This paper has been published in *Journal of Algebra*, 342(2):134–146 (2011).

The final publication is available at www.sciencedirect.com.

http://dx.doi.org/10.1016/j.jalgebra.2011.06.016

http://www.sciencedirect.com/science/article/pii/S0021869311003553

On second minimal subgroups of Sylow subgroups of finite groups

A. Ballester-Bolinches* R. Esteban-Romero† Yangming Li‡

Abstract

A subgroup H of a finite group G is a partial CAP-subgroup of G if there is a chief series of G such that H either covers or avoids its chief factors. Partial cover and avoidance property has turned out to be very useful to clear up the group structure. In this paper, finite groups in which the second minimal subgroups of their Sylow p-subgroups, p a fixed prime, are partial CAP-subgroups are completely classified.

Keywords: finite group, partial CAP-subgroup, second minimal subgroup, supersoluble group.

Mathematics Subject Classification (2000): 20D10, 20D20.

1 Introduction

All groups considered in this paper are finite. Arguably, the study of subgroup embedding properties has been one of the most efficient methods to clear up the structure of the groups. In particular, the embedding properties of 2-maximal and 2-minimal subgroups tend to give additional information about the group ([4, 17, 23, 27]). During the past four decades, the subgroup property known as the cover-avoidance property has gained more and more currency, first in the context of soluble groups ([8–10, 12, 24, 25] and [2, Chapter 4]), and more recently as a way of describing certain classes of soluble and supersoluble groups and their local versions ([3, 7, 11, 13–16, 20–22, 26]).

*Departament d'Àlgebra, Universitat de València, Dr. Moliner, 50, E-46100 Burjassot, València, Spain, email: Adolfo.Ballester@uv.es

†Institut Universitari de Matemàtica Pura i Aplicada, Universitat Politècnica de València, Camí de Vera, s/n, E-46022 València, Spain, email: resteban@mat.upv.es

‡Department of Mathematics, Guangdong University of Education, Guangzhou, 510310, People's Republic of China; Departament d'Àlgebra, Universitat de València, email: liyangming@gdei.edu.cn

Let A be a subgroup of a group G and H/K a section of G. We say that A *covers* H/K if $HA = KA$ and A *avoids* H/K if $A \cap H = A \cap K$. If A either covers or avoids every chief factor of G, then we say that A has the cover and avoidance property in G or A is a *CAP-subgroup* of G. Unfortunately the cover and avoidance property is not hereditary in intermediate subgroups, that is, if A is a CAP-subgroup of G and A is contained in a subgroup B of G, it does not follow in general that A has the cover and avoidance property in B (see [4, Example 3]). The failure of the cover and avoidance property to hold in intermediate subgroups leads to the following weaker property, which is persistent in subgroups and is also extremely useful in the structural study of the groups:

Definition 1.1. A subgroup A of a group G is called a *partial CAP-subgroup* of G if there exists a chief series Γ_A of G such that A either covers or avoids each factor of Γ_A (see [11, 21] for alternative terminologies).

Clearly, every CAP-subgroup is a partial CAP-subgroup, but the converse does not hold ([4, Example 3]). In [4], the authors considered the effect of imposing the partial cover and avoidance property to the second maximal subgroups of the Sylow p-subgroups, p a fixed prime. In the present paper the emphasis is on second minimal subgroups, and we consider what might be considered an opposite extreme, where the second minimal subgroups (2-minimal subgroups for short) of the Sylow p-subgroups are partial CAP-subgroups. In one result, we characterise the groups with this property, identifying a remarkable analogy between the partial cover and avoidance property of the subgroups of index p^2 and the partial cover and avoidance property of the subgroups of order p^2.

Main theorem. *Let p be a prime number, let G be a group, and let $G^+ = G/\mathrm{O}_{p'}(G)$. Then every subgroup of G of order p^2 is a partial CAP-subgroup of G if and only if one of the following statements holds:*

1. *the order of the Sylow p-subgroups of G is at most p;*
2. *G is a p-supersoluble group;*
3. *$\Phi(G^+) = 1$ and, if P is a Sylow p-subgroup of G, $P^+ = \mathrm{Soc}(G^+) = V_1 \times \cdots \times V_r$, where $V_1, \ldots, V_r$ are minimal normal subgroups of G^+ which are G^+-isomorphic to a 2-dimensional irreducible G^+-module V over the Galois field $\mathrm{GF}(p)$. Furthermore, V is not an absolutely irreducible G^+-module when $r > 1$.*

It seems desirable now to give an example of a group satisfying condition 3 of our main theorem. Existence of such groups was already shown in [1]. For the sake of completeness, we reproduce here the example of that paper.

Example 1.2. Consider an elementary abelian group

$$H = \langle a, b \mid a^5 = b^5 = 1, ab = ba\rangle$$

of order 25 and let α be an automorphism of H of order 3 satisfying that $a^\alpha = b$, $b^\alpha = a^{-1}b^{-1}$. Let $H_1 = H$, $H_2 = \langle a', b'\rangle$ be a copy of H_1 and $G = [H_1 \times H_2]\langle\alpha\rangle$. For any subgroup A of G of order 25, there exists a minimal normal subgroup N such that $A \cap N = 1$. Then A covers or avoids the factors of the chief series of G

$$1 < N < AN < G.$$

In other words, A is a partial CAP-subgroup of G. However, G is not 5-supersoluble. Note that H is not an absolutely irreducible G-module over the Galois field of 5 elements.

This example also shows that a group in which the second minimal subgroups of the Sylow subgroups are partial CAP-subgroups is not supersoluble in general. The best we are able to say is the following:

Corollary 1.3. *A group in which the second minimal subgroups of the Sylow subgroups are partial CAP-subgroups is soluble.*

2 Preliminaries

We begin with some preparatory lemmas before coming to the main result of the paper. The main basic properties of partial CAP-subgroups are listed in the following result appeared in [11]. They are particularly useful when induction arguments are applied.

Lemma 2.1. *Let S be a partial CAP-subgroup of a group G.*

1. *If $S \le K \le G$, then S is a partial CAP-subgroup of K.*
2. *If $N \le S$ and $N \trianglelefteq G$, then S/N is a partial CAP-subgroup of G/N.*
3. *If $N \trianglelefteq G$ and $(|S|, |N|) = 1$, then SN/N is a partial CAP-subgroup of G/N.*

The information given in the following lemma comes in extremely useful when studying the partial cover and avoidance property.

Lemma 2.2 ([1, Lemma 2.2])**.** *Let H be a partial CAP-subgroup of a group G. Suppose that Q is a normal subgroup of G such that H is contained in Q. Then there exists a chief series Ω_H of G passing through Q such that H either covers or avoids each chief factor in Ω_H.*

Let r be a positive integer and let H be a subgroup of G. Then H is called an *r-minimal (respectively r-maximal)* subgroup of G if there exists a subgroup chain $1 = H_0 < H_1 < \cdots < H_r = H$ (respectively $H = H_0 < H_1 < \cdots < H_r = G$) such that H_i is a maximal subgroup of H_{i+1} for all $0 \leq i \leq r-1$.

In the present paper we investigate the effect of imposing the partial cover and avoidance property on the 2-minimal subgroups of the Sylow subgroups, and once more we get a sense of why the partial cover and avoidance property has such bearing in the study of soluble groups. In fact, we use a local approach and characterise the groups G enjoying the following property:

> (†) *Every 2-minimal subgroup of every Sylow p-subgroup of G is a partial CAP-subgroup of G, p a fixed prime.*

In the following *p will be a fixed prime.*

Since 2-minimal subgroups of p-groups have order p^2, every group with Sylow p-subgroups of order p satisfies property (†). All p-supersoluble groups, or p-soluble groups whose p-chief factors have order p, also satisfy (†). Therefore we must think about groups whose order is divisible by p^2 which are not p-supersoluble.

