ON FINITE MINIMAL NON-NILPOTENT GROUPS

A. BALLESTER-BOLINCHES, R. ESTEBAN-ROMERO, AND DEREK J. S. ROBINSON

(Communicated by Jonathan I. Hall)

ABSTRACT. A critical group for a class of groups \(\mathfrak{X}\) is a minimal non-\(\mathfrak{X}\)-group. The critical groups are determined for various classes of finite groups. As a consequence, a classification of the minimal non-nilpotent groups (also called Schmidt groups) is given, together with a complete proof of Gol’fand’s theorem on maximal Schmidt groups.

1. Introduction

Given a class of groups \(\mathfrak{X}\), we say that a group \(G\) is a minimal non-\(\mathfrak{X}\)-group, or an \(\mathfrak{X}\)-critical group, if \(G \not\in \mathfrak{X}\), but all proper subgroups of \(G\) belong to \(\mathfrak{X}\). It is clear that detailed knowledge of the structure of minimal non-\(\mathfrak{X}\)-groups can provide insight into what makes a group belong to \(\mathfrak{X}\). All groups considered in this paper are finite.

Minimal non-\(\mathfrak{X}\)-groups have been studied for various classes of groups \(\mathfrak{X}\). For instance, minimal non-abelian groups were analysed by Miller and Moreno [10], while Schmidt [14] studied minimal non-nilpotent groups. The latter are now known as Schmidt groups. Itô [9] considered the minimal non-\(p\)-nilpotent groups for \(p\) a prime, which turn out to be just the Schmidt groups. Finally, the third author [12] characterised the minimal non-\(T\)-groups (\(T\)-groups are groups in which normality is a transitive relation). He also characterised in [13] the minimal non-\(PST\)-groups, where a \(PST\)-group is a group in which Sylow permutability is a transitive relation.

The aim of this paper is to give more precise information about the structure of Schmidt groups and show how to construct them in an efficient way. As a consequence of our study, a new proof of a classical theorem of Gol’fand is given.

Our approach depends on the classification of critical groups for the class of \(PST\)-groups given in [13]. Recall that a subgroup \(H\) is said to be Sylow-permutable, or S-permutable, in a group \(G\) if \(H\) permutes with every Sylow subgroup of \(G\). We mention a similar class \(\mathfrak{Y}_p\), which was introduced in [2]. If \(p\) is a prime, a group \(G\) belongs to the class \(\mathfrak{Y}_p\) if \(G\) enjoys the following property: if \(H\) and \(K\) are \(p\)-subgroups of \(G\) such that \(H\) is contained in \(K\), then \(H\) is S-permutable in \(N_G(K)\).

Clearly every \(PST\)-group is a \(\mathfrak{Y}_p\)-group.

There is a close relation between the class of groups just introduced and \(p\)-nilpotence, as shown by the following result, which was proved in [2, Theorem 5].
Theorem 1. A group $G$ is a $\mathcal{Y}_p$-group if and only if either it is $p$-nilpotent or it has an abelian Sylow $p$-subgroup $P$ and every subgroup of $P$ is normal in $N_G(P)$.

Our first main result is:

Theorem 2. The minimal non-$\mathcal{Y}_p$-groups are just the minimal non-PST-groups with a non-trivial normal Sylow $p$-subgroup. Such groups are of the types described in 1 to IV below. Let $p$ and $q$ be distinct primes.

Type I: $G = [P]Q$, where $P = \langle a,b \rangle$ is an elementary abelian group of order $p^2$, $Q = \langle z \rangle$ is cyclic of order $q^r$, with $q$ a prime such that $q^r$ divides $p-1$, $q^f > 1$ and $r \geq f$, and $a^z = a^i$, $b^z = b^i$, where $i$ is the least positive primitive $q^f$-th root of unity modulo $p$ and $j = 1 + kq^f-1$, with $0 < k < q$.

