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# ON FINITE MINIMAL NON-NILPOTENT GROUPS

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ABSTRACT. A *critical group* for a class of groups $\mathfrak{X}$ is a minimal non-$\mathfrak{X}$-group. The critical groups are determined for various classes of finite groups. As a consequence, a classification of the minimal non-nilpotent groups (also called Schmidt groups) is given, together with a complete proof of Gol'fand's theorem on maximal Schmidt groups.

## 1. INTRODUCTION

Given a class of groups $\mathfrak{X}$, we say that a group $G$ is a *minimal non-$\mathfrak{X}$-group*, or an *$\mathfrak{X}$-critical group*, if $G \notin \mathfrak{X}$, but all proper subgroups of $G$ belong to $\mathfrak{X}$. It is clear that detailed knowledge of the structure of minimal non-$\mathfrak{X}$-groups can provide insight into what makes a group belong to $\mathfrak{X}$. All groups considered in this paper are finite

Minimal non-$\mathfrak{X}$-groups have been studied for various classes of groups $\mathfrak{X}$. For instance, minimal non-abelian groups were analysed by Miller and Moreno [10], while Schmidt [14] studied minimal non-nilpotent groups. The latter are now known as *Schmidt groups*. Itô [9] considered the minimal non-$p$-nilpotent groups for $p$ a prime, which turn out to be just the Schmidt groups. Finally, the third author [12] characterised the minimal non-$T$-groups (*$T$-groups* are groups in which normality is a transitive relation). He also characterised in [13] the minimal non-$PST$-groups, where a *$PST$-group* is a group in which Sylow permutability is a transitive relation.

The aim of this paper is to give more precise information about the structure of Schmidt groups and show how to construct them in an efficient way. As a consequence of our study, a new proof of a classical theorem of Gol'fand is given.

Our approach depends on the classification of critical groups for the class of $PST$-groups given in [13]. Recall that a subgroup $H$ is said to be *Sylow-permutable*, or S-*permutable*, in a group $G$ if $H$ permutes with every Sylow subgroup of $G$. We mention a similar class $\mathcal{Y}_p$, which was introduced in [2]. If $p$ is a prime, a group $G$ belongs to *the class* $\mathcal{Y}_p$ if $G$ enjoys the following property: if $H$ and $K$ are $p$-subgroups of $G$ such that $H$ is contained in $K$, then $H$ is S-permutable in $\mathrm{N}_G(K)$. Clearly every $PST$-group is a $\mathcal{Y}_p$-group.

There is a close relation between the class of groups just introduced and $p$-nilpotence, as in shown by the following result, which was proved in [2, Theorem 5].

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**Theorem 1.** *A group $G$ is a $\mathcal{Y}_p$-group if and only if either it is $p$-nilpotent or it has an abelian Sylow $p$-subgroup $P$ and every subgroup of $P$ is normal in $\mathrm{N}_G(P)$.*

Our first main result is:

**Theorem 2.** *The minimal non-$\mathcal{Y}_p$-groups are just the minimal non-$PST$-groups with a non-trivial normal Sylow $p$-subgroup. Such groups are of the types described in* I *to* IV *below. Let $p$ and $q$ be distinct primes.*

