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Baire-Type Properties in Metrizable $c_0(\Omega, X)$

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Abstract: Ferrando and Lüdkovsky proved that for a non-empty set Ω and a normed space X, the normed space $c_0(\Omega, X)$ is barrelled, ultrabornological, or unordered Baire-like if and only if X is, respectively, barrelled, ultrabornological, or unordered Baire-like. When X is a metrizable locally convex space, with an increasing sequence of semi-norms $\{\|.\|_n \in \mathbb{N}\}$ defining its topology, then $c_0(\Omega, X)$ is the metrizable locally convex space over the field $\mathbb{K}$ (of the real or complex numbers) of all functions $f : \Omega \to X$ such that for each $\varepsilon > 0$ and $n \in \mathbb{N}$ the set $\{\omega \in \Omega : \|f(\omega)\|_n > \varepsilon\}$ is finite or empty, with the topology defined by the semi-norms $\|f\|_n = \sup\{\|f(\omega)\|_n : \omega \in \Omega\}$, $n \in \mathbb{N}$. Kąkol, López-Pellicer and Moll-López also proved that the metrizable space $c_0(\Omega, X)$ is quasi barrelled, barrelled, ultrabornological, bornological, unordered Baire-like, totally barrelled, and barrelled of class p if and only if X is, respectively, quasi barrelled, barrelled, ultrabornological, bornological, unordered Baire-like, totally barrelled, and barrelled of class p. The main result of this paper is that the metrizable $c_0(\Omega, X)$ is baireled if and only if X is baireled, and its proof is divided in several lemmas, with the aim of making it easier to read. An application of this result to closed graph theorem, and two open problems are also presented.

Keywords: Banach disk; Baire-like; barrelled; metrizable; p-barrelled; ultrabornological; unordered Baire-like

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1. Introduction

Let Ω be a non-empty set, X a locally convex space over the field $\mathbb{K}$ (of real or complex numbers), $cs(X)$ the family of all continuous seminorms in X, $\ell_1(X)$ the space of all absolutely summable sequences in X, namely

$$\ell_1(X) := \left\{(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}} : \left\|(x_n)_{n\in\mathbb{N}}\right\|_p = \sum_{n=1}^{\infty} p(x_n) < \infty, \text{ for all } p \in cs(X)\right\}$$

endowed with the family of seminorms $\left\{\|\cdot\|_p : p \in cs(X)\right\}$, and $c_0(\Omega, X)$ the locally convex space over $\mathbb{K}$ of all functions $f : \Omega \to X$ such that for each $\varepsilon > 0$ and $p \in cs(X)$ the set $\{\omega \in \Omega : p(f(\omega)) > \varepsilon\}$ is finite or empty, with the topology defined by the semi-norms $\|f\|_p = \sup\{p(f(\omega)) : \omega \in \Omega\}$, $p \in cs(X)$.

In particular, $c_0(\Omega) := c_0(\Omega, \mathbb{K})$ and for $\Omega = \mathbb{N}$, $c_0(X) := c_0(\mathbb{N}, X)$ and $c_0 := c_0(\mathbb{N}, \mathbb{K})$. It was proved in [1] that $c_0(X)$ is quasibarrelled if and only if X is quasibarrelled and its strong dual satisfies the condition (B) of Pietsch and that if, in addition, X is complete in the sense of Mackey, then $c_0(X)$ is barrelled if and only if X is quasibarrelled and its strong dual satisfies condition (B) of Pietsch. In this case, X is barrelled. Through a clever use of a sliding-hump technique, it was proved in [2] that, even in the absence of completeness in the sense of Mackey, $c_0(X)$ is barrelled if and only if X is barrelled and its strong dual satisfies condition (B) of Pietsch. Recall that X has the property (B) of Pietsch if for any bounded set $\mathcal{B}$ in $\ell_1(X)$ there exists an absolutely convex bounded set B in X such that the normed space X_B formed by the linear hull of B endowed with Minkowski functional p_B of B verifies that $\mathcal{B}$ is contained in the unit ball of the normed space $\ell_1(X_B)$, i.e.,

$$\mathcal{B} \subset \left\{ (x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}} : \sum_{n=1}^{\infty} p_B(x_n) < \infty \right\}$$

Metrizable locally convex spaces as well as dual metric locally convex spaces verify the property (B) of Pietsch ([3]).

Ferrando and Lüdkowsky proved in [4] that for a normed space X the space $c_0(\Omega, X)$ is barrelled, ultrabornological, or unordered Baire-like (see [5]) if and only if X is, respectively, barrelled, ultrabornological, or unordered Baire-like. It was proved in [6] that for a locally convex metrizable space X the space $c_0(\Omega, X)$ is quasi barrelled, barrelled, ultrabornological, bornological, unordered Baire-like, totally barrelled, and barrelled of class p if and only if X is, respectively, quasi barrelled, barrelled, ultrabornological, bornological, unordered Baire-like, totally barrelled, and barrelled of class p. The normed space of all continuous functions vanishing at infinity defined on a locally compact topological space with values in a normed space and endowed with the supremum norm topology is barrelled if and only if X is barrelled; this result was obtained in [7], answering a question posed by J. Horváth.

