times X_n \arrow[r, "T"] \arrow[d, dotted, "u_1\times\cdots\times u_n"'] & Y \\ E_1 \times \cdots \times E_n \arrow[ur, dotted, "\widehat{T}"] & \end{tikzcd}$$

where $E_1, \ldots, E_n$ are Banach spaces and $u_1, \ldots, u_n$ are φ semi-integral linear operators.

Proof. Let us prove (i) $\Rightarrow$ (ii). Suppose that T is φ-dominated. Then, for each $1 \le j \le n$ there exists a Borel probability measure μ_j on $B_{X_j^*}$ endowed with the weak* topology, such that for all $(x_1, \ldots, x_n) \in X_1 \times \cdots X_n$,

$$\|T(x_1, \ldots, x_n)\|_Y \le C \|\langle x_1, \cdot\rangle\|_{L_\varphi(\mu_1)} \cdots \|\langle x_n, \cdot\rangle\|_{L_\varphi(\mu_n)}\,.$$

We point out that the "inclusion" maps $i_1, \ldots, i_n$ are not injective in general. However, it is still possible to define a multilinear operator that closes the commutative diagram.

For each $1 \le j \le n$, we consider the subspace of X_j defined by

$$N_j := \{x_j \in X_j;\, \|\langle x_j, \cdot\rangle\|_{L_\varphi(\mu_j)} = 0\}\,.$$

Now define the quotient spaces X_j/N_j endowed with the quotient norm associated to the seminorm $X_j \ni x_j \mapsto \|\langle x_j, \cdot\rangle\|_{L_\varphi(\mu_j)}$ for each $1 \le j \le n$. The domination with the product of the seminorms shows that in fact the value $T(x_1, \ldots, x_n)$ does not depend on the equivalence class of each x_j, that is,

$$T(x_1, \ldots, x_n) = T(x_1', \ldots, x_n')$$

whenever $x_j' \in [x_j]$ for each $1 \le j \le n$. Therefore, we can define the multilinear mapping $\widehat{T}\colon L_\varphi(\mu_1) \times \cdots \times L_\varphi(\mu_n) \to Y$ by

$$\widehat{T}(\langle x_1, \cdot\rangle, \ldots, \langle x_1, \cdot\rangle) = T(x_1, \ldots, x_n)$$

for all $(x_1, \dots, x_n) \in L_\varphi(\mu_1) \times \cdots \times L_\varphi(\mu_n)$. Now observe that the continuity of $\widehat{T}$ follows from the inequality shown above for T. Clearly, $T = \widehat{T} \circ (i_1 \times \cdots \times i_n)$ and so T admits the required factorization.

(ii) $\Rightarrow$ (iii). We only have to take into account that by Lemma 3.7, the operators i_j are φ semi-integral.

(iii) $\Rightarrow$ (i). Since $u_1, \dots, u_n$ are φ semi-integral, a direct calculation using the factorization yields that T is φ-dominated. $\square$

In what follows we will present general examples of φ-dominated operators. We need a minor definition. For any finite m-dimensional Banach lattice E on $[m]$ we define an associate lattice space E' equipped with the dual norm

$$\|\{\xi_j\}\|_{E'} := \sup\left\{\Big|\sum_{j=1}^{m} \xi_j\eta_j\Big|;\ \|\{\eta_j\}\|_E \le 1\right\}.$$

Clearly, by reflexivity we have $(E')' = E$ with equality of norms.

Lemma 4.2. *Let $1 \le p < \infty$ and let φ be an Orlicz function. Define ϕ by $\phi(t) = \varphi(t^{1/p})$ for all $t \ge 0$, and suppose that ϕ is also an Orlicz function. Then every p-summing operator $T\colon X \to Y$ between Banach spaces is also φ-summing with $\pi_\varphi(T) \le \pi_p(T)$, and so is φ semi-integral.*

Proof. For a given $C > \pi_p(T)$, by Pietsch's domination theorem there exists a Borel probability measure μ on B_{X^*} endowed with the weak* topology so that

$$\|Tx\|_Y \le C\Big(\int_{B_{X^*}} |\langle x, x^*\rangle|^p \, d\mu\Big)^{1/p}, \quad x \in X.$$