An interesting special case is when the Sylow p-subgroups of G have order p^2. In this case, the structure of G is quite restricted as the following lemma shows.

Lemma 2.3. *Let G be a group whose Sylow p-subgroups have order p^2. Suppose that G satisfies property (†). Then G is p-soluble and either G is p-supersoluble or* $\mathrm{Soc}(G/\mathrm{O}_{p'}(G)) = P\mathrm{O}_{p'}(G)/\mathrm{O}_{p'}(G)$ *is an elementary abelian group of order p^2 for each Sylow p-subgroup P of G.*

Proof. We can assume without loss of generality that $\mathrm{O}_{p'}(G) = 1$. Suppose that G is not p-soluble. Then every minimal normal subgroup of G is non-abelian and its order is divisible by p by [4, Theorem 7]. It is clear then that a Sylow p-subgroup P of G neither covers nor avoids any minimal normal subgroup of G, a contradiction which shows that G is p-soluble. In that case, $S = \mathrm{Soc}(G)$ is a minimal normal subgroup of G contained in P by [4, Theorem 7]. Consequently, either S is of order p and G is p-supersoluble or $S = P$ is the Sylow p-subgroup of G. □

The next lemmas will be applied to the consideration of groups satisfying property (†).

Lemma 2.4 ([5, Proposition 1]). *Let $\mathfrak{F}$ be a saturated formation. Assume that G is group such that G does not belong to $\mathfrak{F}$ and there exists a maximal* *subgroup M of G such that $M \in \mathfrak{F}$ and $G = M\mathrm{F}(G)$. Then $G^{\mathfrak{F}}/(G^{\mathfrak{F}})'$ is a chief factor of G, $G^{\mathfrak{F}}$ is a p-group for some prime p, $G^{\mathfrak{F}}$ has exponent p if $p > 2$ and exponent at most 4 if $p = 2$. Moreover, either $G^{\mathfrak{F}}$ is elementary abelian or $(G^{\mathfrak{F}})' = \mathrm{Z}(G^{\mathfrak{F}}) = \Phi(G^{\mathfrak{F}})$.*

As an important deduction we have the

Lemma 2.5 ([5, Theorem 6])**.** *Let $\mathfrak{F}$ be a saturated formation and G a group with a normal subgroup K such that $G/K \in \mathfrak{F}$. If for some prime p, every subgroup of order p of K is contained in the $\mathfrak{F}$-hypercentre $\mathrm{Z}_{\mathfrak{F}}(G)$ of G, then $G/\mathrm{O}_{p'}(K) \in \mathfrak{F}$.*

There are some places where we use a known criterion for a normal p-subgroup to be contained in the hypercentre. For convenience, this is stated here as:

Lemma 2.6. *Suppose that P is a normal p-subgroup of G. Then $P \leq \mathrm{Z}_\infty(G)$ if and only if $\mathrm{O}^p(G) \leq \mathrm{C}_G(P)$.*

3 Main results

In this section we analyse the structure of the groups satisfying property (†), and prepare the way for the proof of the main result. We begin with a theorem about the minimal normal subgroups of the groups satisfying property (†).

Theorem 3.1. *Let G be a group satisfying property (†) whose order is divisible by p^2. Then every minimal normal subgroup of G is either a p'-group or a p-group. The minimal normal p-subgroups of G are of the same order, and it is at most p^2.*

Proof. Let N be a minimal normal subgroup of G, and suppose that N is not a p'-group. Let $1 \neq N_p$ be a Sylow p-subgroup of N, and let Q be a subgroup of G of order p^2 such that $Q \cap N_p \neq 1$. We consider a chief series of G

$$(\Gamma): \quad 1 = G_0 < \cdots < G_i < \cdots < G_j < G_{j+1} < \cdots < G_m = G$$

such that Q either covers of avoids each chief factor of G in (Γ). Then there exists an index $i \in \{1, \ldots, m\}$ such that $N \cap G_{i-1} = 1$ and $G_i = G_{i-1}N$. In that case, G_i/G_{i-1} is G-isomorphic to N. Suppose that Q avoids G_{i+1}/G_i, then $Q \cap N_p \leq Q \cap G_i \cap N = Q \cap G_{i-1} \cap N = 1$, against the choice of Q. Consequently, Q covers G_i/G_{i-1}. Then G_i/G_{i-1} is of order at most p^2 and N is a p-group of order at most p^2.

Next we prove that all minimal normal p-subgroups of G are of the same order. Suppose, arguing by contradiction, that G has two minimal normal p-subgroups, N_1 and N_2 say, such that $|N_1| = p$ and $|N_2| = p^2$. Let H be a subgroup of N_2 of order p. Then N_1H is a subgroup of G of order p^2 which is a partial CAP-subgroup of G. By Lemma 2.2, there exists a chief series of G,

$$(\Delta): \quad 1 \leq N_3 \leq N_1N_2 \leq \cdots \leq G,$$

passing through N_1N_2 such that N_1H either covers or avoids each chief factor of G in (Δ). In addition, the order of N_3 is p or p^2. Assume that N_3 is of order p. If $N_1H \cap N_3 = 1$, then $N_1HN_3 = N_1N_2$ and $N_2 = H(N_2 \cap N_1N_3)$. It means that either $N_2 = H$ or $N_2 = HN_1N_3$. This contradiction shows that N_3 is a subgroup of N_1H. In particular, N_1H cannot cover N_1N_2/N_3. Hence $N_1H = N_1N_2 \cap N_1H = N_1H \cap N_3 = N_3$, a contradiction which shows that N_3 must be of order p^2. Since N_1N_2 is of order p^3 and N_1H is of order p^2, it follows that N_1H covers N_3. Thus $N_1H = N_3$, which contradicts the fact that N_1 and N_3 are two different minimal normal subgroups of G. This proves the result. □

We now touch the question of the p-length of p-soluble groups satisfying property (†). We prove that these groups belong to the saturated formation $\mathfrak{F}$ of all p-soluble groups whose p-length is at most one.

Theorem 3.2. *Let G be a p-soluble group satisfying property* (†). *Then the p-length of G is at most* 1.

Proof. We will obtain a contradiction by supposing that the result is false and choosing a counterexample G of least order. For the ease of reading, we break the argument into separately-stated steps.

1. $\mathrm{O}_{p'}(G) = 1$.

   Assume that $\mathrm{O}_{p'}(G) \neq 1$. By Lemma 2.1 the hypothesis holds in the group $G/\mathrm{O}_{p'}(G)$. The minimal choice of G implies that $G/\mathrm{O}_{p'}(G)$ belongs to $\mathfrak{F}$. Hence G is an $\mathfrak{F}$-group, against the choice of G. Thus $\mathrm{O}_{p'}(G) = 1$.

2. *Any proper subgroup of G belongs to $\mathfrak{F}$.*

   Let H be a proper subgroup of G. If a Sylow p-subgroup H_p of H is of order at most p, we have that $H \in \mathfrak{F}$ by [3, Lemma 3.1]. Assume that p^2 divides $|H_p|$ and let L be a subgroup of H of order p^2. Then L is a partial CAP-subgroup of H by Lemma 2.1, and so H satisfies property (†). The minimality of G yields $H \in \mathfrak{F}$. This confirms Step 2.

Let $G^{\mathfrak{F}}$ denote the $\mathfrak{F}$-residual of G, that is, the smallest normal subgroup of G with quotient in $\mathfrak{F}$.