The linear subspace l_0^∞ of the sequence space l_∞ of finite-valued sequences in the field $\mathbb{K}$ is of the first Baire category [8]. Independently, Dieudonné ([9], p. 133) and Saxon [10] proved that l_0^∞ is barrelled. Schachermayer extended this result by proving that the linear hull $l_0^\infty(\mathcal{A})$ of the characteristic functions $\mathcal{X}_A$, with $A \in \mathcal{A}$, and where $\mathcal{A}$ is a ring of subsets of Ω, endowed with the supremum norm topology, is barrelled if and only if the vector space $ba(A)$, of all bounded finitely additive scalar measures defined on $\mathcal{A}$ equipped with the supremum norm topology, verifies the Nikodým boundedness theorem, see ([11], p. 80).

Furthermore, if $\mathcal{A}$ is a σ-algebra, the space $l_0^\infty(\mathcal{A})$ is barrelled, see ([11], p. 80) and [12]. Valdivia [13] improved this result: If $(E_n)_n$ is an increasing sequence of vector subspaces of $l_0^\infty(\mathcal{A})$ covering $l_0^\infty(\mathcal{A})$, then there is an E_n barrelled and dense in $l_0^\infty(\mathcal{A})$. From this property, *suprabarrelled* spaces are defined, also known as (db) spaces in [14,15]. Interesting applications of suprabarrelled spaces can be found in [13,16] and ([17], Chapter 9). A natural generalization of suprabarrelled spaces are p-barrelled spaces. Let $\mathbb{N}^{\leq p} := \bigcup_{k=1}^{p} \mathbb{N}^k$, $\mathbb{N}^{<\infty} := \bigcup_{k=1}^{\infty} \mathbb{N}^k$ and recall, see [18] and ([19], Definition 3.2.1) that a *p-net* in a vector space E is a family $\mathcal{W} = \left\{ E_t : t \in \mathbb{N}^{\leq p} \right\}$ of vector subspaces of E, such that $E = \cup\{E_n : n \in \mathbb{N}\}$, $E_n \subset E_{n+1}$, $E_t = \bigcup\{E_{t,n} : n \in \mathbb{N}\}$, $E_{t,n} \subset E_{t,n+1}$, for $t \in \mathbb{N}^{\leq r}$, $1 \leq r < p$ and $n \in \mathbb{N}$. Analogously, a *linear web* in E is a family $\mathcal{W} = \{E_t : t \in \mathbb{N}^{<\infty}\}$ of vector subspaces of E, such that $E = \cup\{E_n : n \in \mathbb{N}\}$, $E_n \subset E_{n+1}$, $E_t = \bigcup\{E_{t,n} : n \in \mathbb{N}\}$, $E_{t,n} \subset E_{t,n+1}$, for $t \in \mathbb{N}^{<\infty}$ and $n \in \mathbb{N}$.

All topological spaces are supposed to be Hausdorff and *space* will be used as an abbreviation of locally convex space, when misunderstanding is not possible. A locally convex space E is called *p-barrelled* if given a p-net $\mathcal{W} = \left\{ E_t : t \in \mathbb{N}^{\leq p} \right\}$ there is a $t \in \mathbb{N}^p$ such that E_t is barrelled and dense in E (see [19], Definition 3.2.2). Note that suprabarrelled spaces are 1- barrelled spaces. We refer the reader to [20] for several applications of p-barrelled spaces, particularly in vector measures. The locally convex space E is $\aleph_0$*-barrelled* if it is p-barrelled, for each $p \in \mathbb{N}$ (see [19], Definition 4.1.1) and E is *baireled* if each *linear web* $\mathcal{W} = \{E_t : t \in \mathbb{N}^{<\infty}\}$ in E admits a *strand* formed by dense barrelled subspaces of E, i.e., there exists a sequence $(n_i : i \in \mathbb{N})$ such that $E_{n_1 n_2 \cdots n_i}$ is a barrelled and dense subspace of E, for each $i \in \mathbb{N}$ (see [21], Definition 1 and Theorem 1). It was proved in [22] that for a σ-algebra $\mathcal{A}$ the space $l_0^\infty(\mathcal{A})$ is baireled. Other related properties can be found in [23] and references therein.

In this paper, it is assumed that the locally convex space X is metrizable, denoting by $\{\|.\|_n \in \mathbb{N}\}$ an increasing sequence of semi-norms defining the topology of X, i.e., for every $x \in X$, we have that $\|x\|_n \leq \|x\|_{n+1}$, $n \in \mathbb{N}$. Then, the locally convex space $c_0(\Omega, X)$ is metrizable and its topology is defined by the semi-norms $\|f\|_n = \sup\{\|f(\omega)\|_n : \omega \in \Omega\}$, $n \in \mathbb{N}$ and $f \in c_0(\Omega, X)$. Now for every $f \in c_0(\Omega, X)$, its support, i.e., $\text{supp} f := \{w \in \Omega : f(w) \neq 0\}$, is countable since $\{w \in \Omega : f(w) \neq 0\} =$

$\bigcup_{n,m=1}^{\infty}\left\{\omega \in \Omega : \|f(\omega)\|_n > \frac{1}{m}\right\}$ and, by definition, for each $\varepsilon > 0$ and $n \in \mathbb{N}$ the set $\{\omega \in \Omega : \|f(x)\|_n > \varepsilon\}$ is finite or empty.

