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On totally permutable products of finite groups

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Abstract

The behaviour of totally permutable products of finite groups with respect to certain classes of groups is studied in the paper. The results are applied to obtain information about totally permutable products of T, PT, and PST-groups.

1 Introduction

If a group G can be written as a product of two subgroups A and B, then somehow the structure of G is restricted by that of A and B. Can one transform this general statement into concrete results at least in special situations? In this paper we are concerned with finite groups G which are factorised by their subgroups A and B in such way that every subgroup of A permutes with every subgroup of B. In this case we say that G is a *totally permutable product* of A and B. This sort of products arises when finite products of supersoluble groups are considered and they have been extensively studied even in the non-finite case. More precisely, Asaad and Shaalan [2] first introduced these products and proved that totally permutable products of finite supersoluble groups are supersoluble (Theorem 3.1). Maier [11] generalised Asaad and Shaalan's result to saturated formations containing all supersoluble groups, and the first author and Pérez-Ramos [4] were able to remove the restriction "saturated" from Maier's theorem and proved its converse.

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We also refer to Beidleman and Heineken [6] for other interesting facts on totally permutable products of infinite groups.

A key point behind the results about finite totally permutable products is a theorem of Huppert stating that the product of two cyclic groups is supersoluble. It also holds in the general non-finite case ([1; 7.4.6]). This theorem shows, in particular, that totally permutable products of nilpotent groups are not in general nilpotent. Therefore a natural question arises:

> *Suppose that $G = AB$ is a totally permutable product of two nilpotent groups A and B. What can we say about G?*

Applying Asaad and Shaalan's result, if G is finite, then G is supersoluble. We prove in the paper that in this case G is abelian-by-nilpotent, that is, its nilpotent residual is abelian. Therefore

in the sequel, all groups considered are finite and soluble.

Theorem 1. *Let G be the totally permutable product of the nilpotent groups A and B. Then G is abelian-by-nilpotent.*

This result allows us to think that the nilpotent residual of a group which is a totally permutable product of nilpotent groups plays an important role.

Recall that if $\mathfrak{H}$ is a formation, the $\mathfrak{H}$-residual $G^{\mathfrak{H}}$ of a group G is the smallest normal subgroup of G such that $G/G^{\mathfrak{H}} \in \mathfrak{H}$ ([7; II, 2.3]). For each normal subgroup N of G, we have $(G/N)^{\mathfrak{H}} = G^{\mathfrak{H}}N/N$ ([7; II, 2.4]). Our next result describes completely the Sylow subgroups of the nilpotent residual of a group G which is a totally permutable product of the nilpotent subgroups A and B.

Theorem 2. *Let G be as in Theorem 1 and let K be its nilpotent residual. If p divides $|K|$, then a Sylow p-subgroup of K is either A_p, or B_p, or $A_p \times B_p$, where A_p and B_p are the Sylow p-subgroups of A and B, respectively.*

We apply these results to obtain some information of the behaviour of finite totally permutable products with respect to formations $\mathfrak{F}$ of the form $\mathfrak{F} = \mathfrak{X} \circ \mathfrak{N}$, where $\mathfrak{X}$ is a formation of finite groups containing all abelian groups and $\mathfrak{N}$ is the class of all nilpotent groups. It is clear that $\mathfrak{F}$ is composed of all finite groups whose nilpotent residual is in $\mathfrak{X}$. More precisely, we have:

Theorem 3. *Let $\mathfrak{X}$ be a formation containing all abelian groups. Let $G = AB$ be a totally permutable product of groups in $\mathfrak{X} \circ \mathfrak{N}$. Then $G \in \mathfrak{X} \circ \mathfrak{N}$.*

As the symmetric group of degree 3 shows, Theorem 3 is not true if $\mathfrak{X}$ does not contain the formation of all abelian groups. In fact, if p is an odd prime and C_p does not belong to $\mathfrak{X}$, the dihedral group of order $2p$ is a totally permutable product of its Sylow subgroups, both in $\mathfrak{X} \circ \mathfrak{N}$, but the group is not in $\mathfrak{X} \circ \mathfrak{N}$.

For the converse, we do not require that $\mathfrak{X}$ contains all abelian groups.

Theorem 4. *Let $\mathfrak{X}$ be a formation. Let $G = AB$ be a totally permutable product of the subgroups A and B such that G belongs to $\mathfrak{X} \circ \mathfrak{N}$. Then A and B belong to $\mathfrak{X} \circ \mathfrak{N}$.*

Theorem 4 is not true for non-soluble groups (see for instance [7; X, Exercise 1.12]).