Type II: $G = [P]Q$, where $Q = \langle z \rangle$ is cyclic of order $q^r > 1$, with $q$ a prime not dividing $p-1$ and $P$ an irreducible $Q$-module over the field of $p$ elements with centralizer $\langle z^f \rangle$ in $Q$.

Type III: $G = [P]Q$, where $P = \langle a_0, a_1, \ldots, a_{q-1} \rangle$ is an elementary abelian $p$-group of order $p^q$, $Q = \langle z \rangle$ is cyclic of order $q^r$, with $q$ a prime such that $q^f$ is the highest power of $q$ dividing $p-1$ and $r > f$. Define $a_{j+1}^z = a_{j+1}$ for $0 \leq j < q - 1$ and $a_{q-1}^z = a_0$, where $i$ is a primitive $q^f$-th root of unity modulo $p$.

Type IV: $G = [P]Q$, where $P$ is a non-abelian special $p$-group of rank $2m$, the order of $p$ modulo $q$ being $2m$, $Q = \langle z \rangle$ is cyclic of order $q^r > 1$, $z$ induces an automorphism in $P$ such that $P/\Phi(P)$ is a faithful irreducible $Q$-module, and $z$ centralizes $\Phi(P)$. Furthermore, $|P/\Phi(P)| = p^{2m}$ and $|P| \leq p^m$.

Since a group is a soluble PST-group if and only if it belongs to $\mathcal{Y}_p$ for all primes $p$ [2 Theorem 4], Theorem 2 may be regarded as a local approach to the third author’s classification of minimal non-PST-groups [13].

An interesting consequence of Theorem 2 is the following classification of Schmidt groups. In order to describe the classification, we must introduce one further type of group:

Type V: $G = [P]Q$, where $P = \langle a \rangle$ is a normal subgroup of order $p$, $Q = \langle z \rangle$ is cyclic of order $q^r > 1$, and $a^z = a^i$, where $i$ is the least primitive $q$-th root of unity modulo $p$.

Our main result can now be stated as:

Theorem 3. The Schmidt groups are exactly the groups of Type II, Type IV and Type V.

Our next result shows that $p$-soluble groups with Sylow $p$-subgroups isomorphic to a normal subgroup of a minimal non-$\mathcal{Y}_p$-group have a restricted structure.

Theorem 4. Let $G$ be a $p$-soluble group with a Sylow $p$-subgroup $P$. If $P$ is isomorphic to a non-trivial normal Sylow subgroup of a minimal non-$\mathcal{Y}_p$-group, then $G$ has $p$-length 1.

In [4] Gol’fand stated the following result:

Theorem 5. Let $p$ and $q$ be distinct primes, let $r$ be a given positive integer, and let $a$ be the order of $p$ modulo $q$. Then there is a unique minimal non-$p$-nilpotent group $G_0$ of order $p^a q^r$, where $a_0 = a$ if $a$ is odd and $a_0 = 3a/2$ if $a$ is even, such
that all minimal non-$p$-nilpotent groups of order $p^iq^r$ are isomorphic to quotients of $G_0$ by central subgroups.

Only a sketch of a proof of this theorem is given in Golfand’s article. In Section 3 we show how to construct the Schmidt groups of Gol’fand, and we also give a complete proof of Theorem 5. We remark that Rédei [11] has given another construction of the Schmidt groups of maximum order.

2. PROOFS OF THEOREMS 2, 3 AND 4

Proof of Theorem 2. Assume that $G$ is a minimal non-$Y_p$-group and let $P$ be a Sylow $p$-subgroup of $G$. Since $G$ does not belong to $Y_p$, there exist subgroups $H$ and $K$ of $P$ such that $H \leq K$ and $H$ is not $S$-permutable in $N_G(K)$. Consequently, there is an element $z \in N_G(K)$ such that $z$ does not normalise $H$. Here it can be assumed that $z$ has order $q^r$ for some prime $q \neq p$. Then $G = K \langle z \rangle$ because $G$ is a minimal non-$Y_p$-group. This implies that $K = P$ is a normal Sylow $p$-subgroup of $G$ and $Q = \langle z \rangle$ is a cyclic Sylow $q$-subgroup of $G$. Then $G$ is not a $PST$-group, yet every proper subgroup has $Y_p$ and $Y_q$, and thus is a $PST$-group by [2].