The aim of the paper is to characterize those spaces $c_0(\Omega, X)$ which are baireled. We will prove that $c_0(\Omega, X)$ is baireled if and only if X is baireled (Theorem 2). In order to do this, we need the characterization for $c_0(\Omega, X)$ to be barrelled obtained in ([6], Corollary 2.4). For the sake of completeness, we will remind readers of this characterization in Section 2.

If Γ is a subset of Ω, we denote by $c_0(\Gamma, X)$ the linear subspace of $c_0(\Omega, X)$ consisting of all functions f such that $f(\Omega \setminus \Gamma) = \{0\}$. By $\langle V \rangle$, we denote the linear hull of a subset V of a linear space X, and, if V is absolutely convex and bounded, then $\langle V \rangle_V$ is the normed space formed by $\langle V \rangle$, endowed with the norm defined by the functional of Minkowski of V.

Recall that an absolutely convex bounded set V in X is a Banach disk if the normed space $\langle V \rangle_V$ is a Banach space, and that a locally convex space X is *barrelled* (*quasibarrelled*) if every closed absolutely convex and absorbing (and bornivorous) subset of E is a neighborhood of zero. Barrelled spaces are just the locally convex spaces that verify the Banach–Steinhaus boundedness theorem. Todd and Saxon [5] discovered an applicable and natural generalization of Baire spaces to locally convex spaces: A locally convex space X is called *unordered Baire-like*, if every sequence of absolutely convex and closed subsets of X covering X contains a member which is a neighborhood of zero. Finally, a locally convex space X is *totally barrelled* if for every sequence of subspaces $(X_n)_{n\in\mathbb{N}}$ of X covering X, there is some X_p which is barrelled and its closure is finite-codimensional in X, see ([19], Definition 1.4.1) and [24]. Note that Baire $\Rightarrow$ Unordered Baire-like $\Rightarrow$ Totally barrelled $\Rightarrow$Baireled $\Rightarrow$ $\aleph_0$-barrelled $\Rightarrow$ $p+1$-barrelled $\Rightarrow$ p-barrelled $\Rightarrow$ Baire-like $\Rightarrow$ barrelled$\Rightarrow$ quasibarrelled.

Even for metrizable locally convex spaces, $\aleph_0$-barrelled$\not\Rightarrow$Baireled$\not\Rightarrow$Totally barrelled ([21], Theorems 2 and 3).

2. Revisiting Barrelledness in $c_0(\Omega, X)$

It is well known that, if $\varphi : E \to F$ is a continuous linear map from a Banach space E into a locally convex space F and D is the open unit ball of E, then the normed space $\langle \varphi(D) \rangle_{\varphi(D)}$ is isometric to the quotient $E/(\varphi^{-1}(0))$, hence $\varphi(D)$ is a Banach disk. If B is the closed unit ball of E, then the inclusions $D \subset B \subset 2D$ imply that $\varphi(B)$ is also a Banach disk.

This well known property is used in the following lemmas.

Lemma 1 ([6], Lemma 2.1)**.** *Let X be a metrizable locally convex space and $(f_n)_n$ a bounded sequence in $c_0(\Omega, X)$ such that the set $\{n \in \mathbb{N} : f_n(\omega) \neq 0\}$ is finite or empty for every $\omega \in \Omega$. Then, $(f_n)_n$ is contained in a Banach disk. In particular, if $\Omega = \mathbb{N}$ and supp$(f_n) \subset \mathbb{N}\setminus\{1,2,...,n\}$, for each $n \in \mathbb{N}$, then also $(f_n)_n$ is contained in a Banach disk.*

Proof. The boundedness implies that $M_p = \sup\left\{\|f_n\|_p : n \in \mathbb{N}\right\}$ is finite for each $p \in \mathbb{N}$. Then, for each $\{\xi_n : n \in \mathbb{N}\} \in l_1$, the inequality

$$\left\|\sum_{n=1}^{\infty} \xi_n f_n\right\|_p \leq M_p \sum_{n=1}^{\infty} |\xi_n|$$

implies the continuity of the map $\varphi : l_1 \longrightarrow c_0(\Omega, X)$ defined by $\varphi(\{\xi_n : n \in \mathbb{N}\}) := \sum_{n=1}^{\infty} \xi_n f_n$. Hence, if B is the closed unit ball of l_1, then $\varphi(B)$ is a Banach disk that contains the sequence $(f_n)_n$. □

