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# On a graph related to permutability in finite groups

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**Abstract**

For a finite group $G$ we define the graph $\Gamma(G)$ to be the graph whose vertices are the conjugacy classes of cyclic subgroups of $G$ and two conjugacy classes $\mathcal{A}$, $\mathcal{B}$ are joined by an edge if for some $A \in \mathcal{A}$, $B \in \mathcal{B}$ $A$ and $B$ permute. We characterise those groups $G$ for which $\Gamma(G)$ is complete.
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## 1 Introduction

There are many ways in which a graph has been associated with a finite group. Herzog, Longobardi and Maj [8] have defined a graph whose vertices are the conjugacy classes of a group, with two vertices joined by and edge if an element of one vertex commutes with some element of the other vertex. This is a generalisation of the commuting graph of a group, which has the elements of a group as vertices, joined by an edge if they commute (see [9]). In this paper we will consider a generalisation of the graph of Herzog, Longobardi and Maj. For a finite group $G$ we define the graph $\Gamma(G)$ to be the graph whose vertices are the conjugacy classes of cyclic subgroups of $G$ and two conjugacy classes $\mathcal{A}$, $\mathcal{B}$ are joined by an edge if for some $A \in \mathcal{A}$, $B \in \mathcal{B}$ $A$ and $B$ permute. We characterise those groups $G$ for which $\Gamma(G)$ is complete.

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Recall that a subgroup $H$ of a group $G$ is said to be permutable in $G$ if $HL$ is a subgroup of $G$ for every subgroup $L$ of $G$. Permutability, like normality, is not a transitive relation in general. We say that a group $G$ is a $PT$-group if the permutability is transitive in $G$, that is, if $H$ is permutable in $K$ and $K$ is permutable in $G$, then $H$ is permutable in $G$. According to a classical result of Ore [10] permutable subgroups of finite groups are subnormal. Hence a finite group is a PT-group if and only if every subnormal subgroup is permutable.

We prove:

**Theorem 1.** *A finite group $G$ is a soluble* PT*-group if and only if the graph $\Gamma(G)$ is complete.*

## 2 Proof of Theorem 1

Suppose that $G$ is a soluble PT-group. By a result of Zacher [12] $G = AH$ where $A$ is an abelian normal subgroup of $G$, $H$ is a nilpotent modular subgroup of $G$, $A$ and $H$ have coprime orders and every subgroup of $A$ is normal in $G$. If $X$ and $Y$ are two cyclic subgroups of $G$, we can write $X = X_0X_1$ and $Y = Y_0Y_1$, where $X_0$ and $Y_0$ are subgroups of $A$ and $X_1$ and $Y_1$ have orders dividing $|H|$. Since $H$ is a Hall subgroup of $G$ we can, by replacing $X$ and $Y$ by conjugates if necessary, assume that $X_1$ and $Y_1$ are subgroups of $H$. Since every subgroup of $A$ is normal in $G$ we have $Y_0X_1 = X_1Y_0$ and since $H$ is modular we have $X_1Y_1$ is a subgroup of $G$. It now follows that $X_0X_1Y_0Y_1 = Y_0Y_1X_0X_1$ is a subgroup, that is, $X$ and $Y$ permute.

In the other direction, we argue by induction on the order of $G$. We begin by showing that $G$ is a soluble PT-group if $G$ has at least two minimal normal subgroups. If $M$ and $N$ are two minimal normal subgroups of $G$, then $G/M$ and $G/M$ clearly satisfy the hypothesis of the theorem. Hence $G/M$ and $G/N$ are soluble PT-groups. It follows that $NM/M$ is a minimal normal subgroup of a soluble PT-group, and so is cyclic of prime order because every soluble PT-group is supersoluble. Similarly $MN/N$ is cyclic of prime order and hence $M$ and $N$ have prime orders, $p$ and $q$ say (and $G$ is soluble). Then, for any prime $r \neq p$, all $r$-chief factors of $G/M$ are $G$-isomorphic. Further, by Zacher's Theorem [12], Sylow $r$-subgroups of $G/M$ are abelian if $r$-chief factors are noncentral and modular if all $r$-chief factors are central. If $p \neq q$ by considering $G/N$ we have all $p$-chief factors $G$-isomorphic and Sylow $p$-subgroups abelian if $p$-chief factors are noncentral and modular if $p$-chief factors are central. In this case $G$ is a PT-group by [3, Corollary 3] and [4, Theorem 2] (note that $G$ is supersoluble).