Theorem 5. *If $\mathfrak{F} = \mathfrak{X} \circ \mathfrak{N}$, where $\mathfrak{X}$ is a formation containing all abelian groups, and G is a finite totally permutable product of A and B, then:*

1. $[A^{\mathfrak{F}}, B] = [A, B^{\mathfrak{F}}] = 1$*; in particular, $A^{\mathfrak{F}}$ and $B^{\mathfrak{F}}$ are normal subgroups of G, and*

2. $G^{\mathfrak{F}} = A^{\mathfrak{F}} B^{\mathfrak{F}}$.

The methods applied in the proofs of the above results allow us to prove the following general theorem about totally permutable products of groups.

Theorem 6. *If $G = AB$ is a totally permutable product of the subgroups A and B, and a_p (respectively, b_p) is the number of non-isomorphic non-central p-chief factors in A (respectively, B) for a prime p, then the number c_p of non-central non-isomorphic p-chief factors in G is bounded by $a_{0p}+b_{0p}$, where $a_{0p} = \max\{1, a_p\}$ and $b_{0p} = \max\{1, b_p\}$.*

The above theorems allow us to derive information about totally permutable products of finite groups which have some group theoretical properties different from those described by formations. They are the ones described by the classes of T, PT, PST, and PST_c-groups. These classes are defined through permutability properties of subnormal subgroups. Let us give a short description of these classes before stating the corresponding results.

A subgroup of a group G is called *permutable* if it permutes with every subgroup of G. A result of Ore [12] shows that permutable subgroups of a finite group are subnormal in the group, but the converse need not hold. A group is called a *PT-group* (T-group) if permutability (normality) is a transitive relation. By Ore's result, PT-groups are exactly those groups whose subnormal subgroups are permutable. In particular, every T-group is a PT-group. PST-groups are defined in terms of Sylow permutability. A subgroup of a group G is called *S-permutable* if it permutes with all the Sylow subgroups of G. A result of Kegel [10] shows that every S-permutable subgroup is subnormal and hence PST-groups are exactly those groups in which all subnormal subgroups are S-permutable. In particular, PT-groups are PST-groups. Another class containing the beforementioned ones is the class of PST_c-groups, introduced and studied by Robinson in [13], and composed by all groups in which every cyclic subnormal subgroup is S-permutable.

One notable fact of the class of all soluble PST-groups is that it is subgroup-closed. Moreover it is closed under taking totally permutable products in which the factors have coprime indices [3] (see also [5] for other results in this direction). The following example shows, in particular, that the subgroup-closed character is absent from PST_c-groups, even if they are factors of coprime indices of a totally permutable product.

Example 7. Let W, X, Y, and Z be, respectively, non-abelian groups of orders 6, 21, 55, and 253. Let $G = W \times X \times Y \times Z$. For a group H and a prime p, H_p denotes a Sylow p-subgroup of H. Then G is a totally permutable product of $A = W \times X_3 \times Z_{23}$ and $B = X_7 \times Y \times Z_{11}$, but none of them is a PST_c-group, because neither all cyclic subgroups of the Sylow 3-subgroup of A permute with all Sylow 2-subgroups of A, nor all cyclic subgroups of the Sylow 11-subgroup of B permute with all Sylow 5-subgroups of B. Nevertheless, the group G is clearly a PST_c-group.

However, the extension of the "only if" part of [3; Theorem C] holds.

Theorem 8. *Assume that the group $G = AB$ is a totally permutable product of the soluble PST_c-groups A and B and that* $\gcd(|G : A|, |G : B|) = 1$. *Then G is a soluble PST_c-group.*

Corollary 9 ([3])**.** *Assume that $G = AB$ is a totally permutable product of the soluble PST-groups A and B such that* $\gcd(|G : A|, |G : B|) = 1$. *Then G is a soluble PST-group.*

The classes of PT_c-groups and T_c-groups are defined in a similar fashion, by requiring the cyclic subnormals to be permutable or normal, respectively. Theorem 8 also holds for these classes.