From Lemma 1, it follows that, if T is an absolutely convex subset of $c_0(\Omega, X)$ that absorbs its Banach disks, then there exists in Ω a countable subset Δ and a natural number n such that T absorbs $\{f \in c_0(\Omega\setminus\Delta, X) : \|f\|_n \leq 1\}$ because, if this is not the case, there exists a sequence $(f_n)_n$ such that $f_1 \notin T$, $f_n \in c_0(\Omega\setminus \bigcup_{i=1}^{n-1} \Delta_i, X)\setminus nT$, for $n \geq 2$, where $\Delta_i := \mathrm{supp}(f_i)$, $1 \leq i$, and $\|f_n\|_n \leq 1$ for $n = 1, 2, ...$ The boundedness of $\{f_n : n \in \mathbb{N}\}$ and

Lemma 1 implies that there exists $k \in \mathbb{N}$ such that $\{f_n : n \in \mathbb{N}\} \subset kT$, which yields to the contradiction $f_k \in kT$.

Lemma 2 ([6], Lemma 2.1)**.** *Let T be an absolutely convex subset of $c_0(\Omega, X)$ that absorbs its Banach disks. Then, there exists in Ω a finite subset Δ and a natural number n such that T absorbs $\{f \in c_0(\Omega \diagdown \Delta, X) : \|f\|_n \leq 1\}$.*

Proof. By the observation preceding this lemma, it is enough to prove that, if T is an absolutely convex subset of $c_0(\mathbb{N}, X)$ that absorbs its Banach disks, then there exists $m \in \mathbb{N}$ such that T absorbs

$$\{f \in c_0(\mathbb{N} \diagdown \{1, 2, ..., m\}, X) : \|f\|_m \leq 1\}.$$

Otherwise, there exists $f_n \in c_0(\mathbb{N} \diagdown \{1, 2, ..., n\}, X) \diagdown nT$, with $\|f_n\|_n \leq 1$, for each $n \in \mathbb{N}$. By Lemma 1, there is $h \in \mathbb{N}$ such that $\{f_n : n \in \mathbb{N}\} \subset hT$ and we reach the contradiction $f_h \in hT$. □

The above lemmas nicely apply to get the following characterization of barrelled $c_o(\Omega, X)$.

Theorem 1 ([6], Corollary 2.4a)**.** *Let X be a metrizable locally convex space and Ω a non void set. Then, $c_0(\Omega, X)$ is barrelled if and only if X is barrelled.*

Proof. Fix $p \in \Omega$. As the quotient $c_0(\Omega, X) \diagup c_0(\Omega \diagdown \{p\}, X)$ is isomorphic to X and barrelledness property is inherited by quotients, see ([25] [27.1 (4) and 28.4 (2)]), then, if $c_0(\Omega, X)$ is barrelled, we deduce that X is also barrelled.

Conversely, if T is a barrel in $c_0(\Omega, X)$ and B is a Banach disk in $c_0(\Omega, X)$, it is obvious that T contains a neighborhood of zero in the Banach space $\langle B \rangle_B$, hence there exists a $\lambda > 0$ such that $\lambda B \subset T$. Then, by Lemma 2, there exists in Ω a finite subset Δ such that T contains a neighborhood of zero in $c_0(\Omega \diagdown \Delta, X)$. Hence, if X is barrelled, T also contains a neighborhood of $c_0(\Omega, X)$ because the space $c_0(\Omega, X)$ is isomorphic to the product $c_0(\Omega \diagdown \Delta, X) \times X^\Delta$, and X^Δ is barrelled. □

The analogous result of Theorem 1 for quasibarrelled, ultrabornological, bornological, unordered Baire like, totally barrelled, and barrelled spaces of class p are provided in ([6], Corollaries 2.4 and 2.5 and Theorem 3.7). The unordered Baire-like and the totally barrelled results need in their proofs the preceding lemmas and the following nice result ([5], Theorem 4.1): *If the union of two countable families $\mathcal{F}$ and $\mathcal{G}$ of linear subspaces of a linear space E covers E, then one of them covers E.* In fact, assume that there exists $x \in \cup\{F_i : F_i \in \mathcal{F}\}$, with $x \notin \cup\{G_j : G_j \in \mathcal{G}\}$, and there exists $y \in \cup\{G_j : G_j \in \mathcal{G}\}$, with $y \notin \cup\{F_i : F_i \in \mathcal{F}\}$. As the subset $\{x + t(y - x) : t \in \mathbb{R}\}$ is uncountable, we may suppose that there exists $F_{i_m} \in \mathcal{F}$ and $t_1 \neq t_2$ such that $\{x + t_n(y - x) : n = 1, 2\} \subset F_{i_m}$. This inclusion implies that $\{x + t(y - x) : t \in \mathbb{R}\} \subset F_{i_m}$ because F_{i_m} is a linear subspace. In particular, for $t = 1$, we obtain that $y \in F_{i_m}$, in contradiction with $y \notin \cup\{F_i : F_i \in \mathcal{F}\}$.