Conversely, if $G$ is a minimal non-$PST$-group, then $G$ does not have $Y_p$ for some prime $p$. Since all its proper subgroups satisfy $Y_p$, the group $G$ is a minimal non-$Y_p$-group. The classification of minimal non-$PST$-groups given in [13] completes the proof. (Note that the groups of Types IV and V of [13] are both of Type IV above.)

Proof of Theorem 3. Let $G$ be a minimal non-nilpotent group. Then $G$ is a minimal non-$p$-nilpotent group for some prime $p$. Suppose that $G$ is not a $Y_p$-group, so that $G$ is a minimal non-$Y_p$-group. By Theorem 2, the group $G$ is of one of Types I–IV. By examining the group structure, we see that groups of Type I and III are not minimal non-$p$-nilpotent. Therefore $G$ must be of Type II or IV.

Assume now that $G$ belongs to $Y_p$. Then by [11] Theorem A and [3] VII, 6.18, the $p$-nilpotent residual $P$ of $G$ is an abelian minimal normal Sylow subgroup which is complemented in $G$ by a cyclic Sylow $q$-subgroup $Q$. Moreover $Q$ normalises each subgroup of $P$. This implies that $P$ is cyclic of order $p$, say $P = \langle a \rangle$. In addition, $a^z = a^i$ for some $0 < i < p$ and $z^q$ centralizes $a$. This implies that $i$ must be a primitive $q$-th root of unity modulo $p$ and, by taking a suitable power of $z$ as a generator of $Q$, we can assume that $i$ is the least such positive integer. Hence $G$ is of Type V.

Proof of Theorem 4. Assume that $G$ is a $p$-soluble group with $p$-length $> 1$ and $G$ has least order subject to possessing a Sylow $p$-subgroup $P$ which is isomorphic to a non-trivial normal Sylow subgroup of a Schmidt group. By [6] VI, 6.10], we conclude that $P$ is not abelian. Thus $P$ is a Sylow $p$-subgroup of a group of Type IV in Theorem 2. By minimality of order $O_p(G) = 1$ and $O^p(G) = G$. In addition, since the class of groups of $p$-length at most $1$ is a saturated formation, we have $\Phi(G) = 1$ and hence $G$ has a unique minimal normal subgroup which is an elementary abelian $p$-group. Let $D = O_p(G)$; then $D$ is a non-trivial elementary abelian group and $C_G(D) = D$. Moreover $\Phi(P) = Z(P) \leq D$ and so $P/D$ is elementary abelian.

Let $T$ be the subgroup defined by $T/D = O_p(G/D)$. Since $P/D$ is an elementary abelian $p$-group, $G/D$ has $p$-length at most $1$ by [6] VI, 6.10]. It follows that $(T/D)(P/D)$ is a normal subgroup of $G/D$. Therefore $TP$ is a normal subgroup of
G. Assume that \(TP\) is a proper subgroup of \(G\). Now \(O_{p'}(TP) \leq O_{p'}(G) = 1\), so \(P\) is a normal subgroup of \(TP\) and hence of \(G\), a contradiction which shows that \(G = TP\).

