Cosets of normal subgroups and powers of conjugacy classes

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Abstract

Let G be a finite group and $K = x^G$ the conjugacy class of an element x of G. In this paper, it is proved that if N is a normal subgroup of G such that the coset xN is union of K and K^{-1} (the conjugacy class of the inverse of x), then N and the subgroup $\langle K \rangle$ are solvable. As an application, we prove that if there exists a natural number n such that $K^n = K \cup K^{-1}$, then $\langle K \rangle$ is solvable.

Keywords: Characters, Conjugacy classes, Cosets of normal subgroups, Powers of conjugacy classes.

Mathematics Subject Classification (2010): 20E45, 20C15, 20D20.

1 Introduction

Let N be a normal subgroup of a finite group G. In [4] it was proved, by appealing to the Classification of the Finite Simple Groups, that whenever all elements of the coset xN are conjugate to $x \in G$ then N is solvable. In fact, the result goes further and, for example, it is shown that if all elements in xN are p-elements for some odd prime p, then N is solvable, and if in addition they are conjugate, then N has normal p-complement. This is not the case for $p = 2$. In this note, we investigate the case in which a coset xN is the union of the conjugacy class of x and that of its inverse, and our first objective is to prove the following

Theorem A. *Let G be a finite group and let N be a normal subgroup of G. Let $K = x^G$ be the conjugacy class of an element $x \in G$. Suppose that $xN = K \cup K^{-1}$. Then N is solvable. As a consequence, $\langle K \rangle$ is solvable too.*

We employ Theorem A to address a concrete problem on products of conjugacy classes. We recall that Arad and Fisman's conjecture asserts that the product of two non-trivial conjugacy classes cannot be a conjugacy class in a non-abelian finite simple group. Even though it remains unsolved, this subject is of keen interest for many authors, who have tried to find solvability conditions related to the product of conjugacy classes. For instance, a specific case of Arad and Fisman's conjecture is the following: If K is a conjugacy class, then the fact that K^2 is again a conjugacy class implies that $\langle K \rangle$ is solvable (Theorem A of [4]), and likewise, when K^n is a conjugacy class for some $n \geq 3$ (Theorem A of [2]). We study a particular case of the following conjecture, which was posed in [2].

Conjecture. *Let G be a group and let K be a conjugacy class of G. If $K^n = D \cup D^{-1}$ for some $n \geq 2$ where D a conjugacy class of G, then $\langle K \rangle$ is solvable.*

The above hypotheses are not unusual and it is not difficult, for instance with the help of [3], to find numerous examples (see Examples 3 and 4 of [2] for the case $n = 2$ and also see Section 3 for $n = 3$). It turns out that either $|K| = |D|/2$ or $|K| = |D|$. The first case was already solved in [2], and our contribution here concerns the case $K = D$.

Theorem B. *Let G be a finite group and let $K = x^G$ be a conjugacy class of G. If $K^n = K \cup K^{-1}$ for some $n \in \mathbb{N}$ and $n \geq 2$, then $\langle K \rangle$ is solvable.*

2 Cosets and characters

We start by stating two preliminary results on products of conjugacy classes, whose proofs are based on the Classification of the Finite Simple Groups.

Lemma 2.1. *Let G be a group and K, L and D non-trivial conjugacy classes of G such that $KL = D$ with $|D| = |K|$. Then G possesses a proper normal solvable group which is $\langle LL^{-1} \rangle$. In particular, $\langle L \rangle$ is solvable.*

Proof. See Lemma 2 of [2]. □

Theorem 2.2. *Let $K = x^G$ be a conjugacy class of a group G. There exists $n \in \mathbb{N}$ and $n \geq 2$ satisfying that K^n is a conjugacy class if and only if*

$$\chi(x)^n = \chi(1)^{n-1}\chi(x^n)$$

for every $\chi \in \mathrm{Irr}(G)$. In this case, $\langle K \rangle$ is solvable.

Proof. This is Theorem A of [2]. □

Next we study some properties of the character values for the conjugacy classes that we are dealing with.