- **Type I:** *$G = [P]Q$, where $P = \langle a, b\rangle$ is an elementary abelian group of order $p^2$, $Q = \langle z\rangle$ is cyclic of order $q^r$, with $q$ a prime such that $q^f$ divides $p-1$, $q^f > 1$ and $r \geq f$, and $a^z = a^i$, $b^z = b^{i^j}$, where $i$ is the least positive primitive $q^f$-th root of unity modulo $p$ and $j = 1 + kq^{f-1}$, with $0 < k < q$.*
- **Type II:** *$G = [P]Q$, where $Q = \langle z\rangle$ is cyclic of order $q^r > 1$, with $q$ a prime not dividing $p-1$ and $P$ an irreducible $Q$-module over the field of $p$ elements with centralizer $\langle z^q\rangle$ in $Q$.*
- **Type III:** *$G = [P]Q$, where $P = \langle a_0, a_1, \ldots, a_{q-1}\rangle$ is an elementary abelian $p$-group of order $p^q$, $Q = \langle z\rangle$ is cyclic of order $q^r$, with $q$ a prime such that $q^f$ is the highest power of $q$ dividing $p-1$ and $r > f$. Define $a_j^z = a_{j+1}$ for $0 \leq j < q-1$ and $a_{q-1}^z = a_0^i$, where $i$ is a primitive $q^f$-th root of unity modulo $p$.*
- **Type IV:** *$G = [P]Q$, where $P$ is a non-abelian special $p$-group of rank $2m$, the order of $p$ modulo $q$ being $2m$, $Q = \langle z\rangle$ is cyclic of order $q^r > 1$, $z$ induces an automorphism in $P$ such that $P/\Phi(P)$ is a faithful irreducible $Q$-module, and $z$ centralizes $\Phi(P)$. Furthermore, $|P/\Phi(P)| = p^{2m}$ and $|P'| \leq p^m$.*

Since a group is a soluble $PST$-group if and only if it belongs to $\mathcal{Y}_p$ for all primes $p$ [2, Theorem 4], Theorem 2 may be regarded as a local approach to the third author's classification of minimal non-$PST$-groups [13].

An interesting consequence of Theorem 2 is the following classification of Schmidt groups. In order to describe the classification, we must introduce one further type of group:

- **Type V:** $G = [P]Q$, where $P = \langle a\rangle$ is a normal subgroup of order $p$, $Q = \langle z\rangle$ is cyclic of order $q^r > 1$, and $a^z = a^i$, where $i$ is the least primitive $q$-th root of unity modulo $p$.

Our main result can now be stated as:

**Theorem 3.** *The Schmidt groups are exactly the groups of Type* II*, Type* IV *and Type* V*.*

Our next result shows that $p$-soluble groups with Sylow $p$-subgroups isomorphic to a normal subgroup of a minimal non-$\mathcal{Y}_p$-group have a restricted structure.

**Theorem 4.** *Let $G$ be a $p$-soluble group with a Sylow $p$-subgroup $P$. If $P$ is isomorphic to a non-trivial normal Sylow subgroup of a minimal non-$\mathcal{Y}_p$-group, then $G$ has $p$-length* 1*.*

In [4] Gol'fand stated the following result:

**Theorem 5.** *Let $p$ and $q$ be distinct primes, let $r$ be a given positive integer, and let $a$ be the order of $p$ modulo $q$. Then there is a unique minimal non-$p$-nilpotent group $G_0$ of order $p^{a_0}q^r$, where $a_0 = a$ if $a$ is odd and $a_0 = 3a/2$ if $a$ is even, such* *that all minimal non-p-nilpotent groups of order $p^t q^r$ are isomorphic to quotients of $G_0$ by central subgroups.*

Only a sketch of a proof of this theorem is given in Golfand's article. In Section 3, we show how to construct the Schmidt groups of Gol'fand, and we also give a complete proof of Theorem 5. We remark that Rédei [11] has given another construction of the Schmidt groups of maximum order.

## 2. Proofs of Theorems 2, 3 and 4

*Proof of Theorem* 2. Assume that $G$ is a minimal non-$\mathcal{Y}_p$-group and let $P$ be a Sylow $p$-subgroup of $G$. Since $G$ does not belong to $\mathcal{Y}_p$, there exist subgroups $H$ and $K$ of $P$ such that $H \leq K$ and $H$ is not S-permutable in $\mathrm{N}_G(K)$. Consequently there is an element $z \in \mathrm{N}_G(K)$ such that $z$ does not normalise $H$. Here it can be assumed that $z$ has order $q^r$ for some prime $q \neq p$. Then $G = K\langle z\rangle$ because $G$ is a minimal non-$\mathcal{Y}_p$-group. This implies that $K = P$ is a normal Sylow $p$-subgroup of $G$ and $Q = \langle z\rangle$ is a cyclic Sylow $q$-subgroup of $G$. Then $G$ is not a $PST$-group, yet every proper subgroup has $\mathcal{Y}_p$ and $\mathcal{Y}_q$, and thus is a $PST$-group by [2].