Assume now that \(P/D\) is a non-cyclic elementary abelian group. By [8] X. 1.9, we have \(T/D = (C_{T/D}(xD))\) \(x \in P/D\). Let \(x \in P\). Since \(P/D\) centralizes \(x/D\), we have \(P/D \leq N_{G/D}(C_{T/D}(xD))\) and \(T_x/D = C_{T/D}(xD)\). Assume that \(PT_x = G\); then \(T_x = T\) is a normal subgroup of \(G\) and thus \(O_{p'}(G/D) = T_x/D\). This implies that \(\langle x \rangle D/D \leq Z(G/D)\) and \(\langle x \rangle D\) is a normal \(p\)-subgroup of \(G\), so \(\langle x \rangle D\) is contained in \(D\), a contradiction. Consequently \(PT_x\) is a proper subgroup of \(G\) for all \(1 \neq x \in P/D\). Hence \(PT_x\) has \(p\)-length at most 1 by minimality of \(G\). Since \(C_G(D) = D\) and \(O_{p'}(PT_x)\) centralizes \(D\), we conclude that \(O_{p'}(PT_x) = 1\). Therefore \(P\) is a normal subgroup of \(PT_x\), which shows that \(T\) normalizes \(P\) and thus \(P\) is a normal subgroup of \(G\). This contradiction shows that \(P/D\) is cyclic.

Since \(P\) has class 2, we see from [7] IX. 5.5 that, if \(p > 3\), then \(G\) has \(p\)-length at most 1. Therefore \(p \leq 3\). Let \(X\) be a minimal \(p\)-group such that \(P\) is a Sylow \(p\)-subgroup of \(X\). Note that \(P/\Phi(P)\) is an irreducible \(X\)-module. In particular \(D\), the subgroup of the previous paragraphs, is not normal in \(X\) and so \(P = DD^g\) for some \(g \in X\). Since \(D\) is abelian, \(D \cap D^g \leq Z(P) = \Phi(P)\), and it follows that \(P/\Phi(P)\) has order \(p^2\). This implies that \(P\) is an extra-special group of order \(p^3\). If \(p = 2\), then, since \(C_G(D) = D\), we see that \(G\) must be a symmetric group of degree 4. Hence \(P\) is dihedral of order 8, which cannot lead to a group of Type IV since \(\text{Aut}(P)\) is a 2-group. Hence \(p = 3\). But a non-abelian group of order \(3^3\) cannot occur as the normal Sylow 3-subgroup of a Schmidt group, because the only prime divisor of \(3^3 - 1\) is 2 and the order of 3 modulo 2 is 1. This contradiction completes the proof of the theorem. \(\square\)

3. The construction of Gol’fand’s groups and a proof of Gol’fand’s theorem

We begin by constructing groups of Type IV with a Sylow \(p\)-subgroup \(P\) of order \(p^{3m}\) and \(|P/\Phi(P)| = p^{2m}\). These groups were constructed in [13] by a different method, but the present approach is more convenient when \(p = 2\). We will use the following result on linear operators.

**Lemma 6.** Let \(p\) be a prime and let \(r\) be a positive integer such that gcd\((p, r) = 1\). Let \(\beta\) be a linear operator of order \(p^r\) on a vector space \(V\) over the field of \(p\)-elements, where \(u\) is a non-negative integer. If \(\beta\) has irreducible minimum polynomial \(f\), then \(\beta^u\) also has minimum polynomial \(f\).

**Proof.** Let \(g\) be the minimum polynomial of \(\beta^u\). Now \(f(\beta^u) = f(\beta)^u = 0\), so that \(g\) divides \(f\). Since \(f\) is irreducible, \(f = g\). \(\square\)

**Construction 7.** Let \(p\) and \(q\) be distinct primes such that the order of \(p\) modulo \(q\) is \(2m\), \(m \geq 1\). Let \(F\) be the free group with basis \(\{f_0, f_1, \ldots, f_{2m-1}\}\). Write \(R = F^r F^p\) and \(R^* = [F, R] R^p\). Then \(F/R\) is an elementary abelian \(p\)-group of order \(p^{2m}\) and \(H = F/R^*\) is a \(p\)-group such that \(R/R^* = \Phi(H)\) is an elementary abelian \(p\)-group contained in \(Z(H)\). Moreover \(H\) is a non-abelian group because an extra-special group of order \(p^{2m+1}\) is an epimorphic image of \(H\).