Lemma 2.3. *Let G be a finite group and let N be a normal subgroup of G. Let $K = x^G$ be the conjugacy class of an element $x \in G$. Suppose that $xN \subseteq K \cup K^{-1}$. If $\chi \in \mathrm{Irr}(G)$ does not contain N in its kernel, then $\chi(x)$ is a purely imaginary number. Furthermore, if $\chi(x) \neq 0$ with $\chi \in \mathrm{Irr}(G)$, then $\chi(n)$ is integer for every $n \in N$.*

Proof. Let $\mathfrak{X}$ be a representation of G that affords χ. We know that $\mathfrak{X}$ can be linearly extended to $\mathbb{C}[G]$ and for every conjugacy class of G, say T, we denote by $\widehat{T}$ the sum of all elements in T in the group algebra $\mathbb{C}[G]$. Since N is a disjoint union of conjugacy classes of G, the sum

$$\widehat{N} = \sum_{n \in N} n \in \mathbf{Z}(\mathbb{C}[G])$$

and, by Schur's Lemma, $\mathfrak{X}(\widehat{N})$ is a scalar matrix. The trace of $\mathfrak{X}(\widehat{N})$ is

$$\sum_{n \in N} \chi(n) = |N|[\chi_N, 1_N] = 0,$$

so $\mathfrak{X}(\widehat{N}) = O$, where O denotes the zero matrix, and $\mathfrak{X}(\widehat{x^g N}) = \mathfrak{X}(x^g)\mathfrak{X}(\widehat{N}) = O$, for every $g \in G$.

Observe that, by hypothesis, $KN \subseteq K \cup K^{-1}$, and since KN is union of conjugacy classes then $KN = K$ or $KN = K \cup K^{-1}$. Assume first that $KN = K$. Taking the trace of $\mathfrak{X}(\widehat{xN})$, we have

$$0 = \textstyle\sum_{n \in N} \chi(xn) = m\chi(x)$$

for certain positive integer m, so $\chi(x) = 0$ and we have finished. Consider now the case $KN = K \cup K^{-1} = K^{-1}N$. Then $\mathfrak{X}(\widehat{K}\widehat{N}) = \sum_{x \in K} \mathfrak{X}(\widehat{xN}) = O$ and, analogously, $\mathfrak{X}(\widehat{K^{-1}}\widehat{N}) = O$.

On the other hand, we can write $\widehat{K}\widehat{N} = m_1\widehat{K} + m_2\widehat{K^{-1}}$ with m_1 and m_2 positive integers. Taking inverses, we have $\widehat{K^{-1}}\widehat{N} = m_2\widehat{K} + m_1\widehat{K^{-1}}$. We know that

$$\mathfrak{X}(\widehat{K}) = w_\chi(\widehat{K})I,$$

where

$$w_\chi(\widehat{K}) = \frac{|K|\chi(x)}{\chi(1)}$$

and I is the identity matrix. Hence

$$O = \mathfrak{X}(\widehat{K}\widehat{N}) + \mathfrak{X}(\widehat{K^{-1}}\widehat{N}) = \mathfrak{X}((m_1 + m_2)\widehat{K} + (m_1 + m_2)\widehat{K^{-1}}).$$

By taking traces

$$(m_1 + m_2)|K|\chi(x) + (m_1 + m_2)|K|\chi(x^{-1}) = (m_1 + m_2)|K|(\chi(x) + \chi(x^{-1})) = 0.$$

Thus, $\chi(x) = -\chi(x^{-1}) = -\overline{\chi(x)}$, so the first assertion of the lemma is proved.

For proving the last assertion of the statement, suppose that $\chi(x) \neq 0$ with $\chi \in \mathrm{Irr}(G)$. We can trivially assume that N is not contained in the kernel of χ. By Problem 3.12 of [5],

$$\chi(x)\chi(n) = \frac{\chi(1)}{|G|}\sum_{g\in G}\chi(xn^g)$$

and, since $\chi(x)$ is a purely imaginary number and $xN \subseteq K \cup K^{-1}$, which means that $\chi(xn^g)$ is equal to either $\chi(x)$ or $\chi(x^{-1})$, we deduce as a result that $\chi(n)$ is rational (and algebraic integer). So $\chi(n)$ is integer. □

We are ready to prove Theorem A, which we state again.

Theorem 2.4. *Let G be a finite group and let N be a normal subgroup of G. Let $K = x^G$ be the conjugacy class of an element $x \in G$. Suppose that $xN = K \cup K^{-1}$. Then N is solvable. As a consequence, $\langle K\rangle$ is solvable too.*

Proof. We assume that K is non-real, otherwise the theorem is proved by Theorem B(a) of [4]. We will work by induction on $|G|$. It is clear that $xN = x^{-1}N$ and then $x^2N = N = (K\cup K^{-1})(K\cup K^{-1})$, so in particular $K^2 \subseteq N$. Therefore, $|\langle K^2\rangle|$ divides $|N| = 2|K|$. But $|K| \leq |K^2| < |\langle K^2\rangle|$, and this forces $\langle K^2\rangle = N$. Moreover, we have $K^3 \subseteq NK = K \cup K^{-1}$. If $K^3 = K$ or $K^3 = K^{-1}$, it follows that $\langle K\rangle$ is solvable by Theorem 2.2 and then N is solvable as well (notice that $\langle K\rangle/N$ is cyclic of order 2). Thus we can assume that $K^3 = K\cup K^{-1}$. We claim that $K^{2n+1} = K^3$ for every $n \geq 1$. Indeed, since $K^2 \subseteq N$, then $(K^2)^n \subseteq N$ for every $n \geq 1$, so $K^{2n+1} \subseteq KN = K \cup K^{-1}$, and in fact, the equality can be assumed to hold again by Theorem 2.2, so the claim is proved. As a consequence, x must be a 2-element. Let $C = \mathbf{C}_G(x)$ be the centralizer of x in G. We have