3. *There exists a maximal subgroup M of G such that $M \in \mathfrak{F}$ and $G = MG^{\mathfrak{F}}$. Moreover, $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$ is a chief factor of G, and the exponent of $G^{\mathfrak{F}}$ is p or at most 4 if $p = 2$.*

   Since G is not an $\mathfrak{F}$-group and $\mathfrak{F}$ is saturated, it follows that $G/\Phi(G)$ does not belong to $\mathfrak{F}$. Let $N/\Phi(G)$ a non-trivial normal subgroup of $G/\Phi(G)$. Then $N/\Phi(G)$ is supplemented in $G/\Phi(G)$. By Step 2, G/N belongs to $\mathfrak{F}$. Therefore, since $\mathfrak{F}$ is a formation, $G/\Phi(G)$ has a unique minimal normal subgroup, $T/\Phi(G)$ say. Moreover, $T/\Phi(G)$ is not a p'-group. Since G is p-soluble, it follows that $T/\Phi(G)$ is an abelian p-chief factor of G which is complemented in G by a maximal subgroup M of G. Then $G = M\mathrm{F}(G)$ and $T = G^{\mathfrak{F}}\Phi(G)$. Step 2 implies that $M \in \mathfrak{F}$ and so $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$ is a chief factor of G, and the exponent of $G^{\mathfrak{F}}$ is p or at most 4 if $p = 2$ by Lemma 2.4. This proves our claim.

4. $\Phi(G^{\mathfrak{F}}) = 1$ *and* $|G^{\mathfrak{F}}| = p^2$.

   Suppose that $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$ has order p. Since $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$ is G-isomorphic to $\mathrm{Soc}(G/M_G)$, it follows that G/M_G is in $\mathfrak{F}$. This contradiction shows that $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$ has order greater than p. Let H be a subgroup of $G^{\mathfrak{F}}$ of order p^2 such that $H \not\leq \Phi(G^{\mathfrak{F}})$. Then H is a partial CAP-subgroup of G and, by Lemma 2.2, there exists a chief series of G,

   $$(\Gamma_1): \quad 1 = G_0 \leq G_1 \leq \cdots \leq K \leq G^{\mathfrak{F}} \leq \cdots \leq G_n = G,$$

   passing through $G^{\mathfrak{F}}$ such that H either covers or avoids each G-chief factor in (Γ_1). Since $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$ is a chief factor of G by Step 3, it follows that either $K\Phi(G^{\mathfrak{F}}) = \Phi(G^{\mathfrak{F}})$ or $K\Phi(G^{\mathfrak{F}}) = G^{\mathfrak{F}}$. If $K\Phi(G^{\mathfrak{F}}) = G^{\mathfrak{F}}$, then $K = G^{\mathfrak{F}}$, contrary to assumption. Thus $K \leq \Phi(G^{\mathfrak{F}})$. Since $G^{\mathfrak{F}}/K$ is a chief factor of G, we have $K = \Phi(G^{\mathfrak{F}})$ and so $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$ is a chief factor of G in (Γ). It therefore follows that H either covers or avoids $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$. If H avoids $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$, then $H = H \cap G^{\mathfrak{F}} \leq \Phi(G^{\mathfrak{F}})$, contrary to the choice of H. Hence we have that H covers $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$. Consequently $G^{\mathfrak{F}} = H\Phi(G^{\mathfrak{F}}) = H$, $G^{\mathfrak{F}}$ is of order p^2 and $\Phi(G^{\mathfrak{F}}) = 1$.

5. *G is not a primitive group. In particular $M_G \neq 1$.*

   Suppose, arguing by contradiction, that G is primitive. Then $G^{\mathfrak{F}}$ is the unique minimal normal subgroup of G. Since G has p-length greater than 1 and $G^{\mathfrak{F}}$ is a p-group, it follows that p divides $|M|$. Then we can choose an element $a \in G^{\mathfrak{F}}$ and an element $b \in M$ such that $\langle a, b \rangle$ is a subgroup of G of order p^2. Obviously $\langle a, b\rangle$ neither covers nor avoids $G^{\mathfrak{F}}$, a contradiction which proves Step 5.

6. *The final contradiction.*

   By Step 5, $M_G \neq 1$. Let N be a minimal normal subgroup of G contained in M_G. Then N is a p-group by Step 1, and $N \cap G^{\mathfrak{F}} = 1$. Let us choose an element $a \in G^{\mathfrak{F}}$ and an element $b \in N$ such that $\langle a, b\rangle$ is a group of order p^2. By hypothesis, $\langle a, b\rangle$ is a partial CAP-subgroup of G. Applying Lemma 2.2, there exists a chief series of G,

   $$(\Gamma_2): \quad 1 = G_0 \leq K \leq G^{\mathfrak{F}}N \leq \cdots \leq G_n = G,$$

   passing through $G^{\mathfrak{F}}N$ such that $\langle a, b\rangle$ either covers or avoids each chief factor of G in (Γ_2).

   It is clear that $\langle a, b\rangle$ neither covers nor avoids $G^{\mathfrak{F}}$. Hence $K \neq G^{\mathfrak{F}}$. Then K is an $\mathfrak{F}$-central chief factor of G. Since M is an $\mathfrak{F}$-normaliser of G by [2, Theorem 4.2.17], we can apply [2, Theorem 4.2.4] to conclude that $K \leq M$. Assume that $\langle a, b\rangle$ avoids K. Then $\langle a, b\rangle$ must cover $G^{\mathfrak{F}}N/K$, and so $G^{\mathfrak{F}}N \leq \langle a, b\rangle K$. We therefore have that $G^{\mathfrak{F}} = G^{\mathfrak{F}} \cap (\langle a, b\rangle K) = \langle a\rangle(G^{\mathfrak{F}} \cap \langle b\rangle K) = \langle a\rangle$, contrary to Step 4. Hence $\langle a, b\rangle$ must cover K. Thus $K = M \cap \langle a, b\rangle = \langle b\rangle$.

   Now $\langle a, b\rangle$ either covers or avoids $G^{\mathfrak{F}}N/K$. If $\langle a, b\rangle$ covers $G^{\mathfrak{F}}N/K$, then $G^{\mathfrak{F}}N \leq \langle a, b\rangle K = \langle a, b\rangle$. Then $G^{\mathfrak{F}}$ is of order p, against Step 4. Thus $\langle a, b\rangle$ avoids $G^{\mathfrak{F}}N/K$. This gives the final contradiction $\langle a, b\rangle = G^{\mathfrak{F}}N \cap \langle a, b\rangle = \langle a, b\rangle \cap K = K$. □

Our next result shows that a group satisfying property (†) whose order is divisible by p^2 must be p-soluble.

Theorem 3.3. *Let G be a group satisfying property (†). Then either the Sylow p-subgroups of G are of order p or G is a p-soluble group.*

Proof. Suppose the result false, and let the group G provide a counterexample of least possible order. Then p^2 divides the order of G. According to Lemma 2.1, the property of G is inherited by $G^* = G/\mathrm{O}_{p'}(G)$. Hence the minimality of G implies that $\mathrm{O}_{p'}(G) = 1$. We reach a contradiction after the following steps.

1. *If K is a proper subgroup of G and p^2 divides $|K|$, then K is p-soluble. If, in addition, K is normal in G, then a Sylow p-subgroup of K is also normal in G.*

Assume that K is a proper subgroup of G such that p^2 divides $|K|$, and let L be a subgroup of K of order p^2. Then L is a partial CAP-subgroup of G and so is in K by Lemma 2.1. Hence K satisfies property (†). The minimal choice of G implies that K is p-soluble. Suppose that K is normal in G. Since $\mathrm{O}_{p'}(K) \leq \mathrm{O}_{p'}(G) = 1$, we conclude that $\mathrm{O}_p(K)$ is a Sylow p-subgroup of K by virtue of Theorem 3.2.