Then, for every m-dimensional Orlicz space $\ell_\varphi^m(\nu)$ with a probability measure ν on $[m]$ and all $x_j^i \in X_i$ for each $1 \le i \le n$, $1 \le j \le m$, we have

$$\begin{aligned}
\big\|\{\|Tx_j\|_Y\}\big\|_{\ell_\varphi^m(\mu)} &\le C\Big\|\Big\{\Big(\int_{B_{X^*}} |\langle x_j, x^*\rangle|^p\,d\mu\Big)^{1/p}\Big\}\Big\|_{\ell_\varphi^m(\nu)} \\
&= C\Big\|\Big\{\int_{B_{X^*}} |\langle x_j, x^*\rangle|^p\,d\mu\Big\}\Big\|_{\ell_\phi^m(\nu)}^{1/p} \\
&= C\Big(\sup_{\|\{\eta_j\}\|_{\ell_\phi^m(\mu)'}\le 1}\Big|\sum_{j=1}^m \eta_j \int_{B_{X^*}} |\langle x, x^*\rangle|^p\,d\mu\Big|\Big)^{1/p} \\
&\le C\sup_{x^*\in B_{X^*}}\Big(\sup_{\|\{\eta_j\}\|_{\ell_\phi^m(\mu)'}\le 1}\sum_{j=1}^m |\eta_j||\langle x_j, x^*\rangle|^p\Big)^{1/p} \\
&= C\sup_{x^*\in B_{X^*}}\big\|\{|\langle x_j, x^*\rangle|^p\}\big\|_{\ell_\phi^m(\nu)}^{1/p} \\
&= C\sup_{x^*\in B_{X^*}}\big\|\{\langle x_j, x^*\rangle\}\big\|_{\ell_\varphi^m(\nu)}.
\end{aligned}$$

Since $C > \pi_p(T)$ is arbitrary, T is φ-summing with $\pi_\varphi(T) \le \pi_p(T)$. This, together with Theorem 3.2, completes the proof. $\square$

In order to provide some examples of the classes we have just introduced, let us recall a well-known definition. Following Pisier [22], we say that a Banach space X is a *GT space*, or that X satisfies Grothendieck's Theorem, if there exists a constant $K > 0$ such that every operator $u\colon X \to \ell_2$ is 1-summing and satisfies $\pi_1(u) \le K\|u\|$. We denote the least such constant K by $\mathrm{GT}(X)$. By Grothendieck's Theorem, $\mathcal{L}_1$ spaces are GT spaces. It is well known that the quotient $L_1(\mathbb{T})/H_1(\mathbb{T})$ and L_1/R are GT spaces, where $H_1(\mathbb{T})$ is the Hardy space and R is any reflexive subspace of an L_1-space (see [22]).

We note that it follows from Lemma 4.2 that every 1-summing operator $u\colon X \to Y$ between Banach spaces is φ-summing for any Orlicz function φ, with $\pi_\varphi(u) \le \pi_1(u)$. In particular, this implies that for any GT space X, every operator $u\colon X \to \ell_2$ is φ-summing with $\pi_\varphi(u) \le GT(\mathrm{X})\|u\|_{X\to\ell_2}$.

The following corollary can yield some concrete examples of φ-dominated multilinear operators. The proof follows from Theorem 4.1 combined with Lemma 4.2.

Corollary 4.3. *Let φ be an Orlicz function and let $1 \le p < \infty$. Suppose that the function ϕ given by $\phi(t) = \varphi(t^{1/p})$ for all $t \ge 0$ is equivalent to an Orlicz function. Assume that $u_j\colon X_j \to Y_j$ is a p-summing operator between Banach spaces for each $1 \le j \le n$. Then, for every multilinear operator*

$S\colon Y_1 \times \cdots \times Y_n \to Z$, $T = S \circ (u_1, \ldots, u_n)$ is a φ-dominated multilinear operator.