$$|G : C| = |G : NC||NC : C| = |K| = |N|/2.$$

Also, if $n \in \mathbf{C}_N(x)$, then as xn has the same order as x, we have that n must be a 2-element too. Accordingly, $\mathbf{C}_N(x)$ is a 2-group and $|G : NC| = |\mathbf{C}_N(x)|/2$ is a 2-number. We obtain $G = PNC$, for every Sylow 2-subgroup P of G and, by induction, we can consider $G = NP$, that is, the index $|G : N|$ is a power of 2.

On the other hand, as x is a 2 element, it is well-known that

$$\chi(x) \equiv \chi(1) \bmod 2$$

(in the ring of algebraic integers), for every $\chi \in \mathrm{Irr}(G)$. Now, since G/N is a 2-group, the degree of every non-linear irreducible character of G containing N in its kernel is a power of 2. Also, if $\chi \in \mathrm{Irr}(G)$ does not contain N in its kernel and it is real-valued, necessarily $\chi(x) = 0$ by Lemma 2.3, and hence, by the above congruence, $\chi(1)$ is even. Therefore, we conclude that all non-linear real-valued irreducible characters of G have even degree. By Theorem A of [6], this is equivalent to the fact that G has normal 2-complement, so in particular, G is solvable. Then N and $\langle K\rangle$ are solvable as well. □

We wonder if Theorem A will still be true when the hypothesis $xN = K \cup K^{-1}$ is weakened to $xN \subseteq K \cup K^{-1}$ as is the case with Lemma 2.3, however, we have not been able to demonstrate it by using similar methods.

3 Proof of Theorem B

We employ the results of Section 2 to solve a specific case of the conjecture stated in the Introduction. For that purpose, we need to work with the complex group algebra $\mathbb{C}[G]$. Let $D_1, \ldots, D_k$ be the conjugacy classes of a finite group G and let S be a G-invariant set of G, then the sum $\widehat{S} = \sum_{i=1}^{k} n_i \widehat{D_i}$ with $n_i \in \mathbb{N}$ for $1 \leq i \leq k$. We write $(\widehat{S}, \widehat{D_i}) = n_i$ following [1]. We will use the following properties.

Lemma 3.1. *If D_1, D_2 and D_3 are conjugacy classes of a finite group G, then*

1. $(\widehat{D_1}\widehat{D_2}, \widehat{D_3}) = (\widehat{D_1^{-1}}\widehat{D_2^{-1}}, \widehat{D_3^{-1}})$
2. $(\widehat{D_1}\widehat{D_2}, \widehat{D_3}) = |D_2||D_3|^{-1}(\widehat{D_1}\widehat{D_3^{-1}}, \widehat{D_2^{-1}})$
3. $(\widehat{D_1}\widehat{D_2}, \widehat{D_1}) = |D_2||D_1|^{-1}(\widehat{D_1}\widehat{D_1^{-1}}, \widehat{D_2^{-1}}) = (\widehat{D_2}\widehat{D_1^{-1}}, \widehat{D_1^{-1}}) = (\widehat{D_2^{-1}}\widehat{D_1}, \widehat{D_1})$.

Proof. See the proof of Theorem A of [1]. □

We give the proof of Theorem B.

Proof of Theorem B. We can assume that K is non-real, since the real case is a particular case of Theorem 2.2. Also, the case $n = 2$ is already proved in Theorem D of [2]. Henceforth, we will assume $n \geq 3$ and argue by induction on $|G|$. Since K^{n-1} is a G-invariant set, we write $K^{n-1} = L_1 \cup \cdots \cup L_s$ where L_i are distinct conjugacy classes of G (possibly 1) for every $1 \leq i \leq s$. Then

$$K^n = KK^{n-1} = K(L_1 \cup \cdots \cup L_s) = K \cup K^{-1}.$$

Suppose that L_i is a non-trivial conjugacy class. If either $KL_i = K$ or $KL_i = K^{-1}$, by Lemma 2.1, we know that $\langle L_i \rangle$ is solvable. Consider now $\overline{G} = G/\langle L_i \rangle$ and observe from the hypothesis that $\overline{K^n} = \overline{K} \cup \overline{K^{-1}}$. Then $\langle \overline{K} \rangle$ is solvable by induction. Notice that if $\overline{K} = \overline{K^{-1}}$, then $\overline{K^n} = \overline{K}$ and $\langle \overline{K} \rangle$ is solvable again by Theorem 2.2. Consequently, $\langle K \rangle$ is solvable.