2 Proofs

Proof of Theorem 1. Assume that the theorem is false, and let $G = AB$ be a counterexample of minimal order. Then, since the class of all abelian-by-nilpotent groups is a formation, G has a unique minimal normal subgroup M. Since G is supersoluble by [2], M is a p-group for some prime p, the Fitting subgroup $P = \mathrm{F}(G)$ is a Sylow p-subgroup G, p is the largest prime dividing

$|G|$, and $G/\mathrm{F}(G)$ is abelian of exponent dividing $p-1$. Moreover $P = A_pB_p$, where A_p is the Sylow p-subgroup of A and B_p is the Sylow p-subgroup of B by [1; 1.3.3]. If A and B were abelian, we would have G metabelian by Itô's theorem ([1; 2.1.1]), a contradiction. Therefore we may assume that A is not abelian. As $A_{p'}$, the Hall p'-subgroup of A, is abelian, A_p is non-abelian. Let T be a subgroup of A_p. Then the product $TB_{p'}$, where $B_{p'}$ is a Hall p'-subgroup of B, is a supersoluble subgroup of G. Therefore $B_{p'}$ normalises T, that is, p'-elements of B induce power automorphisms in A_p. As A_p is non-abelian, all p'-power automorphisms are trivial ([8; Hilfssatz 5]), and hence $B_{p'}$ centralises A_p. Therefore $B_{p'}$ centralises P and so $B_{p'} = 1$. This means that B is a p-group.

On the other hand, all p'-elements of A induce power p'-automorphisms in B. Since A cannot be a p-group, it follows that $A_{p'} \neq 1$ and $A_{p'}$ cannot centralise B. By [8; Hilfssatz 5], B is abelian. Then $\mathrm{C}_B(A_{p'}) = 1$ and $B = [B, A_{p'}]$. By the minimality of the order of G, G/M is abelian-by-nilpotent. Therefore the nilpotent residual L/M of G/M is abelian. This implies that L/M is complemented in G/M and L/M contains no central chief factors of G/M by [7; IV, 5.18, V, 4.2, and V, 3.2]. Note that $A_{p'}L/L$ centralises BL/L. In particular, B is contained in L. Suppose that $T/M = L/M \cap AM/M$ is non-trivial. Then T/M is a normal subgroup of G/M and contains a central minimal normal subgroup of G/M, a contradiction. Consequently, $L \cap A$ is contained in M and $L = BM$. Then L is a p-group and $\mathrm{Z}(L)$ is a non-trivial normal subgroup of G. As M is the unique minimal normal subgroup of G, it follows that M is contained in $\mathrm{Z}(L)$ and either $L = B$ or $L = B \times M$. In both cases L is abelian and so G is abelian-by-nilpotent, final contradiction. □

The following lemma turns out to be crucial in the proof of our results.

Lemma 10. *Let $G = AB$ be a totally permutable product of the nilpotent subgroups A and B. Let K be the nilpotent residual of G. For a prime p, denote A_p and B_p the Sylow p-subgroup of A and B, respectively. The following statements hold:*

1. *If p is a prime dividing the order of K, then either $A_p \cap K \neq 1$ or $B_p \cap K \neq 1$.*
2. *If $A_p \cap K \neq 1$, then A_p is contained in K and the Hall p'-subgroup of B does not centralise A_p.*
3. *If $B_p \cap K \neq 1$, then B_p is contained in K and the Hall p'-subgroup of A does not centralise B_p.*

Proof. By Theorem 1, K is abelian and, by Asaad and Shaalan's result [2], G is supersoluble. We prove the statements by induction on $|G|$. Let p be a prime dividing $|K|$ and let q be the largest prime dividing $|G|$. Then a Sylow q-subgroup Q of G is normal in G. Suppose $p \neq q$. By induction, G/Q satisfies 1 and either 2 or 3 because G/Q is a totally permutable product of the nilpotent subgroups A_pQ/Q and B_pQ/Q. Moreover KQ/Q is the nilpotent residual of G/Q. Assume that $A_pQ/Q \cap KQ/Q \neq 1$, then it is clear that $A_p \cap K \neq 1$ because $p \neq q$. Moreover A_pQ/Q is contained in KQ/Q and then A_p is contained in a Sylow p-subgroup of K. The same argument applies in the case $B_pQ/Q \cap KQ/Q \neq 1$. Consequently we may assume that p is the largest prime dividing $|G|$. Then $P = A_pB_p$ is a normal Sylow p-subgroup of G and a Hall p'-subgroup of G is abelian of exponent dividing $p-1$. As K does not contain central p-chief factors by [7; IV, 5.18, V, 4.2, and V, 3.2], K is not centralised by any Hall p'-subgroup of G. Hence there exist an element $z \in K$ of p-power order and a p'-element z_1 of G in $A_{p'}B_{p'}$, where $A_{p'}$ is the Hall p'-subgroup of A and $B_{p'}$ is the Hall p'-subgroup of B, such that $z^{z_1} \neq z$. Since $z \in A_pB_p$, we can find an element $a \in A$ and an element $b \in B$ such that $z = ab$. Moreover $z_1 = a_1b_1$ for $a_1 \in A_{p'}$ and $b_1 \in B_{p'}$. Suppose that b_1 does not centralise z. Then $(ab)^{b_1} = a^{b_1}b \in K$, and so $k = a^{b_1}b(ab)^{-1} = a^{b_1}a^{-1}$ is a non-trivial element of K. Since $\langle a\rangle\langle b_1\rangle$ is a supersoluble subgroup of G, it follows that $\langle a\rangle$ is normal in $\langle a\rangle\langle b_1\rangle$ and so $k \in A_p$. Consequently $A_p \cap K \neq 1$. If a_1 does not centralise z, the above argument shows that $B_p \cap K \neq 1$. Hence 1 holds.