2. *G has a unique maximal normal subgroup, M say. In particular, the chief factor G/M appears in every chief series of G.*

   Suppose that G has two different maximal normal subgroups, M and N say. Then $G = MN$. If the Sylow p-subgroups of M and N are normal in G, then G has a normal Sylow p-subgroup and so G is p-soluble, contradicting the choice of G. Hence, by Step 1, we may assume that the order of the Sylow p-subgroups of M is at most p. If the order of the Sylow p-subgroups of N is also at most p, then the order of the Sylow p-subgroups of G is at most p^2. Applying Lemma 2.3, it follows that either p^2 does not divide the order of G or G is p-soluble, contrary to assumption. Hence, we may assume that p^2 divides $|N|$ and a Sylow p-subgroup N_p of N is normal in G. In that case N_pM is a normal subgroup of G containing M. Hence $G = N_pM$. Then G/M is a cyclic group of order p. This implies that the Sylow p-subgroups have order p^2. Applying Lemma 2.3, G is p-soluble. This contradiction proves our claim.

3. *A Sylow p-subgroup of M is a non-trivial normal subgroup of G and M is p-soluble. Furthermore, $G = \mathrm{O}^p(G)$.*

   Assume that the order of a Sylow p-subgroup M_p of M is at most p. Let H be a subgroup of G of order p^2 containing M_p. Then H is a partial CAP-subgroup of G and so H either covers or avoids G/M. If H avoids G/M, then $H = H \cap G = H \cap M = M_p$. This contradiction implies that H covers G/M. Then $G = HM$ and so the Sylow p-subgroups of G are of order p^2. By Lemma 2.3, G is p-soluble, contrary to supposition. It therefore follows that M_p is of order greater or equal than p^2. By Step 1, we conclude that M_p is normal in G and M is p-soluble. If $\mathrm{O}^p(G) \neq G$, then $\mathrm{O}^p(G) \leq M$ by Step 2. Thus $\mathrm{O}^p(G)$ is p-soluble. Since $G/\mathrm{O}^p(G)$ is a p-group, we have that G is a p-soluble, contradicting again our assumption. Hence $G = \mathrm{O}^p(G)$.

4. *Every subgroup of G of order p or p^2 is contained in M.*

   Let $1 \neq M_p$ be the Sylow p-subgroup of M and let G_p be a Sylow p-subgroup of G containing M_p. Let T be a subgroup of order p contained in $M_p \cap \mathrm{Z}(G_p)$. Assume that H is a subgroup of order p which is not contained in M. Then HT is a subgroup of G of order p^2 which either covers or avoids G/M. If G/M were avoided by HT, then we would have $HT = HT \cap G = HT \cap M = T$, and if G/M were covered by HT, it would follow that $G = HTM = HM$. This would mean that G/M had to be cyclic of order p and then G had to be p-soluble. In both cases, we get a contradiction. Hence every subgroup of order p has to be contained in M. Assume now that X is a subgroup of order p^2 which is not contained in M. Exactly similar reasoning shows that X neither covers nor avoids G/M. This however contradicts the hypothesis that X is a partial CAP-subgroup of G.

5. $M = \Phi(G) = \mathrm{O}_p(G)$.

   Since $\mathrm{O}_{p'}(G) = 1$, it follows that $\Phi(G)$ is a p-group. We want to show that M is contained in $\Phi(G)$. Suppose to the contrary that M is not contained in $\Phi(G)$, and therefore that there exists a proper subgroup X of G such that $G = MX$. Since every subgroup of order p and p^2 of G has to be contained in M by Step 4, then G/M would be a p'-group if p^2 did not divide the order of X, and so G would be p-soluble, in contradiction to our assumption. Therefore p^2 divides the order of X and X is p-soluble by Step 1. Hence G/M is p-soluble and so is G. This contradiction shows that $M \leq \Phi(G)$ and $M = \Phi(G) = \mathrm{O}_p(G)$.

6. *No chief factor of G below M has order p^2.*

   Let us denote $S = G/M$. Let H/K be a chief factor of G below M. Then H/K is an elementary abelian p-group and H/K has the structure of an irreducible and faithful $G/\mathrm{C}_G(H/K)$-module over the Galois field $\mathrm{GF}(p)$. Since $M = \mathrm{O}_p(G) \leq \mathrm{C}_G(H/K)$ by [6, A, 13.8] and $\mathrm{C}_G(H/K) \neq G$, we have $M = \mathrm{C}_G(H/K)$. Assume now that the order of H/K is p^2. Then S can be regarded as a subgroup of $\mathrm{GL}_2(p)$. Since S is a non-abelian simple group, it follows that $S \leq (\mathrm{GL}_2(p))' = \mathrm{SL}_2(p)$, and $S \cap \mathrm{Z}\big(\mathrm{SL}_2(p)\big) = 1$. Hence S can be regarded as a subgroup of $\mathrm{PSL}_2(p)$. According to the subgroup structure of $\mathrm{PSL}_2(p)$ (see [18, II, 8.27]), either $S \cong A_5$, where $p = 5$ or $p^2 - 1 \equiv 0 \pmod 5$ or $S \cong \mathrm{PSL}_2(p)$. Suppose that $S \cong A_5$. Since p is a prime divisor of $\Phi(G)$, $p \in \pi\big((G/\Phi(G)\big) = \pi(G/M) = \pi(A_5) = \{2, 3, 5\}$, and so we must have $p = 5$. This is contrary to the fact that the dimensions of the irreducible and faithful representations of A_5 over $\mathrm{GF}(5)$ are 3 and 5 (see [19, VII, 3.10]). Hence S must be isomorphic to $\mathrm{PSL}_2(p)$. In that case, $p \geq 5$ and since the index of S in $\mathrm{SL}_2(p)$ is 2 and $\mathrm{SL}_2(p)$ is perfect, it therefore follows that $\mathrm{SL}_2(p) = \big(\mathrm{SL}_2(p)\big)' \leq S$. In this case we are also led to a contradiction, and therefore conclude that the result as stated is true.

7. *Every chief factor of G of order p is central in G.*

   Suppose that H/K is a chief factor of G of order p. Then, by [6, A, 13.8], $M = \mathrm{O}_p(G) \leq \mathrm{C}_G(H/K) \leq G$, and consequently we have that either $\mathrm{C}_G(H/K) = M$ or G. If $\mathrm{C}_G(H/K) = M$, then $G/M = G/\mathrm{C}_G(H/K)$ is a p'-group and G is p-soluble. This is in contradiction to the choice of G. Hence $\mathrm{C}_G(H/K) = G$, that is, H/K is central in G.

8. *M is contained in the hypercentre $\mathrm{Z}_\infty(G)$ of G.*

   From Step 7, it suffices to prove that every chief factor of G below M has order p. Assume to the contrary that there exists a chief factor A/B of G below M whose order is greater than p. We choose A of minimal order. Then every chief factor H/K of G below M with $|H| < |A|$ is of order p. Let L be a subgroup of A of order p^2. Then L is a partial CAP-subgroup of G. Applying Lemma 2.2, there exists a chief series of G

   $$1 \leq \cdots \leq T < A < \cdots < G$$

   passing through A such that L either covers or avoids each chief factor of this series. The choice of A implies that every chief factor of G below T is of order p. Consequently, $T \leq \mathrm{Z}_\infty(G)$ by Step 7. If the order of A/T were p, then A would be contained in $\mathrm{Z}_\infty(G)$. This would imply that $|A/B| = p$, in contradiction to the hypothesis that A/B has order greater than p. Therefore the order of A/T is greater than p^2 by Step 6, and so L cannot cover A/T. Hence L avoids A/T. Then $L \leq T \leq \mathrm{Z}_\infty(G)$. We therefore conclude that every subgroup of A of order p^2 is contained in $\mathrm{Z}_\infty(G)$. By Lemma 2.6, $G = \mathrm{O}^p(G) \leq \mathrm{C}_G\big(\Omega_2(A)\big)$, that is, $\Omega_2(A) \leq \mathrm{Z}(G)$. Therefore every p'-element of G centralises every element of A of order p or p^2. Applying [18, IV, 5.12], every p'-element of G centralises A. It therefore follows that $G = \mathrm{O}^p(G) \leq \mathrm{C}_G(A)$. Consequently A/B must be of order p, contrary to the choice of A/B.