Conversely, if $G$ is a minimal non-$PST$-group, then $G$ does not have $\mathcal{Y}_p$ for some prime $p$. Since all its proper subgroups satisfy $\mathcal{Y}_p$, the group $G$ is a minimal non-$\mathcal{Y}_p$-group. The classification of minimal non-$PST$-groups given in [13] completes the proof. (Note that the groups of Types IV and V of [13] are both of Type IV above.) □

*Proof of Theorem* 3. Let $G$ be a minimal non-nilpotent group. Then $G$ is a minimal non-$p$-nilpotent group for some prime $p$. Suppose that $G$ is not a $\mathcal{Y}_p$-group, so that $G$ is a minimal non-$\mathcal{Y}_p$-group. By Theorem 2, the group $G$ is of one of Types I–IV. By examining the group structure, we see that groups of Type I and III are not minimal non-$p$-nilpotent. Therefore $G$ must be of Type II or IV.

Assume now that $G$ belongs to $\mathcal{Y}_p$. Then by [1, Theorem A] and [3, VII, 6.18], the $p$-nilpotent residual $P$ of $G$ is an abelian minimal normal Sylow subgroup which is complemented in $G$ by a cyclic Sylow $q$-subgroup $Q$. Moreover $Q$ normalises each subgroup of $P$. This implies that $P$ is cyclic of order $p$, say $P = \langle a\rangle$. In addition, $a^z = a^i$ for some $0 < i < p$ and $z^q$ centralizes $a$. This implies that $i$ must be a primitive $q$-th root of unity modulo $p$ and, by taking a suitable power of $z$ as a generator of $Q$, we can assume that $i$ is the least such positive integer. Hence $G$ is of Type V. □

*Proof of Theorem* 4. Assume that $G$ is a $p$-soluble group with $p$-length $> 1$ and $G$ has least order subject to possessing a Sylow $p$-subgroup $P$ which is isomorphic to a non-trivial normal Sylow subgroup of a Schmidt group. By [6, VI, 6.10], we conclude that $P$ is not abelian. Thus $P$ is a Sylow $p$-subgroup of a group of Type IV in Theorem 2. By minimality of order $\mathrm{O}_{p'}(G) = 1$ and $\mathrm{O}^{p'}(G) = G$. In addition, since the class of groups of $p$-length at most 1 is a saturated formation, we have $\Phi(G) = 1$ and hence $G$ has a unique minimal normal subgroup which is an elementary abelian $p$-group. Let $D = \mathrm{O}_p(G)$; then $D$ is a non-trivial elementary abelian group and $\mathrm{C}_G(D) = D$. Moreover $\Phi(P) = \mathrm{Z}(P) \leq D$ and so $P/D$ is elementary abelian.

Let $T$ be the subgroup defined by $T/D = \mathrm{O}_{p'}(G/D)$. Since $P/D$ is an elementary abelian $p$-group, $G/D$ has $p$-length at most 1 by [6, VI, 6.10]. It follows that $(T/D)(P/D)$ is a normal subgroup of $G/D$. Therefore $TP$ is a normal subgroup of

$G$. Assume that $TP$ is a proper subgroup of $G$. Now $\mathrm{O}_{p'}(TP) \leq \mathrm{O}_{p'}(G) = 1$, so $P$ is a normal subgroup of $TP$ and hence of $G$, a contradiction which shows that $G = TP$.