Assume now that $A_p \cap K \neq 1$. Then $K \cap P$ is a non-trivial normal subgroup of G. Let M be a minimal normal subgroup of G contained in $K \cap P$. By induction, the lemma holds in G/M because G/M is a totally permutable product of the nilpotent subgroups AM/M and BM/M. Moreover K/M is the nilpotent residual of G/M. Assume that p divides $|K/M|$. If $A_pM/M \cap K/M \neq 1$, then A_pM/M is contained in K/M by induction. Hence $A_p \leq K$. Therefore we may assume that $A_pM/M \cap K/M = 1$ and $A_p \cap K \leq M$. Since $A_p \cap K \neq 1$ and M is of prime order, we have $M = A_p \cap K$. On the other hand, $B_{p'}$ acts as a power automorphism group on A_p because $CB_{p'}$ is a supersoluble subgroup of G for each subgroup C of A_p. Consequently either A_p is abelian or $B_{p'}$ centralises A_p by [8; Hilfssatz 5]. Suppose that $B_{p'}$ centralises A_p. Then M is central in G. This contradicts [7; IV, 5.18, V, 4.2, and V, 3.2]. Thus $B_{p'}$ cannot centralise A_p. This implies that A_p is abelian and $B_{p'}$ acts as a non-trivial universal power automorphism group on A_p by [8; Hilfssatz 5] and [14; 13.4.3]. It follows that $[A_p, B_{p'}] = A_p$ by [7; A, 12.5]. Now A_pK/K is centralised by $B_{p'}K/K$ because G/K is nilpotent. Therefore $A_p = [A_p, B_{p'}] \leq K$ as desired. Similar arguments to those used above yield $B_p \leq K$ if $B_p \cap K \neq 1$.

Finally, suppose that A_p is contained in K and $B_{p'}$ centralises A_p. Let $1 \neq x \in A_p$. Then there exists a chief factor E/F of G below K such that $E/F = \langle xF \rangle$. It is clear that E/F is central in G, a contradiction. Consequently $B_{p'}$ does not centralise A_p and 2 holds. Analogously $A_{p'}$ does not centralise B_p if $B_p \cap K \neq 1$. The proof of the lemma is now complete. □

Proof of Theorem 2. Let p be a prime dividing $|K|$. Then, by Lemma 10, the Sylow p-subgroup K_p of K must contain A_p or B_p. Assume that B_p is a proper subgroup of K_p. Then, since K_p is normal in G, K_p is contained in the Sylow p-subgroup A_pB_p of G. Hence there exists an element $ab \in K$ with $a \in A_p$ and $b \in B_p$ and $a \neq 1$. Since B_p is contained in K, it follows that $A_p \cap K \neq 1$. By Lemma 10, A_p is contained in K and $K_p = A_pB_p$. Assume that $Z = A_p \cap B_p$. Then Z is centralised by a Hall p'-subgroup of G and by a Sylow p-subgroup of G (note that K is abelian). Since K contains no central chief factors of G by [7; IV, 5.18, V, 4.2, and V, 3.2], it follows that $Z = 1$ and $K_p = A_p \times B_p$. □

It is known that if $G = AB$ is a totally permutable product of A and B, then $[A, B^{\mathfrak{N}}] = [B, A^{\mathfrak{N}}] = 1$ ([6; Theorem 1]), but, in general, $G^{\mathfrak{N}} \neq A^{\mathfrak{N}}B^{\mathfrak{N}}$. Our next lemma analyses this case.