9. *Final contradiction.*

   From Step 8 and Lemma 2.6, we have that $M \leq \mathrm{Z}(G)$. Therefore, by Step 4, every element of G of order p or p^2 is contained in $\mathrm{Z}(G)$. Applying [18, IV, 5.5], we have G is p-nilpotent. This last contradiction establishes the theorem. □

Theorem 3.4. *Let G be a p-soluble group satisfying property (†). Then either G is p-supersoluble or G satisfies the following two conditions:*

1. *all non-cyclic p-chief factors of G are G-isomorphic and have order p^2;*

2. *all complemented p-chief factors of G are not cyclic.*

Proof. We proceed by induction on $|G|$. We may assume without loss of generality that $\mathrm{O}_{p'}(G) = 1$. Then $\mathrm{F}(G) = \mathrm{O}_p(G)$ is a Sylow p-subgroup of G by Theorem 3.2. According to [6, A, 10.6], $\mathrm{F}(G)/\Phi(G) = N_1/\Phi(G) \times \cdots \times N_r/\Phi(G)$, where $N_i/\Phi(G)$ is a minimal normal subgroup of $G/\Phi(G)$ which is complemented in G for all i.

Suppose first that $r = 1$, that is, $\mathrm{F}(G)/\Phi(G)$ is a chief factor of G. If $\mathrm{F}(G)/\Phi(G)$ has order p, it can be seen without difficulty that G is p-supersoluble, as desired. Hence we can assume that $\mathrm{F}(G)/\Phi(G)$ has order at least p^2. Let us denote $M = \Phi(G)G_{p'}$. Then $G = \mathrm{F}(G)M$, and $\mathrm{F}(G) \cap M = \Phi(G)$. Let A be a normal subgroup of G such that $\mathrm{F}(G)/A$ is a chief factor of G. If A were not contained in $\Phi(G)$, it would follow that $\mathrm{F}(G) = A\Phi(G)$. This would mean that $G = \mathrm{F}(G)G_{p'} = A\Phi(G)G_{p'} = AG_{p'}$. Consequently, $\mathrm{F}(G) = A$. This is in contradiction to the definition of A. Therefore $\mathrm{F}(G)/\Phi(G)$ is a chief factor of G which appears in every chief series of G passing through $\mathrm{F}(G)$.

Suppose we have an element x of $\mathrm{F}(G)$ of order p which is not in $\Phi(G)$. Let y be an element of $\Phi(G)$ of order p such that $H = \langle x, y\rangle$ has order p^2. Then H either covers or avoids $\mathrm{F}(G)/\Phi(G)$ by Lemma 2.2. If H covers $\mathrm{F}(G)/\Phi(G)$, then $\mathrm{F}(G) = H\Phi(G) = \langle x\rangle\Phi(G)$. Then $\mathrm{F}(G)/\Phi(G)$ is of order p, and if H avoids $\mathrm{F}(G)/\Phi(G)$, it follows $H = H \cap \mathrm{F}(G) = H \cap \Phi(G)$. In each case we are led to a contradiction, and therefore conclude that $\Phi(G)$ contains every element of order p of $\mathrm{F}(G)$.

If $\Phi(G) \leq \mathrm{Z}_{\mathfrak{U}_p}(G)$, the p-supersoluble hypercentre of G, we can apply Lemma 2.5 to the saturated formation $\mathfrak{U}_p$ of all p-supersoluble groups to conclude that G is p-supersoluble. This contradicts that $\mathrm{F}(G)/\Phi(G)$ is a chief factor of G of order at least p^2.

Therefore $\Phi(G) \not\leq \mathrm{Z}_{\mathfrak{U}_p}(G)$. Hence we have a chief series of G passing through $\Phi(G)$:

$$(*) : \quad 1 = L_0 \leq L_1 \leq \cdots \leq L_{k-1} \leq L_k \leq \cdots \leq L_n = \Phi(G) \leq \cdots \leq G$$

such that all chief factors of G below L_{k-1} are of order p and L_k/L_{k-1} has order greater than p. Since $\mathrm{F}(G)$ centralises every chief factor of G, it follows that L_k/L_{k-1} is a non-cyclic chief factor of $L_kG_{p'}$. Hence $L_kG_{p'}$ is not p-supersoluble and L_{k-1} is contained in the p-supersoluble hypercentre of $L_kG_{p'}$. By Lemma 2.5, there exists an element x of L_k of order p such that x does not belong to L_{k-1}. Then, as above, we can consider an element y of L_{k-1} such that $H = \langle x, y\rangle$ is of order p^2. The hypothesis on G implies that

H is a partial CAP-subgroup of G. Applying Lemma 2.2, we know there exists a chief series of G,

$$(**): \quad 1 \leq \cdots \leq T \leq L_k \leq \cdots \leq G,$$

passing through L_k such that H either covers or avoids each chief factor of G in $(**)$. If $T \leq L_{k-1}$, then $T = L_{k-1}$ and H neither covers nor avoids L_k/L_{k-1}. Hence $T \nleq L_{k-1}$. Then $L_k = TL_{k-1}$, and $L_k/T \cong L_{k-1}/L_{k-1} \cap T$. Since $L_{k-1} \leq \mathrm{Z}_{\mathfrak{U}_p}(G)$, we have L_k/T is of order p. Moreover $T/T \cap L_{k-1} \cong L_k/L_{k-1}$ is of order p^2. Therefore $(**)$ has a unique chief factor of order greater than p below T. In particular, $TG_{p'}$ is not p-supersoluble, $T \cap L_{k-1}$ is contained in the p-supersoluble hypercentre of $TG_{p'}$ and $|T| < |L_k|$. Repeating this argument, we finally get a chief series of G,

$$(\Delta): \quad 1 \leq R \leq \cdots \leq G,$$

such that the minimal normal subgroup R of G is of order greater than p. By Theorem 3.1, we have every minimal normal subgroup of G is of order p^2. However L_1 is a minimal normal subgroup of G of order p. This contradiction yields $L_{k-1} = 1$. In particular, $\mathrm{Z}_{\mathfrak{U}_p}(G) = 1$.

Let A/B be a chief factor of G with $A \leq \Phi(G)$. Since the chief factors of $AG_{p'}$ are chief factors of G and $AG_{p'}$ is not supersoluble because it contains a minimal normal subgroup of order greater than p, by minimality of G the complemented chief factors of $AG_{p'}$ are non-cyclic and all non-cyclic chief factors of $AG_{p'}$ are $AG_{p'}$-isomorphic and have order p^2. Since A/B is a complemented chief factor of $AG_{p'}$, it follows that $|A/B| = p^2$ and then, by taking $A = \Phi(G)$, all chief factors of G below $\Phi(G)$ are G-isomorphic and have order p^2. Assume now that $\Phi(G) \neq 1$. Let c be an element of order p^2 of $\mathrm{F}(G)$ and write $S = \langle c \rangle$. Applying Lemma 2.2, there exists a chief series of G passing through $\mathrm{F}(G)$ such that every chief factor in this series is either covered or avoided by S. But no chief factor of G below $\mathrm{F}(G)$ can be cyclic, so that no chief factor of G below $\mathrm{F}(G)$ can be covered by S. It follows that all such chief factors must be avoided by S, contrary to the choice of S. Therefore the exponent of $\mathrm{F}(G)$ is p. Since all elements of $\mathrm{F}(G)$ of order p must be contained in $\Phi(G)$, it follows $\mathrm{F}(G) = \Phi(G)$. This contradiction shows that $\Phi(G) = 1$. Since $\mathrm{F}(G)$ is a minimal normal subgroup of G, it has order p^2 by Theorem 3.1. This completes the proof for $r = 1$.