Denote by \(g_i\) the image of \(f_i\) under the natural epimorphism of \(F\) onto \(H = F/R^*, 0 \leq i \leq 2m - 1\). Since \(H\) has class 2, we know that \(\Phi(H)\) is generated by all \([g_i, g_j], \) with \(i < j\), and \(g_i^p\). Therefore \(\Phi(H)\) has dimension as \(GF(p)\)-vector space
at most $\frac{1}{2}(2m(2m-1)) + 2m = m(2m + 1)$. Then there exists an element

$$r = \prod_{j} (f_j^p)^{\lambda_j} \prod_{j < k} [f_j, f_k]^{\mu_{jk}} \in R^*$$

with some $\lambda_j$ or $\mu_{jk}$ not divisible by $p$. It is clear that $p \mid \lambda_j$ for all $j$ since $F^p F'/F'$ is a free abelian group with basis $\{f_j^p F' \mid 0 \leq j \leq 2m - 1\}$. Suppose that $p \nmid \mu_{jk}$ for some $i < k$ and let $\rho_i$ be the endomorphism of $F$ defined by $f_j^\rho_i = f_j^2$, $f_j^\rho_i = f_j$ for $l \neq i$. Then $r^\rho_i R^* = R^*$ and so $r^{\rho_i} r^{-1} R^* = R^*$. This implies that

$$w = \prod_{j < i} [f_j, f_i]^{\mu_{ji}} \prod_{i < l} [f_i, f_l]^{\mu_{il}} \in R^*.$$  

On the other hand, by applying $\rho_k$ we find that

$$w^{\rho_k} w^{-1} R^* = [f_i, f_k]^{\mu_{ik}} R^* = R^*.$$  

Since $p \nmid \mu_{ik}$, it follows that $\mu_{ik}$ has an inverse modulo $p$. This means that $[f_i, f_k] \in R^*$. Now since permutations of the generators of $F$ induce endomorphisms in $F$ and $R^*$ is fully invariant, it follows that $F' \leq R^*$ and $H$ is abelian, a contradiction. Therefore $\Phi(H)$ has dimension $m(2m + 1)$ and so $|\Phi(H)| = p^{m(2m+1)}$.

Let $f(t) = c_0 + c_1 t + \cdots + c_{2m-1} t^{2m-1} + t^{2m}$ be an irreducible factor of the cyclotomic polynomial of order $q$ over $GF(p)$ and let $\alpha$ be the endomorphism of $F$ given by $f_j^\alpha = f_{j+1}$ for $0 \leq i \leq 2m - 2$, $f_0^\alpha = f_0 c_0 f_1^{-c_1} \cdots f_{2m-1}^{-c_{2m-1}}$. Since $R^*$ is a fully invariant subgroup of $F$, it follows that $\alpha$ induces an endomorphism $\beta$ on $H = F/R^*$. In turn, $\beta$ induces an automorphism $\beta$ of $H/\Phi(H)$. Since $H/\Phi(H) = (H/\Phi(H))/\beta \leq H^\beta \Phi(H)/\Phi(H)$, it follows that $H = H^\beta \Phi(H)$, whence $H = H^\beta$. Consequently $\beta$ is an automorphism of $H$.

It is clear that $\beta$ induces the linear automorphism $\beta$, with minimum polynomial $f$, on the vector space $H/\Phi(H)$. Now by [6, III.18], we conclude that $\beta$ has order $p^m$ for some $u$ and hence $\beta$ has order $p^m q$. By Lemma 6 there is a $GF(p)$-basis $\{g_0', g_1', \ldots, g_{2m-1}'\}$ of $H/\Phi(H)$, where $g_i' = g_i \Phi(H)$, such that $g_i^{\beta^m} = g_i^{c+1}$ for $0 \leq i \leq 2m - 2$ and $g_{2m-1}^{\beta^m} = g_0 c_0 g_1^{-c_1} \cdots g_{2m-1}^{-c_{2m-1}}$. Hence we can replace $\beta$ by $\beta^m$ and assume without loss of generality that $\beta$ has order $q$.