Lemma 11. *Let $G = AB$ be a totally permutable product of two subgroups A and B. Denote M, N, and K the nilpotent residuals of A, B, and G, respectively. Suppose that $K \neq MN$, a Sylow p-subgroup A_p of A is contained in K for some prime p, and $[A_p, B_{p'}]$ is not contained in MN for a Hall p'-subgroup $B_{p'}$ of B. Then $B_{p'}$ acts as a group of power automorphisms on A_pM/M, M is a p'-group, and A_p is subnormal in G.*

Proof. First of all we will prove that $B_{p'}$ normalises A_pM. Denote by T the nilpotent residual of $B_{p'}$. Note that MT is a normal subgroup of $AB_{p'}$ by [6; Theorem 1]. Let H/MT be the nilpotent residual of $AB_{p'}/MT$. Since $[A_p, B_{p'}]$ is not contained in MN, it follows that $[A_p, B_{p'}]$ is not contained in MT. Hence $A_pMT/MT \cap H/MT \neq 1$. Now $AB_{p'}/MT$ is a totally permutable product of the nilpotent subgroups AMT/MT and $B_{p'}MT/MT$. By Lemma 10, A_pMT/MT is contained in H/MT and since A_pMT/MT is the Sylow p-subgroup of the abelian subgroup H/MT, it follows that A_pMT is normal in $AB_{p'}$. On the other hand, $AB_{p'}/M$ is a totally permutable product of the subgroups AM/M and $B_{p'}M/M$. By Beidleman and Heineken's result [6; Theorem 1], $[A_pM/M, TM/M] = 1$. Hence A_pMT/M has A_pM/M as a unique Sylow p-subgroup. Therefore A_pM is normal in $AB_{p'}$. This implies that if X is a subgroup of A_pM/M, then $X(B_{p'}M/M)$ is a subgroup of G/M and $B_{p'}M/M$ normalises X. This means that $B_{p'}$ acts as a group of power automorphisms on A_pM/M. Assume that $A_pM/M \neq 1$. We prove by induction on $|M|$ that $M_p = A_p \cap M = 1$. If $M = 1$, the result is clear. Suppose that $M \neq 1$ and let U be a minimal normal subgroup of $(A_pM)B_{p'}$ contained in M. By induction, M/U is a p'-group. If U is a p'-group, there is nothing to be proved. Therefore we may assume that U is a p-group. On the other hand, A_pM/M is abelian because $A_pM/M \leq K/M$. Since $B_{p'}M/M$ acts as a group of power automorphisms on A_pM/M and $[B_{p'}, A_p]$ is not contained in M, it follows that $B_{p'}M/M$ acts as a group of non-trivial universal power automorphisms on A_pM/M by [8; Hilfssatz 5] and [14; 13.4.3]. Applying [7; A, 12.5], $\mathrm{C}_{A_pM/M}(B_{p'}M/M) = 1$, $A_pM/M = [A_pM/M, B_{p'}M/M]$, and $\langle (B_{p'}M/M)^{A_pM/M} \rangle = (A_pB_{p'})M/M$. This means that $\langle (B_{p'})^{A_p} \rangle M = A_pB_{p'}M$ and $A_pB_{p'} = \langle (B_{p'})^{A_p} \rangle (M \cap A_pB_{p'})$. Denote $Y = \langle (B_{p'})^{A_p} \rangle$ and $Z = M \cap A_pB_{p'}$. Without loss of generality, we can assume that $A_p = Y_pZ_p$ by [1; 1.3.3], where Y_p and Z_p are Sylow p-subgroups of Y and Z, respectively. Then $[A_p, M] = [Y_pZ_p, M] = [Z_p, M]$ because $[Y, M] = 1$ by [6; Theorem 1]. Since $Z_p \leq U$, it follows that $[A_p, M] \leq U$. In particular, $MA_p/U = M/U \times A_pU/U$. This implies that A_p is normalised by $B_{p'}$. Since A_p is a Sylow p-subgroup of $A_pB_{p'}$, we have that $B_{p'}$ acts as a group of power automorphisms on A_p. Moreover $B_{p'}$ does not centralise A_p. By [8; Hilfssatz 5] and [14; 13.4.3], $B_{p'}$ must act as a group of universal power automorphisms on A_p. This contradicts the fact that $1 \neq U$ and $B_{p'}$ centralises U.