Suppose now that $r > 1$. Let $N/\Phi(G)$ be an arbitrary abelian minimal normal subgroup of $G/\Phi(G)$ and let M be a maximal subgroup of G such that $G = \mathrm{F}(G)M$ and $N \cap M = \Phi(G)$. Consider a chief series of G passing through $\Phi(G)$ and N:

$$(\alpha): \quad 1 \leq \cdots \leq \Phi(G) \leq N \leq \cdots \leq G$$

The series $(\alpha)\cap M$ which is obtained by intersecting the series (α) term-by-term with M is, after deleting repetitions, a chief series of M. In particular, every chief factor of G below $\Phi(G)$ is a chief factor of M. Let A/B a chief factor of G in (α) such that $N \leq B$. Then $A \cap M/B \cap M$ is a chief factor of M which is M-isomorphic to A/B. Moreover A/B is complemented in G if and only if $A \cap M/B \cap M$ is complemented in M (see [6, III, 6.5 and 6.6]). Since by Lemma 2.1, M inherits the property of G, we can apply the induction hypothesis to M to conclude that either M is p-supersoluble or M satisfies the properties enunciated in the statement of the theorem, that is, all non-cyclic p-chief factors of M are M-isomorphic and have order p^2 and every complemented p-chief factor of M is non-cyclic.

Assume that $r = 2$. Then $\mathrm{F}(G)/\Phi(G) = N_1/\Phi(G) \times N_2/\Phi(G)$. Let $G_{p'}$ be a Hall p'-subgroup of G. Then $G = N_1N_2G_{p'}$ and $M_i = N_{3-i}G_{p'}$ is a maximal subgroup of G complementing the chief factor $N_i/\Phi(G)$, $i = 1, 2$, and $X = \Phi(G)G_{p'}$ is a maximal subgroup of M_i, $i = 1, 2$. We distinguish two possibilities:

1. $\Phi(G) = 1$.

   In this case, N_1, N_2 are two minimal normal subgroups of G. By Theorem 3.1, the orders of N_1, N_2 are simultaneously p or p^2. If N_1, N_2 are of order p, then it can be easily seen that G is p-supersoluble, as desired. Assume that N_1, N_2 are of order p^2. We shall prove that in this case N_1, N_2 are G-isomorphic. Suppose that this is false and derive a contradiction. If the number of the minimal normal subgroups of G were 2, then N_1 and N_2 would be exactly the minimal normal subgroups of G. Let a be an element of N_1 of order p and let b and element of N_2 of order p. Then $L = \langle a, b\rangle$ is of order p^2 and L either covers or avoids every chief factor of a chief series of G passing through $\mathrm{F}(G)$ by Lemma 2.2. But L neither covers nor avoid the minimal normal subgroups of G. This contradiction shows that the number of the minimal normal subgroups of G is greater than 2. Let N be a minimal normal subgroup of G different from N_1 and N_2. Consider the following two chief series of G:

   $$\begin{aligned}(\beta_1):&\quad 1 < N_1 < N_1N_2 = \mathrm{F}(G) < \cdots < G,\\ (\beta_2):&\quad 1 < N < NN_2 = \mathrm{F}(G) < \cdots < G.\end{aligned}$$

   By [2, 1.2.36], N_1 is G-isomorphic to N as N_1 is not G-isomorphic to NN_2/N. Similarly, if we consider the following two chief series of G:

   $$\begin{aligned}(\beta_1'):&\quad 1 < N_2 < N_1N_2 = \mathrm{F}(G) < \cdots < G,\\ (\beta_2):&\quad 1 < N < NN_2 = \mathrm{F}(G) < \cdots < G,\end{aligned}$$

it follows that N is G-isomorphic to N_2. Hence N_1 is G-isomorphic to N_2. This contradiction, together with [2, 1.2.36], allow us to conclude that all non-cyclic p-chief factors of G are G-isomorphic and of order p^2, and they are all complemented in G. The theorem holds in this case.

2. $\Phi(G) \neq 1$.

   Consider the following two chief series of G:

$$(\gamma_1): \quad 1 \leq \cdots \leq \Phi(G) \leq N_1 \leq N_1N_2 \leq \cdots \leq G,$$
$$(\gamma_2): \quad 1 \leq \cdots \leq \Phi(G) \leq N_2 \leq N_1N_2 \leq \cdots \leq G.$$

   Intersecting the series (γ_i) term-by-term with M_i, $i = 1, 2$, and deleting repetitions, we get the chief series of M_1 and M_2, respectively:

$$(\gamma_1) \cap M_1: \quad 1 \leq \cdots \leq \Phi(G) = N_1 \cap M_1 \leq N_2 = N_1N_2 \cap M_1 \leq \cdots \leq M_1,$$
$$(\gamma_2) \cap M_2: \quad 1 \leq \cdots \leq \Phi(G) = N_2 \cap M_2 \leq N_1 = N_1N_2 \cap M_2 \leq \cdots \leq M_2.$$

   Now we intersect these two chief series with X and delete repetitions.

$$(\gamma_1) \cap X = (\gamma_2) \cap X: \quad 1 \leq \cdots \leq \Phi(G) = N_1 \cap X = N_2 \cap X \leq \cdots \leq X.$$

   It is clear that $(\gamma_1) \cap X = (\gamma_2) \cap X$ is a chief series of X and the X-chief factors of this series below $\Phi(G)$ are M_i-chief factors for all $i = 1, 2$. We know that either M_i is p-supersoluble or all non-cyclic p-chief factors of M_i are M_i-isomorphic and have order p^2, and every complemented M_i-chief factor of M_i is non-cyclic, $i = 1, 2$. Hence the orders of $N_1/\Phi(G)$ and $N_2/\Phi(G)$ are either p or p^2. Suppose that $|N_1/\Phi(G)| = p^2$ and $|N_2/\Phi(G)| = p$. Since $N_2/\Phi(G)$ is a cyclic complemented p-chief factor of M_1, it follows that M_1 is p-supersoluble. Therefore every p-chief factor of M_1 in $(\gamma_1) \cap M_1$ is of order p. Hence every G-chief factor below $\Phi(G)$ in (γ_1) is of order p. On the other hand, $N_1/\Phi(G)$ is a complemented chief factor of M_2 of order p^2. Hence M_2 is not p-supersoluble and so every non-cyclic chief factor of M_2 is of order p^2. But every chief factor of M_2 below $\Phi(G)$ is of order p. Consequently, $N_1/\Phi(G)$ is the unique complemented chief factor of M_2 in the chief series $(\gamma_2) \cap M_2$ of M_2. It follows that $\Phi(G) \leq \Phi(M_2)$. Since $\Phi(M_2)$ is a nilpotent group, we obtain by order considerations that $\Phi(G) =$

$O_p(\Phi(M_2))$. However, the same arguments of the proof for $r = 1$ show now that $\Phi(G) = 1$, contrary to supposition. Therefore $N_1/\Phi(G)$ and $N_2/\Phi(G)$ have the same order. If $|N_1/\Phi(G)| = |N_2/\Phi(G)| = p$, then G is p-supersoluble, and the theorem holds.

Assume that $|N_1/\Phi(G)| = |N_2/\Phi(G)| = p^2$. Since $N_{3-i}/\Phi(G)$ is a chief factor of M_i, M_i is not p-supersoluble, $i = 1, 2$. Then all non-cyclic chief factors of M_i are M_i-isomorphic and have order p^2, and every complemented chief factor of M_i is non-cyclic. Assume that X is p-supersoluble. Then every chief factor of X below $\Phi(G)$ is cyclic. Certainly these chief factors are also chief factors of G. Since M_i has no cyclic complemented chief factors, exactly similar arguments to those used above show that $\Phi(G) = \Phi(M_i) = 1$. This contradiction shows that X is not p-supersoluble and so X satisfies the properties enunciated in the statement of the theorem. In particular, $N_i/\Phi(G)$ is M_i-isomorphic to an X-chief factor of the form $\Phi(G)/C$, which is also a G-chief factor. It implies that $N_1/\Phi(G)$ and $N_2/\Phi(G)$ are G-isomorphic, and G satisfies properties 1 and 2 by [2, 1.2.36].