Now we introduce a useful definition. Let $K_1, \ldots, K_n$ be compact Hausdorff topological spaces, and let $\mu_1, \ldots, \mu_n$ be a set of regular Borel measures on them, respectively. Let μ be a regular Borel measure on the compact space $K_1 \times \cdots \times K_n$. We say that μ is *Riesz representable via* $\mu_1, \ldots, \mu_n$ if for every function $f \in C(K_1 \times \cdots \times K_n)$, we have that

$$\int_{K_1\times\cdots\times K_n} f\,d\mu = \int_{K_n} \Big(\cdots \Big(\int_{K_1} f\,d\mu_1 \Big) \cdots \Big)\,d\mu_n\,.$$

We will need the following observation. For the sake of completeness we include a proof.

Proposition 4.4. *Let $G_1, \ldots, G_n$ be compact topological groups. Then the normalized Haar measure on $G_1\times\cdots\times G_n$ is Riesz representable via the normalized Haar measures $\mu_1, \ldots, \mu_n$ on each group $G_1, \ldots, G_n$, respectively.*

Proof. Observe that the positive linear functional F on $C(G_1 \times \cdots \times G_n)$ given by an iterated integral:

$$F(f) := \int_{G_n} \Big(\cdots \Big(\int_{G_1} f\,d\mu_1 \Big) \cdots \Big)\,d\mu_n, \quad f \in C(G_1 \times \cdots \times G_n)$$

has norm 1. Thus, it follows from the Riesz representation theorem that there is a regular Borel probability measure μ on $G_1 \times \cdots \times G_n$ such that, for all $f \in C(G_1 \times \cdots \times G_n)$, we have

$$\int_{G_1\times\cdots\times G_n} f\,d\mu = \int_{G_n} \Big(\cdots \Big(\int_{G_1} f\,d\mu_1 \Big) \cdots \Big)\,d\mu_n\,.$$

Clearly, F is a translation invariant functional and so μ is the (unique) normalized Haar measure on $G_1\times\cdots\times G_n$ having the required property. $\square$

Proposition 4.5. *Let $T\colon X_1\times\cdots\times X_n \to Y$ be a φ semi-integral multilinear operator, where φ is a submultiplicative Orlicz function. Fix $C > 0$ such that $\varphi(st) \leq C\varphi(s)\varphi(t)$ for all $s, t > 0$. Suppose that the domination is given by a Riesz representable measure. Then T is φ-dominated with $\pi_\varphi^d(T) \leq 2C\pi_\varphi(T)$.*

Proof. By hypothesis there exists a Borel probability measure $\mu \in \mathcal{M}(B_{X_1^*}\times \cdots \times B_{X_n^*})$ such that for all $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$,

$$\|T(x_1, \ldots, x_n)\|_Y \leq 2\pi_\varphi(T)\|\langle x_1, \cdot\rangle \cdots \langle x_n, \cdot\rangle\|_{L_\varphi(\mu)}\,.$$

For each $1 \leq j \leq n$, we take $x_j \in X_j$ such that $\|\langle x_j, \cdot\rangle\|_{L_\varphi(\mu_j)} \leq 1$. Then we have

$$\int_{B_{X_j^*}} \varphi(|\langle x_j, \cdot\rangle|)\,d\mu_j \leq 1, \quad 1 \leq j \leq n\,,$$

and so by the fact that φ is a submultiplicative function, it follows that

$$\begin{aligned}\int_{B_{X_1^*}\times\cdots\times B_{X_n^*}} &\varphi(|\odot(\langle x_1,\cdot\rangle,\ldots,\langle x_n,\cdot\rangle)|)\,d\mu \\ &= \int_{B_{X_n^*}}\cdots\int_{B_{X_1^*}} \varphi(|\langle x_1,\cdot\rangle|\cdots|\langle x_n,\cdot\rangle|)\,d\mu_1\ldots\,d\mu_n \\ &\le C\prod_{j=1}^n \int_{B_{X_j^*}} \varphi(|\langle x_j,\cdot\rangle|)\,d\mu_j \le C\,,\end{aligned}$$

where $\mu_1,\ldots,\mu_n$ are the measures that represent μ.