Consequently M is a p'-group and $A_p \cong A_pM/M$ is abelian because $B_{p'}M/M$ does not centralise A_pM/M. Note that the arguments used above show that A_p is contained in $Y = \langle (B_{p'})^{A_p} \rangle$ and so $[A_p, M] = 1$ because $[Y, M] = 1$. Therefore $[MN, A_p] = 1$ (note that $[N, A] = 1$ by [6; Theorem 1]) and A_p is normal in A_pMN. Now we distinguish two cases. If no non-trivial elements of B_pMN/MN belong to K/MN, then, by Lemma 10, A_pMN/MN is a normal Sylow subgroup of the abelian normal subgroup K/MN of G/MN, and so A_pMN is normal in G. If a non-trivial element of B_pMN/MN belongs to K/MN, then, by Lemma 10, we have that $B_pMN/MN \leq K/MN$ and thus $P/MN = A_pMN/MN \times B_pMN/MN$ is a Sylow p-subgroup of K/MN. Since K/MN is abelian, we have that P/MN is normal in K/MN, and so A_pMN is a subnormal subgroup of G. Hence, in both cases, A_p is a subnormal subgroup of G. □

Proof of Theorem 3. Assume the result is false and let G be a counterexample of minimal order. Then G has a unique minimal normal subgroup because $\mathfrak{X} \circ \mathfrak{N}$ is a formation. Denote $M = A^{\mathfrak{N}}$, $N = B^{\mathfrak{N}}$, and $K = G^{\mathfrak{N}}$. Then M and N are normal subgroups of G because $[M, B] = 1 = [N, A]$ by [6; Theorem 1]. Since A and B are $(\mathfrak{X} \circ \mathfrak{N})$-groups, it follows that M and N are

$\mathfrak{X}$-groups. Therefore $MN \in \mathfrak{X}$ because formations are closed under taking central products ([7; A, 19.4]). This implies that $K \neq MN$. By Lemma 10, there exists a prime p such that either $A_pMN/MN \leq K/MN$ and $B_{p'}$ does not centralise A_pMN/MN or $B_pMN/MN \leq K/MN$ and $A_{p'}$ does not centralise B_pMN/MN. Suppose that $A_pMN/MN \leq K/MN$. By Lemma 11, $B_{p'}$ acts as a group of power automorphisms on A_pMN/MN, M is a p'-group and A_p is a subnormal subgroup of G. Since $A_p \neq 1$, it follows that $\mathrm{O}_p(G) \neq 1$ and so $\mathrm{Soc}(G)$ is a p-group. This implies that $M = 1$. If $N = 1$, then K is abelian and so $K \in \mathfrak{X}$, a contradiction. Consequently $N \neq 1$ and $\mathrm{Soc}(G)$ is contained in N. In particular, N is not a p'-group. By Lemmas 10 and 11, $B_pN/N \cap K/N = 1$ and, by Theorem 2, A_pN/N is the Sylow p-subgroup of K/N. If q is another prime dividing $|K/N|$, then either $B_{q'}$ does not centralise A_qN/N and A_q is subnormal in G, or $A_{q'}$ does not centralise B_qN/N and B_q is subnormal in G. In both cases, $\mathrm{O}_q(G) \neq 1$, a contradiction. Therefore K/N is a p-group and $K/N = A_pN/N$. Now $B_{p'}$ acts as a group of power automorphisms on A_p because A_p is subnormal in G and the product $A_pB_{p'}$ is totally permutable. Since $B_{p'}$ does not centralise A_p, it follows that A_p is abelian. Consequently $K = A_pN$ is a central product of an abelian group and an $\mathfrak{X}$-group. By [7; A, 19.4], this implies that $K \in \mathfrak{X}$, final contradiction. □

The following result is needed in the proof of Theorem 4.

Lemma 12. *Let $\mathfrak{F}$ be a formation and let G be a group in $\mathfrak{F}$ which is a central product of the subgroups A and B. Then A and B belong to $\mathfrak{F}$.*

Proof. Assume that the result is false. Choose for G a group of least order such that G is a central product of the subgroups A and B, but A does not belong to $\mathfrak{F}$. Among all these pairs of subgroups (A, B), we can choose one with $|A| + |B|$ minimal. By [7; IV, 1.14], B cannot be nilpotent. Let M be a maximal subgroup of B such that $B = M\,\mathrm{F}(B)$. Then $G = AM\,\mathrm{F}(B)$. Since AM is a supplement to $\mathrm{F}(B)$ in G, we have that AM belongs to $\mathfrak{F}$ by [7; IV, 1.14]. On the other hand, AM is a central product of A and M. The minimality of (G, A, B) yields that $A \in \mathfrak{F}$, a contradiction. □

Proof of Theorem 4. Assume that the result is false. Let $G \in \mathfrak{X} \circ \mathfrak{N}$ be a group of least order such that G is a totally permutable product of two groups A and B, but $B \notin \mathfrak{X} \circ \mathfrak{N}$. Let L be a normal subgroup of G. Since G/L is a totally permutable product of the subgroups AL/L and BL/L, we have that $BL/L \in \mathfrak{X} \circ \mathfrak{N}$. Since $BL/L \cong B/B \cap L$, we have that $B^{\mathfrak{X} \circ \mathfrak{N}} \leq B \cap L$. If G has two minimal normal subgroups, then $B \in \mathfrak{X} \circ \mathfrak{N}$, a contradiction. Hence G has a unique minimal normal subgroup, L say. Set $M = A^{\mathfrak{N}}$, $N = B^{\mathfrak{N}}$, and $K = G^{\mathfrak{N}}$. Then K/MN is the nilpotent residual of G/MN.