The fact that $c_0(\Omega, X)$ is barrelled of class p if and only if X is barrelled of class p, for each $p \in \mathbb{N}$, implies directly that $c_0(\Omega, X)$ is $\aleph_0$-barrelled if and only if X is $\aleph_0$-barrelled.

3. Baireledness

In this section, we prove that the space $c_0(\Omega, X)$ is baireled if and only if X is baireled. Recall that a locally convex space E is baireled if each linear web in E contains a strand formed by Baire-like spaces [26] and that, if E is metrizable, then E is baireled if each linear web in E contains a strand formed by barrelled spaces.

Let $\mathbb{T}$ be a non-void subset of $\mathbb{N}^{<\infty} := \cup\{\mathbb{N}^s : s \in \mathbb{N}\}$ and let $t = (t_1, t_2, \ldots, t_p)$ be an element of $\mathbb{T}$. The element $t(i) := (t_1, t_2, \ldots, t_i)$, if $1 \leqslant i \leqslant p$, and $t(i) := \emptyset$ if $i > p$, and the set $T(i) := \{t(i) : t \in T\}$ are named the *section of length i* of t and T, respectively. With this notation, a sequence $(t^n : n \in \mathbb{N})$ formed by elements of $\mathbb{N}^{<\infty}$ is a *strand* if $t^{n+1}(n) = t^n(n)$, for each $n \in \mathbb{N}$. A non-void subset $\mathbb{T}$ of $\mathbb{N}^{<\infty}$ is *increasing* if, for each

$t = (t_1, t_2, \ldots, t_p) \in \mathbb{T}$, there exists p scalars t_i^i verifying $t_i < t_i^i$, for $1 \leqslant i \leqslant p$, such that $(t_1^1) \in \mathbb{T}(1)$ and $(t_1, t_2, \ldots, t_{i-1}, t_i^i) \in \mathbb{T}(i)$, $1 < i \leq p$. If $s = (s_1, s_2, \ldots, s_q) \in \mathbb{N}^{<\infty}$ then $(t, s) := (t_1, t_2, \ldots, t_p, s_1, s_2, \ldots, s_q)$

The following definition provides a particular type of increasing subsets U of $\mathbb{N}^{<\infty}$ considered in ([27], Definition 1) and named NV-trees, reminding readers of O.M. Nikodým and M. Valdivia.

Definition 1. *An NV-tree is a non-void increasing subset $\mathbb{T}$ of $\mathbb{N}^{<\infty}$ without strands and such that, for each $t = (t_1, t_2, \ldots, t_p) \in \mathbb{T}$, the set $\{s \in \mathbb{N}^{<\infty} : (t, s) \in \mathbb{T}\}$ is empty.*

The last condition means that elements of an NV-tree $\mathbb{T}$ do not have proper continuation in $\mathbb{T}$. An NV-tree $\mathbb{T}$ is an infinite subset of $\mathbb{N}$ if and only if $\mathbb{T} = \mathbb{T}(1)$. The sets $\mathbb{N}^i$, $i \in \mathbb{N} \backslash \{1\}$, and the set $\cup\{(i, \mathbb{N}^i) : i \in \mathbb{N}\}$ are non trivial NV-trees.

If $\mathbb{T}$ is an increasing subset of $\mathbb{N}^{<\infty}$ and $\{E_u : u \in \mathbb{N}^{<\infty}\}$ is a linear web in a space E, then $(E_{u(1)})_{u \in \mathbb{T}}$ is an increasing covering of B, and for each $u = (u_1, u_2, \ldots, u_p) \in \mathbb{T}$ and each $i < p$ the sequence $(B_{u(i) \times n})_{u(i) \times n \in \mathbb{T}(i+1)}$ is an increasing covering of $B_{u(i)}$. In particular, if $\mathbb{T}$ is an NV-tree, then $E = \cup\{E_t : t \in \mathbb{T}\}$ because $\mathbb{T}$ does not contain strands.

By definition, a locally convex space E is non baireled if there exists a linear web $\{E_t : t \in \mathbb{N}^{<\infty}\}$ without a strand formed by Baire-like spaces. In particular, a metrizable barrelled locally convex space E is non baireled if there exists a linear web without a strand formed by barrelled spaces because a metrizable space is barrelled if and only if it is Baire-like.

Note that, if $(E_{n_1}, n_1 \in \mathbb{N})$ is an increasing covering of a metrizable barrelled space E then, since E is Baire-like, we may suppose, without loss of generality that all subspaces E_{n_1}, $n_1 \in \mathbb{N}$, are dense in E. Consequently, again because of denseness, if E_{n_1} is barrelled, then every E_{m_1}, with $m_1 \geq n_1$, is barrelled.