Since $C\ge 1$, it follows by convexity of φ that

$$\int_{B_{X_1^*}\times\cdots\times B_{X_n^*}} \varphi(|\odot(\langle x_1,\cdot\rangle,\ldots,\langle x_n,\cdot\rangle)|/C)\,d\mu \le 1\,,$$

so

$$\|\langle x_1,\cdot\rangle\cdots\langle x_n,\cdot\rangle\|_{L_\varphi(\mu)} \le C\,.$$

Combining the inequalities above yields that for all $x_j\in X_j$ with $\|\langle x_j,\cdot\rangle\|_{L_\varphi(\mu_j)} > 0$,

$$\Big\|T\Big(\frac{x_1}{\|\langle x_1,\cdot\rangle\|_{L_\varphi(\mu_1)}},\ldots,\frac{x_n}{\|\langle x_n,\cdot\rangle\|_{L_\varphi(\mu_n)}}\Big)\Big\|_Y \le 2C\pi_\varphi(T)\,.$$

This completes the proof. □

5. The Haar measure for φ semi-integral multilinear operators

We start this section with the remark that in general little can be said about the measures which appear in the domination theorem for φ-summing multilinear operators (Theorem 3.1). The reason is that, as in the case of the Pietsch's measures that appear in Pietsch's domination theorem for p-absolutely summing operators, the measure comes into the proof from abstract existence results: a combination of the Hahn–Banach theorem with the Riesz representation theorem for the dual of $C(K)$-spaces.

The aim of this section is to study some classes of φ semi-integral operators on products of a special type of subspaces of $C(K)$-spaces, and the Pietsch's measures that appear in the Domination Theorem for these operators. As a by-product of some general results, we obtain that for translation invariant φ semi-summing multilinear operators defined on a finite product of $C(G)$-spaces on compact topological groups, the normalized Haar measure is a Pietsch's measure.

We recall that a compact topological group $G=(G,\cdot)$ is said to act as a group of homeomorphisms of a compact topological space K if to every

$g \in G$ corresponds a homeomorphism $i_g \colon K \to K$ such that

$$i_{g\cdot h} = i_g \circ i_h, \quad g \in G, \ h \in G\,,$$

and also the mapping $(g,k) \mapsto i_g(k)$ of (G,K) into K is continuous. For a function $f \in C(K)$ and $g \in G$, we define a map $I_g \colon C(K) \to C(K)$ by $I_g f = f \circ i_g$ for all $f \in C(K)$. A closed subspace X of $C(K)$ is said to be *invariant* if, for all $g \in G$, we have $I_g(X) \subset X$.

Before we state and prove the main result, we fix some notation to simplify the presentation. Let $G_1, \ldots, G_n$ be topological compact groups and let $K_1, \ldots, K_n$ be compact spaces. Suppose that G_j acts as a group of homeomorphisms of K_j for each $1 \leq j \leq n$. For all $g_j \in G_j$ and each $1 \leq j \leq n$, we denote by i_{g_j} all corresponding homeomorphisms on the compact space K_j and all the maps $I^j_{g_j} \colon C(K_j) \to C(K_j)$.

Clearly, the compact group $G := G_1 \times \cdots \times G_n$ acts as a group of homeomorphisms on the compact space $K := K_1 \times \cdots \times K_n$. To see this, observe that for every $g = (g_1, \ldots, g_n) \in G$, the map $i^\times_g \colon K \to K$ given by

$$i^\times_g(k_1, \ldots, k_n) := (i_{g_1}(k_1), \ldots, i_{g_n}(k_n)), \quad (k_1, \ldots, k_n) \in K_1 \times \cdots \times K_n$$

is a homeomorphism. Observe that $i^\times_{g\cdot h} = i^\times_g \circ i^\times_h$ for all $g \in G$, $h \in G$ and also the mapping $(g,k) \mapsto i^\times_g(k)$ of $G \times K$ into K is continuous.

In what follows, we assume that $F_j \subset C(K_j)$ is a closed invariant subspace with respect to $I^j_{g_j}$ for all $g_j \in G_j$ and for each $1 \leq j \leq n$. In this setting, a multilinear operator T from $F_1 \times \cdots \times F_n$ into a Banach space X is called *translation invariant* if, for all $(g_1, \ldots, g_n) \in G_1 \times \cdots \times G_n$ and for all $(f_1, \ldots, f_n) \in F_1 \times \cdots \times F_n$, we have

$$\left\| T(I^1_{g_1}(f_1), \ldots, I^n_{g_n}(f_n)) \right\|_Y = \|T(f_1, \ldots, f_n)\|_X\,.$$

Using this notation and assuming these requirements on the subspaces F_i we are ready to state the following theorem.