It follows that $\Phi(H)$ is a $GF(p)T$-module, where $T = (\beta)$ is a cyclic group of order $q$. By Maschke’s Theorem $\Phi(H)$ is a direct sum of irreducible $T$-modules. Let $N$ be the sum of all non-trivial irreducible submodules in the direct decomposition and write $P = H/N$. It is clear that $N$ is $\beta$-invariant and therefore $\beta$ induces an automorphism $\gamma$ of order $q$ in $P$. Let $Q = (z)$ be a cyclic group of order $q^*$ acting on $P$ via $z \mapsto \gamma$. Let $G = |P|Q$ be the corresponding semidirect product.

It is easily checked that $G$ is a Schmidt group. Next we show that $P$ has order $p^{2m}$. From Theorem 3 we see that $\Phi(P)$ has order at most $p^m$, where $|P/\Phi(P)| = p^{2m}$. On the other hand, $|\Phi(H)| = p^{m(2m+1)}$, and $N$ has order a power of $p^{2m}$ because every faithful irreducible $(\beta)$-module over $GF(p)$ has dimension $2m$. Therefore $|\Phi(P)| = p^m$.

**Remark 8.** In the group of Construction [7] we may assume that $g_{2m-1}^z = g_0^{-c_0} g_1^{-c_1} \cdots g_{2m-1}^{-c_{2m-1}}$, where $g_i = g_i N$.

**Proof.** We know that $g_{2m-1}^z = g_0^{-c_0} g_1^{-c_1} \cdots g_{2m-1}^{-c_{2m-1}} \bar{w}$, where $\bar{w} \in \Phi(P)$. Since $f(t)$ is irreducible, 1 is not a root of $f(t)$ and it follows that $c = c_0 + c_1 + \cdots + c_{2m-1} + 1 \neq 0$.
(mod \(p\)). Consequently there exists an integer \(d\) such that \(cd \equiv -1 \pmod{p}\). Put \(\bar{w}_0 = \bar{w}^d\) and consider the automorphism \(\delta\) of \(P\) defined by \(\bar{g}^i = \bar{g}_i \bar{w}_i\) for \(0 \leq i \leq 2m - 1\). If we write \(\gamma_0 = \delta \gamma \delta^{-1}\), it is easily checked by an elementary calculation that \(\bar{g}^i_0 = \bar{g}_{i+1}\) for \(0 \leq i \leq 2m - 2\), and \(\bar{g}_{2m-1} = \bar{g}_0^{-c_0} \bar{g}_1^{-c_1} \cdots \bar{g}_{2m-1}^{-c_{2m-1}}\). Let \(\langle z_0 \rangle\) be a cyclic group of order \(q^{'}\), with \(z_0\) acting on \(P\) via \(z_0 \mapsto \gamma_0\). Since \(\langle z_0 \rangle\) and \(\langle z \rangle\) are conjugate in \(\text{Aut}(P)\), it follows by [3, B, 12.1] that the groups \(P\langle z \rangle\) and \(P\langle z_0 \rangle\) are isomorphic. \(\square\)

**Remark 9.** The group in Construction [7] does not depend on the choice of irreducible factor \(f(t)\).

**Proof.** Assume that the group \(G_1 = \langle P_1 \rangle \langle z_1 \rangle\) has been constructed by using another irreducible factor \(g(t)\) of the cyclotomic polynomial of order \(q\) over \(\text{GF}(p)\). Since \(G\) and \(G_1\) have the same order, it will be enough to find a set of generators of \(G_1\) for which the relations of \(G\) hold. Since \(z\) centralizes \(\Phi(P)\) and \(z_1\) centralizes \(\Phi(P_1)\), we have \(G/\Phi(P) \cong [P/\Phi(P)]/\langle z \rangle\) and \(G_1/\Phi(P_1) \cong [P_1/\Phi(P_1)]/\langle z_1 \rangle\). But \(P/\Phi(P)\) and \(P_1/\Phi(P_1)\) are faithful irreducible modules for a cyclic group of order \(q\). Therefore \([P/\Phi(P)]/\langle z \rangle\) is isomorphic to \([P_1/\Phi(P)]/\langle z_1 \rangle\) by [3, B, 12.4]. Let \(\phi\) be an isomorphism between these groups. Then it is clear that \(\phi\) induces an isomorphism \(\psi\) between \(G/\Phi(P)\) and \(G_1/\Phi(P_1)\).