Therefore, we can assume that $KL_i = K \cup K^{-1}$ for every non-trivial class L_i. By Lemma 3.1(2), we know that

$$0 \neq (\widehat{K}\widehat{L_i}, \widehat{K^{-1}}) = \frac{|L_i|}{|K|}(\widehat{K^2}, \widehat{L_i^{-1}}).$$

Thus, $L_i^{-1} \subseteq K^2$ and then $L_i \subseteq K^{-2}$. We deduce that $K^{n-1} \subseteq K^{-2} \cup \{1\}$. On the other hand, $|K^2| = |K^{-2}| \leq |K^{n-1}| \leq |K^2| + 1$ (in the first inequality we are using that $n \geq 3$), thus either $K^{n-1} = K^{-2}$ or $K^{n-1} = K^{-2} \cup \{1\}$. In both cases, $K^{-2} \subseteq K^{n-1}$. Moreover, if L_i is a non-trivial conjugacy class, again by Lemma 3.1(2), we have

$$0 \neq (\widehat{K}\widehat{L_i}, \widehat{K}) = \frac{|L_i|}{|K|}(\widehat{K}\widehat{K^{-1}}, \widehat{L_i^{-1}}).$$

Hence $L_i^{-1} \subseteq KK^{-1}$ and then $L_i \subseteq KK^{-1}$. In addition, $K^{-1}L_i \subseteq K^{-1}K^{-1}K \subseteq K^{n-1}K = K^n$ and consequently, $K^{-1}L_i \subseteq K \cup K^{-1}$. If $K^{-1}L_i = K$ or $K^{-1}L_i = K^{-1}$, then $\langle K \rangle$ is solvable by arguing as before. We can assume then that $K^{-1}L_i = K \cup K^{-1}$ for every non-trivial L_i. In particular, by applying Lemma 3.1(1) and (2)

$$0 \neq (\widehat{K^{-1}}\widehat{L_i}, \widehat{K}) = (\widehat{K}\widehat{L_i^{-1}}, \widehat{K^{-1}}) = \frac{|L_i|}{|K|}(\widehat{K^2}, \widehat{L_i}),$$

which means that $L_i \subseteq K^2$. Therefore, $K^{n-1} \subseteq K^2 \cup \{1\}$. Analogously as above, taking cardinalities we obtain that either $K^{n-1} = K^2$ or $K^{n-1} = K^2 \cup \{1\}$. In both cases, $K^2 \subseteq K^{n-1}$. Hence $K^3 \subseteq K^n = K \cup K^{-1}$. By applying Lemma 2.1 again, it can be assumed that $K^3 = K \cup K^{-1}$. Taking into account that $K^2 = K^{-2}$, we obtain

$$K^5 = K^2K^3 = K^2(K \cup K^{-1}) = K^3 \cup K^{-2}K^{-1} = K^3 \cup K^{-3} = K \cup K^{-1}.$$

Inductively, we easily get $K^{2k+1} = K \cup K^{-1}$ for every $k \geq 1$, and as a consequence, $K\langle K^2 \rangle = K \cup K^{-1}$. The fact that $K \cup K^{-1}$ is union of cosets of the normal subgroup $\langle K^2 \rangle$ shows that $|\langle K^2 \rangle|$ divides $2|K|$. Now, note that $|K| \leq |K^2| < 1 + |K^2| \leq |\langle K^2 \rangle|$, so we conclude that $|\langle K^2 \rangle| = 2|K|$. By cardinalities, it follows that $x\langle K^2 \rangle = K \cup K^{-1}$, and then, we apply Theorem A to get that $\langle K^2 \rangle$ and $\langle K \rangle$ are solvable. $\square$

Example 3.2. *We give an example of a group satisfying the hypotheses of Theorem B with $n = 3$, in which the order of the elements in K is not a prime. Let $G = \langle a, x \mid a^8 = x^2 = 1, a^{x^{-1}} = a^3 \rangle$ the semidihedral group of order 16 and $K = a^G$ which satisfies $K^3 = K \cup K^{-1}$.*

Acknowledgements

This research is partially supported by the Spanish Government, Proyecto PGC2018-096872-B-I00. The first author is also supported by Proyecto UJI-B2019-03.

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