Assume now that $P/D$ is a non-cyclic elementary abelian group. By [8, X, 1.9], we have $T/D = \langle \mathrm{C}_{T/D}(xD) \mid xD \in P/D, xD \neq D\rangle$. Let $x \in P \backslash D$. Since $P/D$ centralizes $xD$, we have $P/D \leq \mathrm{N}_{G/D}\big(\mathrm{C}_{T/D}(xD)\big)$. Let $T_x/D = \mathrm{C}_{T/D}(xD)$. Assume that $PT_x = G$; then $T_x = T$ is a normal subgroup of $G$ and thus $\mathrm{O}_{p'}(G/D) = T_x/D$. This implies that $\langle x\rangle D/D \leq \mathrm{Z}(G/D)$ and $\langle x\rangle D$ is a normal $p$-subgroup of $G$, so that $\langle x\rangle D$ is contained in $D$, a contradiction. Consequently $PT_x$ is a proper subgroup of $G$ for all $1 \neq xD \in P/D$. Hence $PT_x$ has $p$-length at most 1 by minimality of $G$. Since $\mathrm{C}_G(D) = D$ and $\mathrm{O}_{p'}(PT_x)$ centralizes $D$, we conclude that $\mathrm{O}_{p'}(PT_x) = 1$. Therefore $P$ is a normal subgroup of $PT_x$, which shows that $T$ normalizes $P$ and thus $P$ is a normal subgroup of $G$. This contradiction shows that $P/D$ is cyclic.

Since $P$ has class 2, we see from [7, IX, 5.5] that, if $p > 3$, then $G$ has $p$-length at most 1. Therefore $p \leq 3$. Let $X$ be a minimal non-$\mathcal{Y}_p$-group such that $P$ is a Sylow $p$-subgroup of $X$. Note that $P/\Phi(P)$ is an irreducible $X$-module. In particular $D$, the subgroup of the previous paragraphs, is not normal in $X$ and so $P = DD^g$ for some $g \in X$. Since $D$ is abelian, $D \cap D^g \leq Z(P) = \Phi(P)$, and it follows that $P/\Phi(P)$ has order $p^2$. This implies that $P$ is an extra-special group of order $p^3$. If $p = 2$, then, since $C_G(D) = D$, we see that $G$ must be a symmetric group of degree 4. Hence $P$ is dihedral of order 8, which cannot lead to a group of Type IV since $\mathrm{Aut}(P)$ is a 2-group. Hence $p = 3$. But a non-abelian group of order $3^3$ cannot occur as the normal Sylow 3-subgroup of a Schmidt group, because the only prime divisor of $3^2 - 1$ is 2 and the order of 3 modulo 2 is 1. This contradiction completes the proof of the theorem. □

## 3. The construction of Gol'fand's groups and a proof of Gol'fand's theorem

We begin by constructing groups of Type IV with a Sylow $p$-subgroup $P$ of order $p^{3m}$ and $|P/\Phi(P)| = p^{2m}$. These groups were constructed in [13] by a different method, but the present approach is more convenient when $p = 2$. We will use the following result on linear operators.

**Lemma 6.** *Let $p$ be a prime and let $r$ be a positive integer such that* $\gcd(p, r) = 1$. *Let $\beta$ be a linear operator of order $p^u r$ on a vector space $V$ over the field of $p$-elements, where $u$ is a non-negative integer. If $\beta$ has irreducible minimum polynomial $f$, then $\beta^{p^u}$ also has minimum polynomial $f$.*

*Proof.* Let $g$ be the minimum polynomial of $\beta^{p^u}$. Now $f(\beta^{p^u}) = f(\beta)^{p^u} = 0$, so that $g$ divides $f$. Since $f$ is irreducible, $f = g$. □

**Construction 7.** Let $p$ and $q$ be distinct primes such that the order of $p$ modulo $q$ is $2m$, $m \geq 1$. Let $F$ be the free group with basis $\{f_0, f_1, \ldots, f_{2m-1}\}$. Write $R = F'F^p$ and $R^* = [F, R]R^p$. Then $F/R$ is an elementary abelian $p$-group of order $p^{2m}$ and $H = F/R^*$ is a $p$-group such that $R/R^* = \Phi(H)$ is an elementary abelian $p$-group contained in $\mathrm{Z}(H)$. Moreover $H$ is a non-abelian group because an extra-special group of order $p^{2m+1}$ is an epimorphic image of $H$.