Theorem 5.1. *Suppose that $T \colon F_1 \times \cdots \times F_n \to X$ is a translation invariant φ-summing multilinear operator, where φ is a normalized Orlicz function. Then there exists a regular Borel probability measure μ on $K_1 \times \cdots \times K_n$ such that, for all $(f_1, \ldots, f_n) \in F_1 \times \cdots \times F_n$,*

$$\|T(f_1, \ldots, f_n)\|_X \leq \pi_\varphi(T)\, \| \odot (f_1, \ldots, f_n)\|_{L_\varphi(\mu)}\,,$$

and $(i^\times_g)\mu = \mu$ for all $g \in G_1 \times \cdots \times G_n$, where $(i^\times_g)\mu$ is the image measure of μ via $i^\times_g$.

Proof. Without loss of generality we may assume that $\pi_\varphi(T) = 1$. By Theorem 3.1, there exists a regular Borel probability measure ν on $K := K_1 \times$

$\cdots \times K_n$ such that for all $(f_1, \ldots, f_n) \in F_1 \times \cdots \times F_n$ with $\|T(f_1, \ldots, f_n)\|_X \neq 0$, we have

$$1 \leq \int_K \varphi\Big(\frac{|(f_1 \cdots f_n)(k)|}{\|T(f_1, \ldots, f_n)\|_X}\Big)\, d\nu\,.$$

Let $G := G_1 \times \cdots \times G_n$ and let $\phi\colon G \times K \to K$ be the continuous map given by

$$\phi(g, k) := i_g^{\times}(k), \quad (g, k) \in G \times K\,.$$

Since the operator T and the spaces $F_1, \ldots, F_n$ are translation invariant, for all $g = (g_1, \ldots, g_n) \in G$ and all $(f_1, \ldots, f_n) \in F_1 \times \cdots \times F_n$, we have

$$1 \leq \int_K \varphi\Big(\frac{|f_1 \circ i_{g_1}(k_1) \cdots f_n \circ i_{g_n}(k_n)|}{\|T(f_1 \circ i_{g_1}, \ldots, f_n \circ i_{g_n})\|_X}\Big)\, d\nu = \int_K \varphi\Big(\frac{|(f_1 \cdots f_n) \circ \phi(g, k)|}{\|T(f_1, \ldots, f_n)\|_X}\Big)\, d\nu\,.$$

Clearly, the non-negative function $G \times K \ni (g, k) \mapsto \varphi\Big(\frac{|(f_1 \cdots f_n) \circ \phi(g,k)|}{\|T(f_1, \ldots, f_n)\|_X}\Big)$ is continuous. Thus, it follows from Fubini's Theorem that $G \ni g \mapsto \int_K \varphi\Big(\frac{|(f_1 \cdots f_n) \circ \phi(g,k)|}{\|T(f_1, \ldots, f_n)\|_X}\Big)\, d\nu$ is Borel measurable and

$$\int_G \bigg(\int_K \varphi\Big(\frac{|(f_1 \cdots f_n) \circ \phi(g, k)|}{\|T(f_1, \ldots, f_n)\|_X}\Big)\, d\nu\bigg)\, dm = \int_{G \times K} \varphi\Big(\frac{|(f_1 \cdots f_n) \circ \phi|}{\|T(f_1, \ldots, f_n)\|_X}\Big)\, d(m \times \nu)\,,$$

where m is the normalized Haar measure on G. Combining with the above inequality yields

$$1 \leq \int_{G \times K} \varphi\Big(\frac{|(f_1 \cdots f_n) \circ \phi|}{\|T(f_1, \ldots, f_n)\|_X}\Big)\, d(m \times \nu)\,.$$

Now we use the image measure $\phi(m \times \nu)$ on Borel sets of $K_1 \times \cdots \times K_n$ to get that

$$1 \leq \int_{K_1 \times \cdots \times K_n} \varphi\Big(\frac{|\odot(f_1, \ldots, f_n)|}{\|T(f_1, \ldots, f_n)\|_X}\Big)\, d\phi(m \times \nu)\,,$$

where $\phi(m \times \nu)(A) := (m \times \nu)(\phi^{-1}(A))$ for all A in the product of Borel σ-algebras in G and K. For these A, we have $\phi^{-1}(A)_g = (i_g^{\times})^{-1}(A)$ for all $g \in G$. Combining with the formula for the product measure $m \times \nu$, we get

$$\phi(m \times \nu)(A) = \int_G \nu\big(\phi^{-1}(A)_g\big)\, dm(g) = \int_G (i_g^{\times})\nu(A)\, dm(g)\,.$$