If $K = MN$, then M, $N \in \mathfrak{X}$ by Lemma 12. Hence $B \in \mathfrak{X} \circ \mathfrak{N}$, a contradiction. Therefore $K \neq MN$. By Lemma 10, there exists a prime p dividing $|K/MN|$ such that either $A_pMN/MN \leq K/MN$ and $B_{p'}$ does not centralise A_pMN/MN, or $B_pMN/MN \leq K/MN$ and $A_{p'}$ does not centralise B_pMN/MN. Suppose that the first possibility holds. Then, arguing as in the above theorem, $M = 1$ and L is a p-group. Therefore $A_{p'}$ centralises B_pMN/MN and $B_p/MN \cap K/MN = 1$ by Lemma 10. This implies that A_pN/N is a Sylow p-subgroup of K/N. If K/N is not a p-group and q is another prime dividing its order, then $\mathrm{O}_q(G) \neq 1$ by Lemmas 10 and 11, a contradiction. Consequently $K = A_pN$ is a central product of A_p and N. By Lemma 12, $N \in \mathfrak{X}$, a contradiction. □

Proof of Theorem 5. Let G be a totally permutable product of the subgroups A and B. Since $A^{\mathfrak{F}} \leq A^{\mathfrak{N}}$ and $B^{\mathfrak{F}} \leq B^{\mathfrak{N}}$, from [6; Theorem 1] it follows that $[A^{\mathfrak{F}}, B] = [A, B^{\mathfrak{F}}] = 1$, and so $A^{\mathfrak{F}}$ and $B^{\mathfrak{F}}$ are normal subgroups of G.

On the other hand, since $A^{\mathfrak{F}}B^{\mathfrak{F}}$ is a normal subgroup of G and $G/A^{\mathfrak{F}}B^{\mathfrak{F}}$ is a totally permutable product of the subgroups $AB^{\mathfrak{F}}/A^{\mathfrak{F}}B^{\mathfrak{F}}$ and $BA^{\mathfrak{F}}/A^{\mathfrak{F}}B^{\mathfrak{F}}$, which belong to $\mathfrak{F}$, $G/A^{\mathfrak{F}}B^{\mathfrak{F}}$ belongs to $\mathfrak{F}$ by Theorem 3. It follows that $G^{\mathfrak{F}} \leq A^{\mathfrak{F}}B^{\mathfrak{F}}$.

Now $G/G^{\mathfrak{F}} \in \mathfrak{F}$ is a totally permutable product of the subgroups $AG^{\mathfrak{F}}/G^{\mathfrak{F}}$ and $BG^{\mathfrak{F}}/G^{\mathfrak{F}}$. Hence both factors belong to $\mathfrak{F}$ by Theorem 4. But $AG^{\mathfrak{F}}/G^{\mathfrak{F}} \cong A/(A \cap G^{\mathfrak{F}}) \in \mathfrak{F}$, which implies that $A^{\mathfrak{F}} \leq G^{\mathfrak{F}}$. Analogously, $B^{\mathfrak{F}} \leq G^{\mathfrak{F}}$. It follows that $A^{\mathfrak{F}}B^{\mathfrak{F}} \leq G^{\mathfrak{F}}$, and so $A^{\mathfrak{F}}B^{\mathfrak{F}} = G^{\mathfrak{F}}$, as desired. □

Proof of Theorem 6. Let $G = AB$ be a totally permutable product of the subgroups A and B. As usual, denote M, N, and K the nilpotent residuals of A, B, and G, respectively. Let a_p (respectively, b_p) be the number of non-isomorphic non-central p-chief factors in A (respectively, B) for a prime p. Then all p-chief factors of G above K are central and every chief factor of G between MN and K is non-central because K/MN is the nilpotent residual of G/MN and K/MN is abelian by Theorem 1 and [7; IV, 5.18, V, 4.2, and V, 3.2]. On the other hand, note that the p-chief factors of G covered by M are centralised by B, so they are indeed chief factors of A. A similar argument shows that the p-chief factors of G covered by N are chief factors of B. Now a non-central p-chief factor of A covered by M cannot be G-isomorphic to a non-central p-chief factor of B covered by N. Consequently the number of non-central non-G-isomorphic chief factors of G covered by MN is $a_p + b_p$.