Let \(\bar{k}_i = h_i \Phi(P), 0 \leq i \leq 2m - 1\). Put \(k_i = \bar{k}_i\) and \(\bar{u} = \bar{z}^\phi\). We show how to extend the isomorphism \(\psi\) to an isomorphism between \(G\) and \(G_1\). In order to do so, we choose representatives \(k_i\) of \(k_i\) and \(u\) of \(\bar{u}\) such that the order of \(u\) is \(q^{'}\). There is no loss of generality in assuming that \(k_i^u = k_{i+1}\) for \(0 \leq i \leq 2m - 2\). Indeed, if \(k_i^u = k_{i+1}w_{i+1}^{-1}\) with \(w_{i+1} \in \Phi(P_1)\), then \(k_i^u = k_i w_i \cdots w_1\) for \(1 \leq i \leq 2m - 1\), \(k_0^u = k_0\) are representatives of \(k_i\) and \(k_i^u = k_i^{u+1}\) for \(1 \leq i \leq 2m - 1\) because \(u\) centralizes \(\Phi(P_1)\). By using the same argument as in Remark [8], we may also assume that \(k_{2m-1}^u = k_0^{-c_0} k_1^{-c_1} \cdots k_{2m-1}^{-c_{2m-1}}\). Therefore \(G\) and \(G_1\) satisfy the same relations and by Von Dyck’s theorem they are isomorphic. \(\square\)

**Remark 10.** In Construction [7] it is not necessary to assume that \(\beta\) has order \(q\). Indeed, it can be proved that \(\beta^q\) fixes all elements of \(\Phi(H)\) and that the automorphism \(\gamma\) induced by \(\beta\) in \(H/N\) has order \(q\).

Gol’fand’s result (Theorem [5]) can be recovered with the help of Construction [7] and Theorem [3].

**Proof of Theorem [5].** Let \(p\) and \(q\) be distinct primes and let \(a\) be the order of \(p\) modulo \(q\). Then \(a\) is the dimension of each non-trivial irreducible module for a cyclic group of order \(q\) over \(\text{GF}(p)\). Assume that \(a\) is odd. Then every Schmidt group \(G\) with a normal Sylow \(p\)-subgroup \(P\) such that \(|P/\Phi(P)| = p^a\) is of Type II or Type V. Then the theorem holds in this case because all Schmidt groups of the same type with isomorphic Sylow \(q\)-subgroups are actually isomorphic.

Assume now that \(a\) is even, with say \(a = 2m\). Then we are dealing with Schmidt groups of Type II or Type IV. Let \(G_0\) be the group of Construction [7]. Then \(|G_0| = p^{3m} q^a\) and \(|P_0/\Phi(P_0)| = p^{3m}\), where \(P_0\) is a normal Sylow \(p\)-subgroup of \(G_0\). It is clear that \(G_0/\Phi(P_0)\) is a Schmidt group of Type II. Therefore, if \(G\) is a Schmidt group of Type II with order \(p^t q^r\) and a normal Sylow \(p\)-subgroup, then \(G \cong G_0/\Phi(P_0)\) and \(\Phi(P_0) \leq \text{Z}(G_0)\). Consequently, we need only show that all Schmidt groups of Type IV and order \(p^t q^r\), \(t \leq 3m\), which have a normal Sylow \(p\)-subgroup are isomorphic to quotients of \(G_0\) by central subgroups.
ON FINITE MINIMAL NON-NILPOTENT GROUPS