Since m is the normalized Haar measure on G and the homeomorphisms $i_s^{\times}$ satisfy $i_s^{\times} \circ i_t^{\times} = i_{st}^{\times}$ for all $s,\ t \in G$, it follows that the measure μ that is defined for any Borel set A in $K_1 \times \cdots \times K_n$ by

$$\mu(A) = \int_G (i_g^{\times})\nu(A)\, dm(g)\,,$$

satisfies $(i_s^\times)\mu = \mu$ for all $s \in G$. To summarize, we conclude that

$$1 \leq \int_{K_1 \times \cdots \times K_n} \varphi\left(\frac{|\odot(f_1, \ldots, f_n)|}{\|T(f_1, \ldots, f_n)\|_X} \right) d\mu \,.$$

This implies

$$\|T(f_1, \ldots, f_n)\|_Y \leq \|\odot(f_1, \ldots, f_n)\|_{L_\varphi(\mu)}$$

and so the desired statement follows. □

As an application, we obtain the following multilinear version of Pietsch's domination theorem.

Corollary 5.2. *Let X be a Banach space, $G_1, \ldots, G_n$ compact topological groups and $F_1, \ldots, F_n$ closed translation invariant subspaces of $C(G_1), \ldots, C(G_n)$, respectively. Suppose that $T\colon F_1 \times \cdots \times F_n \to X$ is a translation invariant φ-summing multilinear operator generated by the normalized Orlicz function φ. Then the normalized Haar measure μ on $G_1 \times \cdots \times G_n$ is a Pietsch's measure for T, i.e., for every $(f_1, \ldots, f_n) \in F_1 \times \cdots \times F_n$,*

$$\|T(f_1, \ldots, f_n)\|_X \leq \pi_\varphi(T)\, \|\odot(f_1, \ldots, f_n)\|_{L_\varphi(\mu)} \,.$$

Proof. For each $1 \leq j \leq n$ and every $g_j \in G_j$, we define the homeomorphism i_{g_j} on G_j, by $i_{g_j}(h) = g_j h$ for all $h \in G_j$. Clearly, the map $(g_j, h_j) \mapsto i_{g_j} h_j$ is continuous from $G_j \times G_j$ to G_j for each $1 \leq j \leq n$. From the definition of the topological group $G_1 \times \cdots \times G_n$, for all $g = (g_1, \ldots, g_n)$, $h = (h_1, \ldots, h_n) \in G_1 \times \cdots \times G_n)$, we have

$$gh := (g_1 h_1, \ldots, g_n h_n) = (i_{g_1}(h_1), \ldots, i_{g_n}(h_n)) = i_g^\times(h) \,.$$

Observe that $i_{g\cdot h}^\times = i_g^\times \circ i_h^\times$ for all $g \in G$ and $h \in G$. This shows that a compact group $G := G_1 \times \cdots \times G_n$ acts as a group of homeomorphisms on G. By Theorem 5.1, there is a probability Borel measure μ on G, which satisfies the required estimate, and such that

$$(i_g^\times)\mu = \mu \quad \text{for all} \quad g \in G \,.$$

This equality ensures that μ is the normalized Haar measure. □

In the spirit of this section, we finish with the following factorization result.