Therefore, for a linear web $\{E_t : t \in \mathbb{N}^{<\infty}\}$ in a metrizable barrelled locally convex space E that is not baireled, we may suppose that every E_{n_1} is dense and barrelled or that every E_{n_1} is dense and not barrelled, for each $n_1 \in \mathbb{N}$. The preceding process continues inductively only when we get barrelled spaces, i.e., if the dense subspace E_{n_1} is barrelled, then we may suppose that $E_{n_1 n_2}$, $n_2 \in \mathbb{N}$, is a sequence of dense subspaces such that for all $E_{n_1 n_2}$, $n_2 \in \mathbb{N}$, are not barrelled, or all $E_{n_1 n_2}$, $n_2 \in \mathbb{N}$, are barrelled; in the first case, the inductive process stops and, in the second case, we continue with the increasing sequence $(E_{n_1 n_2 n_3}, n_3 \in \mathbb{N})$. As the linear web $\{E_t : t \in \mathbb{N}^{<\infty}\}$ does not contain a strand formed by barrelled spaces, then this natural induction produces a NV-tree $\mathbb{T}$, such that, for each $t = (n_1, n_2, \cdots, n_p) \in \mathbb{T}$ the space E_t is dense in E and not barrelled, and $E_{t(i)}$ is barrelled, for each $i < p$.

The following lemmas are part of the proof of Theorem 2. Therefore, those lemmas consider that $E = c_0(\Omega, X)$, with X metrizable. Moreover, we will suppose that the metrizable space $c_0(\Omega, X)$ is barrelled and not baireled, hence $c_0(\Omega, X)$ has a linear web $\{E_t : t \in \mathbb{N}^{<\infty}\}$ without a strand formed by Baire-like spaces. With the preceding induction, we obtain a NV-tree $\mathbb{T}$, such that, for each $t = (n_1, n_2, \cdots, n_p) \in \mathbb{T}$, we have that $E_{n_1 n_2 \ldots n_p}$ is a non barrelled dense subspace of $c_0(\Omega, X)$, hence there exists a barrel $T_{n_1 n_2 \ldots n_p}$ in $E_{n_1 n_2 \ldots n_p}$ that it is no neighborhood of zero in $E_{n_1 n_2 \ldots n_p}$. With the barrels $T_{n_1 n_2 \ldots n_p}$, with $(n_1, n_2, \cdots, n_p) \in \mathbb{T}$, we form

$$Z_{n_1 n_2 \ldots n_p} := \left\langle \overline{T_{n_1 n_2 \ldots n_p}}^E \right\rangle \quad \text{and} \quad S_{n_1 n_2 \ldots n_p} = \bigcap_{m=n_p}^{\infty} Z_{n_1 n_2 \ldots n_{p-1} m}, \tag{1}$$

and

$$Z_{n_1 n_2 \ldots n_{p-1}} := \bigcup_{n_p=1}^{\infty} S_{n_1 n_2 \ldots n_{p-1} n_p} \quad \text{and} \quad S_{n_1 n_2 \ldots n_{p-1}} := \bigcap_{m=n_{p-1}}^{\infty} Z_{n_1 n_2 \ldots n_{p-2} m}, \tag{2}$$

$\cdots$ and finally

$$Z_{n_1} = \bigcup_{n_2=1}^{\infty} S_{n_1 n_2} \quad \text{and} \quad S_{n_1} = \bigcap_{m=n_1}^{\infty} Z_m. \tag{3}$$

A NV-tree $\mathbb{T}_1$ contained in a NV-tree $\mathbb{T}$ is cofinal in $\mathbb{T}$ if $\mathbb{T}_1(1)$ is a cofinal subset of $\mathbb{T}(1)$ and for each $(n_1, n_2, \cdots, n_i) \in \mathbb{T}_1(i)$ the set $\{m : (n_1, n_2, \cdots, n_i, m) \in \mathbb{T}_1(i+1)\}$ is a cofinal subset of $\{m : (n_1, n_2, \cdots, n_i, m) \in \mathbb{T}(i+1)\}$. Note that, if $\mathbb{T}_1$ is cofinal in $\mathbb{T}$ and $F \subset Z_t$, for every $t \in \mathbb{T}_1$, then $F \subset S_{m_1}$, for every $m_1 \in \mathbb{T}_1(1)$.

In the following four lemmas, we suppose the following conditions hold:

(H): X is a metrizable locally convex space such that $c_0(\Omega, X)$ is barrelled but not baireled, being $\{E_t : t \in \mathbb{N}^{<\infty}\}$ a linear web in $c_0(\Omega, X)$ without a strand formed by barrelled spaces and $\mathbb{T}$ the NV-tree such that for each $t \in \mathbb{T}$ there exists a barrel T_t in E_t which is not a neighborhood of zero in E_t and E_t is a dense subspace of $c_0(\Omega, X)$.

With these barrels T_t, with $t = (n_1, n_2, ..., n_p) \in \mathbb{T}$, we form the sets $Z_{n_1 n_2 ... n_p}$, $S_{n_1 n_2 ... n_p}$, $\cdots$, Z_{n_1} and S_{n_1}, given in (1)–(3).

Lemma 3. *Assume conditions (H) hold and let F be a linear subspace of E, τ a locally convex topology in F finer (or equal) than the topology induced by E, and such that (F, τ) is baireled. Then, there exists $m_1 \in \mathbb{N}$ such that $F \subset S_{n_1}$ for $n_1 \geq m_1$.*

In particular, if D is a Banach disk contained in E, there exists $m_1 \in \mathbb{N}$ such that $\langle D \rangle \subset S_{n_1}$, for $n_1 \geq m_1$.