If p does not divide $|K/MN|$, then the number of non-central non-G-isomorphic p-chief factors of G is a_p+b_p. Assume now that p divides $|K/MN|$. By Lemma 10, either $A_p \leq K$ or $B_p \leq K$, where A_p is a Sylow p-subgroup of A and B_p is a Sylow p-subgroup of B. Assume that $1 \neq A_pMN/MN \leq$

K/MN. By Lemma 11, we have that M is a p'-group and $B_{p'}$ acts as a group of non-trivial power automorphisms on A_pM/M. In particular, $B_{p'}$ acts as a group of non-trivial power automorphisms on A_pMN/MN. Then A_pMN/MN is abelian by [8; Hilfssatz 5] and all chief factors of G between MN and A_pMN are G-isomorphic p-chief factors. If $1 \neq B_pMN/MN \leq K/MN$, the same argument shows that N is a p'-group and all chief factors of G between MN and B_pMN are isomorphic when regarded as G-modules. This gives two G-isomorphism classes of non-central chief factors. If $K/MN \cap B_pMN/MN = 1$, then the number c_p of non-isomorphic non-central chief factors is bounded by $1 + b_p$. Therefore we have the bound $c_p \leq a_{0p} + b_{0p}$, where $a_{0p} = \max\{1, a_p\}$ and $b_{0p} = \max\{1, b_p\}$. □

Proof of Theorem 8. Since PST_c-groups are abelian-by-nilpotent by [13; Theorem 2], we have that $G = AB$ is abelian-by-nilpotent by Theorem 3. Thus the nilpotent residual $L = G^{\mathfrak{N}}$ of G is abelian. Let p be a prime dividing $|L|$. Denote $F_p = \mathrm{O}_p(G)$ and let L_p be a Sylow p-subgroup of L. Then L_p is contained in F_p. Since A and B have coprime indices in G, we have that either F_p is contained in A or F_p is contained in B. Assume, for instance, that F_p is contained in A. Since F_p is a normal subgroup of G and the product is totally permutable, we have that for every subgroup X of F_p and for every p'-subgroup Y of B, XY is a subgroup of G. Hence Y normalises X. Therefore the p'-elements of B induce power automorphisms on F_p. On the other hand, since F_p is a nilpotent normal subgroup of A, we have that F_p is contained in $\mathrm{F}(A)$, the Fitting subgroup of A. Since the nilpotent residual $A^{\mathfrak{N}}$ is abelian, $A = A^{\mathfrak{N}}C$ and $A \cap C = 1$ for a Carter subgroup C of A by [9; VI, 7.15]. Now $A^{\mathfrak{N}}$ is a Hall subgroup of $\mathrm{F}(A)$ by [13; Theorem 2]. This implies that $\mathrm{F}(A) = A^{\mathfrak{N}} \times \big(F(G) \cap C\big)$. Hence either F_p is contained in $A^{\mathfrak{N}}$ or F_p is contained in C. If $F_p \leq A^{\mathfrak{N}}$, then the p'-elements of A induce power automorphisms on F_p by [13; Theorem 2]. If F_p is contained in C, then F_p is centralised by a Hall p'-subgroup of C and by $A^{\mathfrak{N}}$. Therefore, in any case, the p'-elements of A act as power automorphisms on F_p. Let q be a prime number different from p. There exists a Sylow q-subgroup G_q such that $G_q = A_qB_q$ for suitable Sylow q-subgroups A_q of A and B_q of B by [1; 1.3.3]. It follows that every subgroup of F_p is normalised by G_q. Since F_p is normal in G, it follows that all p'-elements of G act as power automorphisms on F_p. Note that F_p cannot be centralised by all p'-elements of G because L does not contain central p-chief factors of G. But if $L_p \neq F_p$, then the p'-elements of G act trivially on F_p/L_p, but non-trivially on L_p, a contradiction. Hence $L_p = F_p$. By [13; Theorem 2], G is a PST_c-group. □

Proof of Corollary 9. Let M be a normal subgroup of G. Since G/M is a totally permutable product of the PST-groups AM/M and BM/M, and both have coprime index in G/M, it follows that G/M is a PST_c-group by Theorem 8. By [13; Theorem 7], we have that G is a PST-group. □

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