Theorem 5.3. *Let $G_1, \ldots, G_n$ be compact topological groups and let φ be a submultiplicative Orlicz function. Consider a translation invariant φ-summing multilinear operator*

$$T\colon F_1 \times \cdots F_n \to X \,,$$

where $F_1 \subset C(G_1), \ldots, F_n \subset C(G_n)$ are closed invariant subspaces. Then T admits the following factorization:

$$\begin{tikzcd} F_1 \times \cdots \times F_n \arrow[rr, "T"] \arrow[d, dotted, "i_1 \times \cdots \times i_n"'] & & X, \\ (F_1)_{L_\varphi(\mu_1)} \times \cdots \times (F_n)_{L_\varphi(\mu_n)} \arrow[urr, dotted, "\widehat{T}"] & & \end{tikzcd}$$

where $\widehat{T}$ is multilinear, $\mu_1, \ldots, \mu_n$ are the Haar measures on $G_1, \ldots, G_n$ and $i_1, \ldots, i_n$ the inclusions into the spaces $(F_1)_{L_\varphi(\mu_1)}, \ldots, (F_n)_{L_\varphi(\mu_n)}$ defined by the functions in $F_1, \ldots, F_n$ with the Orlicz norms $\|\cdot\|_{L_\varphi(\mu_1)}, \ldots, \|\cdot\|_{L_\varphi(\mu_n)}$, respectively.

Proof. If $T\colon F_1 \times \cdots \times F_n \to X$ is φ-summing, we have by Theorem 3.1 and Proposition 4.5 that it is φ semi-integral. This implies that there is a domination as

$$\|T(f_1, \ldots, f_n)\| \leq K \int_{G_1 \times \cdots \times G_n} \varphi(\odot\,(f_1, \ldots, f_n))\, d\nu$$

for a certain regular Borel probability measure ν. By Corollary 5.2, it follows that the Haar measure μ is a Pietsch's measure, and it is also Riesz representable (see Proposition 4.4). Thus, we have a domination with the same integral, however with μ instead of ν. Taking into account also that φ is submultiplicative and μ is Riesz representable, we deduce that T is φ-dominated for the Haar measures $\mu_1, \ldots, \mu_n$ on $G_1, \ldots, G_n$, respectively, that represent μ (Lemma 4.5). Finally, Theorem 4.1 yields the required factorization. □

Motivated by the applications of the class of multiple summing operators (see [4, 9, 20]), we will explain in this section that our arguments allow us to prove that some abstract classes of semi-integral operators are multiple summing in a more general setting. Before stating the results, we recall some geometrical notions from the theory of Banach lattices. A Banach lattice X is said to be *p-convex*, $1 \leq p < \infty$, respectively *q-concave*, $1 \leq q < \infty$, if there are positive constants $C^{(p)}$ and $C_{(q)}$ such that

$$\Big\| \Big(\sum_{j=1}^{n} |x_j|^p \Big)^{1/p} \Big\|_X \leq C^{(p)} \Big(\sum_{j=1}^{n} \|x_j\|_X^p \Big)^{1/p},$$

respectively,

$$\Big(\sum_{j=1}^{n} \|x_j\|_X^q \Big)^{1/q} \leq C_{(q)} \Big\| \Big(\sum_{j=1}^{n} |x_j|^q \Big)^{1/q} \Big\|_X$$

for every finite sequence $(x_j)_{j=1}^n$ in X. The least such $C^{(p)}$ (respectively, $C_{(q)}$) is denoted by $M^{(p)}(X)$ (respectively, $M_{(q)}(X)$). It is well-known that a p-convex Banach (q-concave) lattice can always be renormed with a lattice norm in such a way that $M^{(p)}(X) = 1$ ($M_{(q)}(X) = 1$). We refer to [15, Ch. 1.d] for more information about the classical geometric concepts of p-convexity and q-concavity.

We will use a remarkable result due to Schep [27, Theorem 2.3], which states the following: *Let E and F be Banach lattices in $L^0(\nu)$ and $L^0(\mu)$, respectively. Assume that E and F have the Fatou property and that there exists $1 \leq p \leq \infty$ such that E is p-convex and F is p-concave. Then the following continuous inclusion holds*:

$$[E]F \hookrightarrow E[F]\,,$$

with norm less than or equal to C depending on $M^{(p)}(E)$ and $M_{(p)}(F)$.

The proof of the following theorem is an immediate consequence of Schep's result, and so it is omitted.