Proof. By definition of baireled, it follows that, if $(E_n, n \in \mathbb{N})$ is an increasing covering of a baireled space E, then there exists a set $\mathbb{N}_1$ cofinal in $\mathbb{N}$ such that E_n is baireled and dense in E, for each $n \in \mathbb{N}_1$ (see ([21], Theorem 1) adding the trivial fact that, if a baireled space H is dense in the space G, then G is baireled). Hence, there exists an NV-tree $\mathbb{T}_1$ that is cofinal in $\mathbb{T}$ such that $\{F \cap E_t : t \in \mathbb{T}_1\}$ is a family of baireled dense subspaces of (F, τ). Then, for each $t \in \mathbb{T}_1$, the set $F \cap T_t$ is a neighborhood of zero in $F \cap E_t$ endowed with the topology induced by τ, hence, by denseness, $\overline{F \cap T_t}^{(F,\tau)}$ is a neighborhood of zero in (F, τ), so $F = \left\langle \overline{F \cap T_t}^{(F,\tau)} \right\rangle \subset \left\langle \overline{T_t}^{E} \right\rangle = Z_t$, if $t \in \mathbb{T}_1$. Then, if $m_1 \in \mathbb{T}_1(1)$, we have that $F \subset S_{n_1}$ for $n_1 \geq m_1$. □

Lemma 4. *If conditions (H) hold, there exists in Ω a countable subset Δ (possibly empty) and $m_1 \in \mathbb{N}$ such that $c_0(\Omega \setminus \Delta, X) \subset S_{n_1}$ if $n_1 \geq m_1$.*

Proof. Assume the conclusion fails. Then, we can find $f_1 \in c_0(\Omega, X)$ such that $\|f_1\|_1 \leq 1$ and $f_1 \notin S_1$. Since the set $\Delta_1 = \mathrm{supp}(f_1)$ is countable, we deduce that $c_0(\Omega \setminus \Delta_1, X) \nsubseteq S_2$ and we find $f_2 \in c_0(\Omega \setminus \Delta_1, X)$ with $\|f_2\|_2 \leq 1$ and $f_2 \notin S_2$. Since $\Delta_2 = \mathrm{supp}(f_2)$ is countable, $c_0(\Omega \setminus (\Delta_1 \cup \Delta_2), X) \nsubseteq S_3$, which implies that there exists $f_3 \in c_0(\Omega \setminus (\Delta_1 \cup \Delta_2), X)$ with $\|f_3\|_3 \leq 1$ and $f_3 \notin S_3$.

By induction, we obtain the sequence $(f_n)_n$ such that

$$\{f_n : n \in \mathbb{N}\} \nsubseteq \bigcup_{m=1}^{\infty} S_m$$

and, by Lemma 1, this sequence is contained in a Banach disk D. Then, by Lemma 3, there exists S_p such that

$$\{f_n : n \in \mathbb{N}\} \subset D \subset S_p,$$

in contradiction with $f_p \notin S_p$. □

Lemma 5. *Assume conditions (H) hold. Then, there exists in Ω a finite subset Δ (possibly empty) and $m_1 \in \mathbb{N}$ such that $c_0(\Omega \setminus \Delta, X) \subset S_{n_1}$ if $n_1 \geq m_1$*

Proof. Applying Lemma 4, it is enough to prove this lemma for $\Omega = \mathbb{N}$. It is necessary to prove the existence of an $i \in \mathbb{N}$ such that $c_0(\mathbb{N} \setminus \{1, 2, ..., i\}, X) \subset S_i$. Suppose this is not true. Then, by induction, we find a sequence $(f_n)_n$ in $c_0(\mathbb{N}, X)$ such that $f_i \in c_0(\mathbb{N} \setminus \{1, 2, ..., i\}, X) \setminus S_i$ with $\|f_i\|_i \leq 1$. It is clear that

$$\{f_n : n \in \mathbb{N}\} \not\subseteq \bigcup_{m=1}^{\infty} S_m$$

and, by Lemma 1, this sequence is contained in a Banach disk D. By Lemma 3, there exists S_p such that $\{f_n : n \in \mathbb{N}\} \subset D \subset S_p$, in contradiction with $f_p \notin S_p$. □

Lemma 6. *Let us suppose that conditions (H) hold. If X is baireled, then there exists a NV-tree $\mathbb{T}_1$ cofinal in $\mathbb{T}$ such that $c_0(\Omega, X) = S_{n_1 n_2 \ldots n_p}$, if $(n_1, n_2, ..., n_p) \in \mathbb{T}_1$.*