Suppose that $r \geq 3$. Denote $G = N_1M_1 = N_2M_2$, where M_1, M_2 are maximal subgroups of G such that $N_1 \cap M_1 = N_2 \cap M_2 = \Phi(G)$. Consider the following two chief series of G:

$$(\delta_1): \quad 1 \leq \cdots \leq \Phi(G) \leq N_1 \leq N_1N_2 \leq N_1N_2N_3 \leq \cdots \leq G,$$
$$(\delta_2): \quad 1 \leq \cdots \leq \Phi(G) \leq N_2 \leq N_1N_2 \leq N_1N_2N_3 \leq \cdots \leq G.$$

Intersecting the series (δ_i) term-by-term with M_i, $i = 1, 2$, we get the series:

$$\begin{aligned}(\delta_1) \cap M_1: \quad & 1 \leq \cdots \leq \Phi(G) = N_1 \cap M_1 \leq N_2 = N_1N_2 \cap M_1 \\ & \qquad \leq N_2N_3 = N_1N_2N_3 \cap M_1 \leq \cdots \leq M_1, \\ (\delta_2) \cap M_2: \quad & 1 \leq \cdots \leq \Phi(G) = N_2 \cap M_2 \leq N_1 = N_1N_2 \cap M_2 \\ & \qquad \leq N_1N_3 = N_1N_2N_3 \cap M_2 \leq \cdots \leq M_2.\end{aligned}$$

By induction, we have that M_i is p-supersoluble or all non-cyclic chief factors of M_i are M_i-isomorphic and have order p^2 and every complemented chief factor of M_i is non-cyclic, $i = 1, 2$. We distinguish two possibilities:

1. *M_1 or M_2 is p-supersoluble.*

   Assume that M_1 is p-supersoluble. Then it follows that $N_2/\Phi(G)$ and $N_3/\Phi(G)$ have order p. If M_2 were not p-supersoluble, then it would follow that all complemented p-chief factors of M_2 are M_2-isomorphic and of order p^2. In that case, $N_3/\Phi(G)$ would have order p^2. This contradiction yields that M_2 is p-supersoluble. In that case, every chief factor $N_i/\Phi(G)$ has order p, $i = 1, \dots, r$. In this case G is p-supersoluble, and the result holds.

2. *Neither M_1 nor M_2 is p-supersoluble.*

   Then all non-cyclic M_1-chief factors are M_1-isomorphic and have order p^2 and every complemented p-chief factor of M_1 is non-cyclic. In particular, $N_2/\Phi(G)$, $N_3/\Phi(G)$, ..., $N_r/\Phi(G)$ are M_1-isomorphic (and so G-isomorphic), and have order p^2. Since M_2 is not p-supersoluble, it follows that $N_1/\Phi(G)$, $N_3/\Phi(G)$, ..., $N_r/\Phi(G)$ are M_1-isomorphic (and so G-isomorphic). Consequently, $N_1/\Phi(G)$, $N_2/\Phi(G)$, ..., $N_r/\Phi(G)$ are G-isomorphic. Applying [2, 1.2.36], G satisfies the properties enunciated in the statement of the theorem.

Therefore we conclude that the result as stated is true. □

The proof of Theorem 3.4 leads also to the following result:

Corollary 3.5. *Let G be a p-soluble group satisfying property (†) such that $\mathrm{O}_{p'}(G) = 1$. If G is not p-supersoluble and $\mathrm{F}(G)/\Phi(G)$ is a chief factor of G, then $\Phi(G) = 1$.*

As an interesting deduction we have the

Corollary 3.6. *Let G be a p-soluble group satisfying property (†). Assume that $\mathrm{O}_{p'}(G) = 1$. Then either G is p-supersoluble or $\Phi(G) = 1$ and all complemented p-chief factors of G are G-isomorphic and have order p^2.*

Proof. Applying Theorem 3.3, it follows that $\mathrm{F}(G) = \mathrm{O}_p(G)$ is the unique Sylow p-subgroup of G. We shall proceed by induction on $|G|$. As usual, we write $\mathrm{F}\big(G/\Phi(G)\big) = N_1/\Phi(G) \times \cdots \times N_r/\Phi(G)$, where $N_i/\Phi(G)$ is a minimal normal subgroup of $G/\Phi(G)$ for all i. According to [2, 1.2.36], every complemented p-chief factor is G-isomorphic to $N_i/\Phi(G)$ for some i. Certainly, we can assume that $r \geq 2$ by Theorem 3.4 and Corollary 3.5. Suppose that G is not p-supersoluble. According to Theorem 3.4, $N_i/\Phi(G)$ has order p^2 by Theorem 3.4. Let M_i be a maximal subgroup of G such that $G = N_iM_i$ and $N_i \cap M_i = \Phi(G)$, $i = 1, 2, \dots, r$. Since $N_j/\Phi(G)$ is a p-chief factor of M_i for all $j \neq i$, it follows that M_i is not p-supersoluble. We observe that $\mathrm{C}_{M_i}(N_j/\Phi(G))$ is the Sylow p-subgroup of M_i for all $j \neq i$. Therefore $\mathrm{O}_{p'}(M_i) = 1$. Consequently $\Phi\big(M_i\big) = 1$ by induction. It therefore follows that $\Phi(N_i) = 1$ for all i and N_i is centralised by N_j for all $j \neq i$. Then N_i is elementary abelian and so it has the structure of G-module over the Galois field $\mathrm{GF}(p)$ as a natural way. Since N_i is centralised by

F(G), Maschke's theorem [6, A, 11.5] implies the complete reducibility of the representation space. In particular, there exists a minimal normal subgroup K_i of G of order p^2 such that $N_i = \Phi(G) \times K_i$, $i = 1, 2, \ldots, r$. Consequently, $G = N_1 \cdots N_r G_{p'} = K_1 \cdots K_r G_{p'}$, $\mathrm{F}(G) = K_1 \cdots K_r$ and $\Phi(G) = 1$, as required. □

Proof of the main theorem. We prove the necessity of the condition by induction on the order of G. Certainly, by Lemma 2.1, we may assume that $\mathrm{O}_{p'}(G) = 1$. Let P be a Sylow p-subgroup of G. It may be supposed that $|P|$ is greater than p. Then, applying Theorem 3.3, G is p-soluble and, by Theorem 3.2, the p-length of G is at most 1. Hence $P = \mathrm{O}_p(G) = \mathrm{F}(G)$ is the Sylow p-subgroup of G. By Corollary 3.6, we have that either G is p-supersoluble or $\Phi(G) = 1$. Suppose that G is not p-supersoluble. Then $\Phi(G) = 1$, and $\mathrm{F}(G)$ is elementary abelian and it can be regarded as a completely reducible G-module over the Galois field $\mathrm{GF}(p)$ by [6, A, 11.5]. This means that P is expressible as a direct product of minimal normal subgroups of G, say $P = V_1 \times \cdots \times V_r$, where V_i is an irreducible G-module over $\mathrm{GF}(p)$ ($i = 1, \ldots, r$). By Corollary 3.6, each $|V_i| = p^2$, $i = 1, \ldots, r$, and all of them are G-isomorphic. Now we consider the case that $r > 1$. Consider the submodule $W = V_1 \times V_2$. Write $K = \mathrm{GF}(p)$ and $V = V_1$. Suppose that V is an absolutely irreducible G-module. Then $E = \mathrm{End}_{KG}(V) = K$ and the G-endomorphisms of V are exactly those defined by $\theta_t : V \to V$, given by $v^{\theta_t} = v^t, v \in V, t \in K$. According to [6, B, 8.2], the irreducible submodules of W are $V^t = \{(v, v^t) : v \in V\}$, for $t \in K$, and $V_2 = \{(1, v) : v \in V\}$. On the other hand, V, as a vector K-space, has dimension 2. Let $\{a, b\}$ be a basis of V. With the obvious notation, consider the subgroup $A = \langle (a_1b_1, 1), (1, a_2b_2) \rangle$ of W. Then A has order p^2 and so A is a partial CAP-subgroup of G. But $A \cap V_2 = \langle (1, a_2b_2) \rangle$ and $A \cap V^t = \langle (a_1b_1, (a_2b_2)^t) \rangle$ for any $t \in K$, that is, A neither covers nor avoids any irreducible submodule of W, contrary to Lemma 2.2. Hence V is not an absolutely G-module and the necessity of the condition holds.