Thus we suppose that all minimal normal subgroups have the same prime order $p$. If $M$ and $N$ are minimal normal subgroups and $p$ divides $|G/MN|$, then both $M$ and $N$ are $G$-isomorphic to a (fixed) $p$-chief factor of $G/MN$ and so are $G$-isomorphic. Thus all $p$-chief factors are $G$-isomorphic. Therefore $G$ is supersoluble and all chief factors of the same order are $G$-isomorphic. By [3, Corollary 3] $G$ is a group in which every subnormal subgroup permutes with all Sylow subgroups ($G$ is a PST-group). If the $p$-chief factors are central, then $G$ is a $p$-group with all proper quotients modular and so is itself modular, since by Theorem of Longobardi [7] such a group must have a unique minimal normal subgroup. Applying a result of Agrawal [2], $G$ has an abelian Sylow $p$-subgroup and it then follows that $G$ is a PT-group by [4, Theorem 2]. Assume now that $MN$ is a Sylow $p$-subgroup of $G$. Let $M = \langle m \rangle$, $N = \langle n \rangle$. By hypothesis, given a $p'$-element $y \in G$, $\langle mn \rangle$ permutes with a conjugate $\langle y^g \rangle$ of $\langle y \rangle$. Hence $\langle mn \rangle \langle y^g \rangle \cap MN = \langle mn \rangle$ is normalised by $y^g$. Call $m^g = m^{a_1}$, $n^g = m^{a_2}$ and $m^y = m^{b_1}$, $n^y = n^{b_2}$ and $(mn)^{g^{-1}yg} = (mn)^c$. Hence $(mn)^{g^{-1}yg} = m^{b_1} n^{b_2} = (mn)^c$, which implies that $b_1 \equiv b_2 \equiv c \pmod{p}$. Consequently $M$ and $N$ are $G$-isomorphic. Since $G$ has all Sylow subgroups modular, it follows that $G$ is a PT-group by [3, Corollary 3] and [4, Theorem 2].

We now suppose that $G$ has a unique minimal normal subgroup $N$. If $N$ is not soluble, then $N = S_1 \times \cdots \times S_r$, where the $S_i$ are isomorphic (nonabelian) simple groups. Let $p$ and $q$ be different primes dividing the order of $S_1$ and let $x_1$ and $y_1$ be elements of $S_1$ of orders $p$ and $q$, respectively. For $2 \leq i \leq r$, let $x_i$, $y_i$ be the images of $x_1$, $y_1$ under the isomorphism between $S_1$ and $S_i$. Then $\langle x_1 \cdots x_r \rangle$ permutes with a conjugate $\langle (y_1 \cdots y_r)^g \rangle$ of $\langle y_1 \cdots y_r \rangle$. The projection of $\langle x_1 \cdots x_r \rangle \langle (y_1 \cdots y_r)^g \rangle$ onto $S_1$ is then a subgroup of $S_1$ of order $pq$ and so $S_1$ has subgroups of order $pq$ for every pair of primes dividing its order. A result of Abe and Iiyori [1] shows that this is impossible. Consequently $N$ is a $p$-group for some prime $p$.

If $N$ is not contained in the Frattini subgroup of $G$, then $G$ is a primitive soluble group and $G = NM$, where $M$ is a maximal subgroup of $G$, $N \cap M = 1$ and $N$ is self-centralising. Since $M$ is isomorphic to the soluble PT-group $G/N$, $M$ is the product of its nilpotent residual $F = M^{\mathfrak{N}}$, which is an abelian normal Hall subgroup of odd order, and a complement $C$ which acts on $F$ as power automorphisms ([12]). Let $Q$ be a cyclic normal subgroup of $M$. Suppose that $QP \neq PQ$ for some cyclic subgroup $P$ of $N$. We have $Q^g P = PQ^g$ for some $g \in G$ and we can assume that $g \in M$. Since $Q$ is normal in $M$ we have $Q^g = Q$, giving a contradiction. Thus $P$ is normalised by $Q$ since $P = N \cap PQ$. It now follows that every element of $F$ acts as a power automorphism on $N$ and hence $F$ acts as a power automorphism group on $N$. Since power automorphisms are central in the power automorphism group of $N$ ([5, Theorem 2.2.1]), $F$ is central in $M$ and so $F = 1$. Thus $M$ is a nilpotent modular group. In particular $M$ is a $p'$-group. Let $Q$ be a non-abelian Sylow $q$-subgroup of $M$. By Iwasawa's Theorem ([11, Theorem 2.4.14] $Q$ has an abelian normal subgroup $Q_0$ with cyclic supplement $S$ with $S$ acting as a power automorphism group on $Q_0$ or $Q$ is Hamiltonian. In both cases every cyclic subgroup of $Q_0$ is normal in $Q$ and hence in $M$. Let $U$ be a cyclic subgroup of $N$ and let $R$ cyclic subgroup of $Q_0$. By hypothesis, there exits an element $a \in M$ such that $RU^a$ is subgroup. Since $R^a = R$, we have that $RU$ is also a subgroup and $U$ is normalised by $R$. Further, there exists an element $m \in M$ such that $S$ permutes with $U^m$ and so $S$ and $R$ normalise $U^m$. This implies that $Q$ normalises $U$ and $Q$ acts as power automorphisms on $N$. It now follows that $M$ acts as power automorphisms on $N$ and so $N$ is a cyclic group of order $p$ and $M$ is cyclic of order dividing $p-1$ and $G$ is clearly a PT-group.