Theorem 5.4. *Let $1 < p < \infty$ and let $T\colon X_1 \times \cdots \times X_n \to Y$ be an $F(\mu)$ semi-integral multilinear operator for a p-concave Banach lattice $F(\mu)$ with $\mu \in \mathcal{M}(B_{X_1^*} \times \cdots \times B_{X_n^*})$, and let $(\Omega_j, \Sigma_j, \nu_j)$ be a σ-finite measure space for each $1 \leq j \leq n$. Suppose that a Banach lattice E in $L^0(\Omega_1 \times \cdots \times \Omega_n, \nu_1 \times \cdots \times \nu_n)$ has the Fatou property and is p-convex. Then there exists a constant $C > 0$ depending on $M^{(p)}(E)$ and $M_{(p)}(F)$ such that the following estimates holds,*

$$\big\|\|T(f_1(\cdot), \ldots, f_n(\cdot))\|_Y\big\|_E \leq C \sup_{(x_1^*, \ldots, x_n^*) \in B_{X_1^*} \times \cdots \times B_{X_n^*}} \big\|\langle f_1(\cdot), x_1^*\rangle \cdots \langle f_n(\cdot), x_n^*\rangle\big\|_E\,,$$

for all $(f_1, \ldots, f_n) \in L^0(\nu_1, X_1) \times \cdots \times L^0(\nu_n, X_n)$ such that the value of the expression in the right hand side of the inequality is finite.

As was mentioned, multiple p-summing multilinear mappings are intensively investigated because of interesting applications. We provide a more general definition, however, before we need to introduce some notation. Let E be a Banach sequence lattice modelled on $\mathbb{N}$ and let X be a Banach space. We denote by $E^w(X)$ the Banach space of all sequences $\{x_j\}_{j=1}^\infty$ in X such that

$$\|x\|_{E^w(X)} := \sup_{x^* \in B_{X^*}} \|\{\langle x_j, x^*\rangle\}\|_E < \infty\,.$$

Let $E_1, \ldots, E_n$ be Banach sequence lattices modelled on $\mathbb{N}$ and E a Banach sequence lattice modelled on $\mathbb{N}^n$. A multilinear operator $T\colon X_1 \times \cdots \times X_n \to Y$ is said to be multiple $(E; E_1, \ldots, E_n)$-summing if there exists a constant $C_n > 0$ such that

$$\big\|\big\{T(x_{j_1}^1, \ldots, x_{j_n}^n)\big\}_{j_1, \ldots, j_n = 1}^\infty\big\|_E \leq C_n \big\|\{x_j^1\}\big\|_{E_1^w(X_1)} \cdots \big\|\{x_j^n\}\big\|_{E_n^w(X_n)}$$

for every sequence $\{x_j^i\}_{j=1}^\infty$ in $E_i^w(X_i)$ with $1 \leq i \leq n$. The least constant C_n satisfying the above estimates is a norm on the space of all multiple summing operators from $X_1 \times \cdots \times X_n$ to Y and is denoted by $\pi_{E;E_1,\ldots,E_n}$. If $1 \leq p_1, \ldots, p_n \leq q < \infty$ and $E_j = \ell_{p_j}$ for each $1 \leq j \leq n$, then T is called multiple $(q;p_1,\ldots,p_n)$-summing. Let us mention that the notion of multiple $(q;p_1,\ldots,p_n)$-summing mapping was introduced in [16], under the name of "strictly absolutely summing multilinear mappings". We refer to the survey paper [7] and the references therein related to applications of $(E;F)$-summing operators in the study of eigenvalues, s-numbers and interpolation of linear operators.

Using the techniques provided in the paper, we finish with the following result, which is a consequence of Theorem 5.4.

Corollary 5.5. *Let $1 < p < \infty$ and let $T\colon X_1 \times \cdots \times X_n \to Y$ be an $F(\mu)$ semi-integral multilinear operator for some p-concave Banach lattice $F(\mu)$ with $\mu \in \mathcal{M}(B_{X_1^*} \times \cdots \times B_{X_n^*})$. Suppose that the Banach sequence lattice E is modelled on $\mathbb{N}^n$, has the Fatou property and is p-convex. If the Banach sequence lattices $E_1, \ldots, E_n$ are such that the multilinear multiplication operator $\odot\colon E_1 \times \cdots \times E_n \to E$ is bounded, then T is $(E;E_1,\ldots,E_n)$ multiple summing with*

$$\pi_{E;E_1,\ldots,E_n}(T) \leq C\|\odot\|,$$

where $C > 0$ is a constant depending on $M^{(p)}(E)$ and $M_{(p)}(E)$.

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