To prove the sufficiency, it may be assumed that G satisfies the condition 3 and $\mathrm{O}_{p'}(G) \neq 1$. Hence $\mathrm{F}(G) = \mathrm{Soc}(G) = N_1 \times \cdots \times N_r$, where $N_i \cong_G V$ is a G-irreducible module over $\mathrm{GF}(p)$ of dimension 2, for every $i = 1, \ldots, r$. Moreover, $\mathrm{F}(G)$ is a Sylow p-subgroup of G. We may also assume that $r \geq 2$. Then V is not an absolutely irreducible G-module. Let $Q = \langle a, b \rangle$ be a subgroup of G of order p^2. We prove that Q is a partial CAP-subgroup of G by induction on the order of G. Obviously, we can suppose that Q is not a minimal normal subgroup of G. Let N be a minimal normal subgroup of G. Then $Q \cap N$ is trivial or has order p. Assume that $Q \cap N$ is of order p for all minimal normal subgroups N of G. Then two different minimal normal subgroups produce different subgroups of order p of Q. Since the number of subgroups of order p of Q is exactly $p+1$, it follows that G has exactly $p+1$ minimal normal subgroups. However, according to [6, B, 8.2], G has at least p^k+1 minimal normal subgroups, where p^k is the number of elements in $\mathrm{End}_{KG}(V)$, and $k \geq 2$ since V is not an absolutely irreducible G-module. This contradiction implies that there exists a minimal normal subgroup A of G such that $Q \cap A = 1$. Since the group G/A satisfies the conditions of the theorem, we have that QA/A is a partial CAP-subgroup of G/A. It is clear then that Q is a partial CAP-subgroup of G. We conclude that G satisfies property (†). □

Acknowledgements

The first and the second authors have been supported by the research grant MTM2010-19938-C03-01 from MICINN (Spain). Most of this research was carried out during a visit of the third author to the Departament d'Àlgebra, Universitat de València, Burjassot, València, Spain, during the academic year 2009–10.

References

[1] A. Ballester-Bolinches, R. Esteban-Romero, and Y. Li. A question on partial CAP-subgroups of finite groups. Preprint.

[2] A. Ballester-Bolinches and L. M. Ezquerro. *Classes of Finite Groups*, volume 584 of *Mathematics and its Applications*. Springer, New York, 2006.

[3] A. Ballester-Bolinches, L. M. Ezquerro, and A. N. Skiba. Local embeddings of some families of subgroups of finite groups. *Acta Math. Sinica*, 25:869–882, 2009.

[4] A. Ballester-Bolinches, L. M. Ezquerro, and A. N. Skiba. On second maximal subgroups of Sylow subgroups of finite groups. *J. Pure Appl. Algebra*, 215:705–714, 2011.

[5] A. Ballester-Bolinches and M. C. Pedraza-Aguilera. On minimal subgroups of finite groups. *Acta Math. Hungar.*, 73(4):335–342, 1996.

[6] K. Doerk and T. Hawkes. *Finite Soluble Groups*, volume 4 of *De Gruyter Expositions in Mathematics*. Walter de Gruyter, Berlin, New York, 1992.

[7] L. M. Ezquerro. A contribution to the theory of finite supersolvable groups. *Rend. Sem. Mat. Uni. Padova*, 89:161–170, 1993.

[8] L. M. Ezquerro and X. Soler-Escrivà. Some permutability properties related to $\mathfrak{F}$-hypercentrally embedded subgroups of finite groups. *J. Algebra*, 264:279–295, 2003.

[9] L. M. Ezquerro and X. Soler-Escrivà. Some new permutability properties of hypercentrally embedded groups. *J. Austral. Math. Soc. Ser. A*, 79(2):243–255, 2005.

[10] L. M. Ezquerro and X. Soler-Escrivà. On certain distributive lattices of subgroups of finite soluble groups. *Acta Math. Sinica (Engl. Ser.)*, 23(11):2069–2078, 2007.

[11] Y. Fan, X. Guo, and K. P. Shum. Remarks on two generalizations of normality of subgroups. *Chinese Ann. Math.*, 27A(2):169–176, 2006. Chinese.

[12] J. D. Gillam. Cover-avoid subgroup in finite solvable groups. *J. Algebra*, 29:324–329, 1974.

[13] X. Guo, P. Guo, and K. P. Shum. On semi cover-avoiding subgroups of finite groups. *J. Pure Appl. Algebra*, 209:151–158, 2007.

[14] X. Guo and K. P. Shum. Cover-avoidance properties and the structure of finite groups. *J. Pure Appl. Algebra*, 181:297–308, 2003.

[15] X. Guo, J. Wang, and K. P. Shum. On semi-cover-avoiding maximal subgroups and solvability of finite groups. *Comm. Algebra*, 34:3235–3244, 2006.

[16] X. Guo and L. L. Wang. On finite groups with some semi cover-avoiding subgroups. *Acta Math. Sin. (Engl. Ser.)*, 23(9):1689–1696, 2007.

[17] B. Huppert. Normalteiler und maximale Untergruppen endlicher Gruppen. *Math. Z.*, 60:409–434, 1954.

[18] B. Huppert. *Endliche Gruppen I*, volume 134 of *Grund. Math. Wiss.* Springer Verlag, Berlin, Heidelberg, New York, 1967.

[19] B. Huppert and N. Blackburn. *Finite Groups II*, volume 242 of *Grund. Math. Wiss.* Springer-Verlag, Berlin-Heidelberg-New York, 1982.

[20] Y. Li. On cover-avoiding subgroups of Sylow subgroups of finite groups. *Rend. Sem. Mat. Uni. Padova*, 123:249–258, 2010.

[21] Y. Li, L. Miao, and Y. Wang. On semi cover-avoiding maximal subgroups of Sylow subgroups of finite groups. *Comm. Algebra*, 37(4):1160–1169, 2009.

[22] J. Petrillo. The embedding of CAP-subgroups in finite groups. *Ricerche mat.*, 59:97–107, 2010.

[23] N. S. N. Sastry and W. E. Deskins. Influence of normality conditions on almost minimal subgroups of a finite group. *J. Algebra*, 52:364–377, 1978.

[24] K. U. Schaller. *Über Deck-Meide-Untergruppen in endlichen auflösbaren Gruppen.* PhD thesis, Universität Kiel, 1971.

[25] M. J. Tomkinson. Cover-avoidance properties in finite solvable groups. *Canad. Math. Bull.*, 19(2):213–216, 1976.

[26] J. X. Wang and X. Y. Guo. Semi-cover-avoiding properties and the structure of finite groups. *J. Math. Res. Exposition*, 30(3):527–535, 2010.

[27] J. Zhang. Influence of S-quasinormality condition on almost minimal subgroups of a finite group. *Acta Math. Sin., New Ser.*, 3:125–132, 1987.