Now suppose that $N$ is contained in the Frattini subgroup of $G$. If $G$ is nilpotent, then $G$ is a $p$-group since it has a unique minimal normal subgroup and $G/N$ is an modular group. Assume that $G$ is not modular. Let $M(p)$ denote the nonabelian group of order $p^3$ and exponent $p$ for $p$ odd and the dihedral group of order 8 for $p = 2$. By the Theorem of Longobardi [7] either $G$ is the central product of a subgroup $P$ isomorphic to $M(p)$ and another subgroup or $G$ is isomorphic to

$$G_0 = \langle a, b, w : a^{p^n} = w^p = 1, a^b = a^{1+p^s}, b^{p^j} = a^{p^{n-s}}, a^w = a^{1+p^{n-1}}, b^w = b \rangle,$$

where $0 < s < n$, $s \geq 2$ if $p = 2$ and $j \geq n - s$. In the first case it is clear that if $P$ is generated by $a$ and $b$ of order $p$ no conjugate of $a$ will commute with $b$ and hence will not permute with $b$. Now consider $G_0$ and let $C = \langle a^\alpha b^\beta \rangle$ be a cyclic subgroup of $H = \langle a, b \rangle$. Then $C$ permutes with $\langle w^g \rangle$ for some $g \in G_0$ and so, being of index $p$ in $C\langle w^g \rangle$, is normalised by $w^g$. Since $[w, g] \in \langle a^{p^{n-1}} \rangle$, $C$ is also normalised by $w$. Thus $w$ acts as a power automorphism on $H$. If $p$ is odd, $H$ is regular ([6, III, Satz 11.4]) and so it acts as a universal power automorphism on $H$ by [5, Theorem 5.3.1], a contradiction. Hence we suppose $p = 2$. Since $w$ centralises $b$ and $b$ has order at least 4, $w$ acts as universal power automorphism on $H$ by a theorem of Napolitani [11, Theorem 2.3.24], again a contradiction. Thus $G$ cannot be nilpotent.

If $G$ is not nilpotent, then $E = G^{\mathfrak{N}} \neq 1$ and so $N \leq E$. Since $G/N$ is a PT-group, $G$ supersoluble, $E$ is nilpotent and so it is a $p$-group. Furthermore $E/N$ is abelian and complemented in $G/N$ by a $p'$-subgroup $B/N$ say which acts on $E/N$ as power automorphisms. Then there exists a $p'$-subgroup $D$ of $G$ complementing $E$ in $G$. If $C_D(E/N) \neq 1$ then $C_D(E/N)$ is a nontrivial normal subgroup of $G$, a contradiction. It follows that $D$ is cyclic of order dividing $p-1$. We have that $N \leq \mathrm{Z}(E)$. Consequently $[a^p, b] = [a, b]^p = 1$ for every $a$, $b \in E$ by [6, III, Hilfssatz 1.3]. This implies that $E^p \leq \mathrm{Z}(E)$. Assume that $E^p$ is not trivial. Hence $E^p$ is cyclic because $E^p$ is an abelian normal subgroup of $G$. Let $E^p = \langle a \rangle$, where $a$ has order $p^n$. Suppose that $N$ is a proper subgroup of $E^p$. Given $x \in G$, we have that $a^x = a^i$ and so all chief factors of $G$ below $P^p$ are $G$-isomorphic. Appying again [3, Corollary 3] and [4, Theorem 2], $G$ is a PT-group. Assume now that $E^p = N$. Let $x$ be an element of $E$ not in $N$. If $x$ has order greater than $p$, then $N \leq \langle x^p \rangle \leq E^p$ and $\langle x \rangle / N$ is a normal subgroup of $G/N$ . Since all chief factors of $G$ between $N$ and $E$ are $G$-isomorphic and the same happens with all chief factors of $G$ below $\langle x \rangle$, we conclude again that all chief factors of $G$ are $G$-isomorphic and $G$ is a PT-group. Thus all elements of $E$ are of order $p$. If $E$ is abelian then $E$ is cyclic and so $E = N \leq \Phi(G)$, a contradiction. Suppose that $E$ is non-abelian and $x, y \in E$ do not commute, so that $N = \langle [x, y] \rangle$. Since $\langle x \rangle$ permutes with $\langle y^g \rangle$ for some $g \in G$, we have that $\langle x, y^g \rangle$ is of order $p^2$ and hence abelian. Since $\langle yN \rangle$ is a normal subgroup of $G/N$, we have that $\langle y^g \rangle \leq \langle y, N \rangle$ and it follows that $y^g = y^j n$ for some $n \in N$ and some integer $j$ coprime with $p$. Then $[x, y^g] = [x, y^j n] = [x, y^j] = [x, y]^j \neq 1$, a contradiction. This completes the proof.


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