Proof. It is obvious that we only need to prove that there exists n_1 such that $c_0(\Omega, X) = S_{n_1}$. By Lemma 5, it is enough to show that, given a finite subset Δ of Ω, there exists m_1 such that $c_0(\Delta, X) \subset S_{m_1}$. However, this follows from Lemma 3 and the trivial facts that $c_0(\Delta, X)$ and X^{Δ} are isomorphic and that the finite product of baireled spaces is baireled ([21], Proposition 7). □

Theorem 2. *Let X be a metrizable locally convex space and Ω a non void set. Then, $c_0(\Omega, X)$ is baireled if and only if X is baireled.*

Proof. Assume that X is baireled and that the metrizable space $c_0(\Omega, X)$ is not baireled. Then, by Theorem 1, the space $c_0(\Omega, X)$ is barrelled, hence there exists a linear $\mathbb{T}$-web $\mathcal{W} := \{E_t : t \in \mathbb{T}(i), i \in \mathbb{N}\}$ in $c_0(\Omega, X)$ consisting of dense subspaces such that, for each $t \in \mathbb{T}$, there exists a barrel T_t in E_t which is not a neighborhood of zero in E_t. By Lemma 6, there is $t \in \mathbb{T}$ such that $c_0(\Omega, X) = \left\langle \overline{T_t}^E \right\rangle$ and the barrelledness implies that $\overline{T_t}^E$ is a neighborhood of zero in $c_0(\Delta, X)$. Then, we get the contradiction that $E_t \cap \overline{T_t}^E = T_t$ is a neighborhood of zero in E_t. Therefore, the assumption that X baireled implies that $c_0(\Omega, X)$ is baireled.

The converse follows from the trivial facts that for $p \in \Omega$ the quotient

$$c_0(\Omega, X) \diagup c_0(\Omega \setminus \{p\}, X)$$

is isomorphic to X and that the baireledness is inherited by quotients ([21], 5 Permanence properties of Baireled spaces). □

We apply Theorem 2 to get the following closed graph theorem for baireled spaces.

Theorem 3. *Let X be a metrizable baireled locally convex space and let F be a locally convex space that contains a linear web $\{F_t : t \in \mathbb{N}^{<\infty}\}$ such that F_t admits a topology τ_t finer than the topology induced by F so that (F_t, τ_t) is a Fréchet space, for each $t \in \mathbb{N}^{<\infty}$. Let f be a linear map from $c_0(\Omega, X)$ into F with closed graph. There exists in $\mathbb{N}^{<\infty}$ a strand $(t_n : n \in \mathbb{N})$ such that f is a continuous mapping from $c_0(\Omega, X)$ into (F_{t_n}, τ_{t_n}), for each $n \in \mathbb{N}$.*

Proof. Let $E_t := f^{-1}(F_t)$ for each $t \in \mathbb{N}^{<\infty}$. By Theorem 2, there exists a strand $(t_n : n \in \mathbb{N})$ such that each E_{t_n} is barrelled and dense in $c_0(\Omega, X)$. The map f restricted to $f^{-1}(F_t)$ has closed graph. By ([28], Theorems 1 and 14), this restriction admits a continuous extension U to $c_0(\Omega, X)$ with values in (F_t, τ_t) and clearly $f = U$. □

This theorem is correct if we replace "Fréchet space, for each $t \in \mathbb{N}^{<\infty}$" by "$\Gamma_r$-space, for each $t \in \mathbb{N}^{<\infty}$" (see [28]). Recall that every B_r-space, in particular every Fréchet space, is a Γ_r-space. Reference [29] contains very interesting properties.

4. Open Problems

Problem 1. *Let X be a metrizable Baire locally convex space. Is $c_0(\Omega, X)$ a Baire space?*

The converse is true because, if X is not a Baire space, then X is not a space of the second Baire category, and if $(A_n)_n$ is a sequence of closed subsets of X with empty interior covering X and p is a fixed point of Ω the sets $B_n = \{f \in c_0(\Omega, X) : f(p) \in A_n\}$, $n \in \mathbb{N}$, define a cover of $c_0(\Omega, X)$ of closed sets with empty interior. Hence, for such X, the space $c_0(\Omega, X)$ is not Baire.

Let Ω be a Hausdorff completely regular space and X be a locally convex space. Then, $C_c(\Omega, X)$ denotes the linear space of continuous functions on Ω with values in X, endowed with the compact-open topology. In 1954, Nachbin and Shirota characterized the spaces Ω for which $C_c(\Omega) := C_c(\Omega, \mathbb{R})$ is barrelled and bornological; in 1958, Warner characterized the spaces Ω for which $C_c(\Omega)$ is quasibarrelled ([30], Propositions 2.15 and 2.16). Mendoza solved in [31] the corresponding problems for barrelled and quasibarrelled spaces $C_c(\Omega, X)$, proving that, if X contains an infinite compact subset, then $C_c(\Omega, X)$ is barrelled [resp. quasibarrelled] if and only if $C_c(\Omega)$ and X are barrelled [resp. quasibarrelled] and such that the strong dual of X has the property (B) of Pietsch.

Problem 2. *Characterize when $C_c(\Omega, X)$ is p-barrelled, $\aleph_0$-barrelled or baireled.*

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