Let $\mathcal{G}$ be a Schmidt group of Type IV and order $p^jq^r$ with a normal Sylow $p$-subgroup $P$. Then $G_0/\Phi(P_0)$ and $\mathcal{G}/\Phi(\mathcal{P})$ are isomorphic. Let us choose generators $z$ and $\bar{z}$ of Sylow $q$-subgroups $Q$ of $G_0$ and $\mathcal{G}$ of $\mathcal{G}$ such that the minimum polynomials of the actions of $z$ on $P_0/\Phi(P_0)$ and $\bar{z}$ on $\mathcal{P}/\Phi(\mathcal{P})$ coincide. Also choose generators $g_0, g_1, \ldots, g_{2m-1}$ of the Sylow $p$-subgroup $P_0$ of $G_0$ and generators $\bar{g}_0, \bar{g}_1, \ldots, \bar{g}_{2m-1}$ of the Sylow $p$-subgroup $\mathcal{P}$ of $\mathcal{G}$ such that $g_j^2 = g_{j+1}$ and $\bar{g}_j^2 = \bar{g}_{j+1}$ for $0 \leq j < 2m -2$. Since $\Phi(P_0) = P'_0$ and $\Phi(\mathcal{P}) = \mathcal{P}'$, and both $P_0$ and $\mathcal{P}$ have class 2, the subgroup $\Phi(P_0)$ can be generated by the commutators $[g_i, g_j]$, while $\Phi(\mathcal{P})$ is generated by the commutators $[\bar{g}_i, \bar{g}_j]$. On the other hand, if $u_i = [g_0, g_i^s]$, we have $u_i = u_i^{-1} = [g_k, \bar{g}_k^s]$. It is easy to see that $u_i = [g_0, g_i^s] = [g_0, \bar{g}_0^s]$.

Observe that $q$ is odd since $2q - 1$: write $q = 2s + 1$. By definition of the $g_i$ and $u_i$, and use of the minimum polynomial of the action of $z$ on $P_0/\Phi(P_0)$, it may be shown that for $l \geq 1$,

$$u_{s+m+l} = u_{s+m+l}^{-1} = u_{s-m+l}^{-1} \cdots u_{s-m+1}^{-1} u_{s+m+l}^{-1} \cdots u_{s+m+l}^{-1}.$$  

Now this formula and the relations $u_i = u_i^{-1}$ allow us to show by induction that each $u_{s+m+l}$ can be expressed in terms of elements of the set $B = \{u_{s-m+l}, u_{s-m+2}, \ldots, u_s\}$. Since $\Phi(P_0)$ has dimension $m$ over $\text{GF}(p)$, this expression is unique. It follows that each $u_i$ can be uniquely expressed in terms of the elements of $B$, and so this is also true for each generator of $\Phi(P_0)$. The same argument shows that the generators of $\Phi(\mathcal{P})$ have a similar unique expression subject to the same relations.

The arguments of Remark [3] allow us to assume that $g_{2m-1} = g_0^{-c_0} g_1^{-c_1} \cdots g_{2m-1}^{-c_2}$ and $\bar{g}_{2m-1} = \bar{g}_0^{-c_0} \bar{g}_1^{-c_1} \cdots \bar{g}_{2m-1}^{-c_2}$. Consequently, all relations of $G_0$ are satisfied by $\mathcal{G}$. By Von Dyck's theorem, it follows that $\mathcal{G}$ is an epimorphic image of $G_0$ by a central subgroup of $G_0$.

References


Departament d’Àlgebra, Universitat de València, Dr. Moliner, 50, E-46100 Burjassot, València, Spain
E-mail address: Adolfo.Ballester@uv.es

Departament de Matemàtica Aplicada, Universitat Politècnica de València, Camí de Vera, s/n, E-46022 València, Spain
E-mail address: resteban@mat.upv.es

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, Illinois 61801
E-mail address: robinson@math.uiuc.edu