Denote by $g_i$ the image of $f_i$ under the natural epimorphism of $F$ onto $H = F/R^*$, $0 \leq i \leq 2m-1$. Since $H$ has class 2, we know that $\Phi(H)$ is generated by all $[g_i, g_j]$, with $i < j$, and $g_i^p$. Therefore $\Phi(H)$ has dimension as $\mathrm{GF}(p)$-vector space at most $\frac{1}{2}(2m(2m-1))+2m = m(2m+1)$. Assume that the dimension is less than $m(2m+1)$. Then there exists an element

$$r = \prod_j (f_j^p)^{\lambda_j} \prod_{j<k} [f_j, f_k]^{\mu_{jk}} \in R^*$$

with some $\lambda_j$ or $\mu_{jk}$ not divisible by $p$. It is clear that $p \mid \lambda_j$ for all $j$ since $F^p F'/F'$ is a free abelian group with basis $\{f_j^p F' \mid 0 \le j \le 2m-1\}$. Suppose that $p \nmid \mu_{ik}$ for some $i < k$ and let $\rho_i$ be the endomorphism of $F$ defined by $f_i^{\rho_i} = f_i^2$, $f_l^{\rho_i} = f_l$ for $l \neq i$. Then $r^{\rho_i} R^* = R^*$ and so $r^{\rho_i} r^{-1} R^* = R^*$. This implies that

$$w = \prod_{j<i} [f_j, f_i]^{\mu_{ji}} \prod_{i<l} [f_i, f_l]^{\mu_{il}} \in R^*.$$

On the other hand, by applying $\rho_k$ we find that

$$w^{\rho_k} w^{-1} R^* = [f_i, f_k]^{\mu_{ik}} R^* = R^*.$$

Since $p \nmid \mu_{ik}$, it follows that $\mu_{ik}$ has an inverse modulo $p$. This means that $[f_i, f_k] \in R^*$. Now since permutations of the generators of $F$ induce endomorphisms in $F$ and $R^*$ is fully invariant, it follows that $F' \le R^*$ and $H$ is abelian, a contradiction. Therefore $\Phi(H)$ has dimension $m(2m+1)$ and so $|\Phi(H)| = p^{m(2m+1)}$.

Let $f(t) = c_0 + c_1 t + \cdots + c_{2m-1} t^{2m-1} + t^{2m}$ be an irreducible factor of the cyclotomic polynomial of order $q$ over $\mathrm{GF}(p)$ and let $\alpha$ be the endomorphism of $F$ given by $f_i^\alpha = f_{i+1}$ for $0 \le i \le 2m-2$, $f_{2m-1}^\alpha = f_0^{-c_0} f_1^{-c_1} \cdots f_{2m-1}^{-c_{2m-1}}$. Since $R^*$ is a fully invariant subgroup of $F$, it follows that $\alpha$ induces an endomorphism $\beta$ on $H = F/R^*$. In turn, $\beta$ induces an automorphism $\bar\beta$ on $H/\Phi(H)$. Since $H/\Phi(H) = \big(H/\Phi(H)\big)^{\bar\beta} \le H^\beta \Phi(H)/\Phi(H)$, it follows that $H = H^\beta \Phi(H)$, whence $H = H^\beta$. Consequently $\beta$ is an automorphism of $H$.

It is clear that $\beta$ induces the linear operator $\bar\beta$, with minimum polynomial $f$, on the vector space $H/\Phi(H)$. Now by [6, III, 3.18], we conclude that $\beta^q$ has order $p^u$ for a some $u$ and hence $\beta$ has order $p^u q$. By Lemma 6, there is a $\mathrm{GF}(p)$-basis $\{g_0', g_1', \ldots, g_{2m-1}'\}$ of $H/\Phi(H)$, where $g_i' = g_i \Phi(H)$, such that $g_i'^{\bar\beta^{p^u}} = g_{i+1}'$ for $0 \le i \le 2m-2$ and $g_{2m-1}'^{\bar\beta^{p^u}} = g_0'^{-c_0} g_1'^{-c_1} \cdots g_{2m-1}'^{-c_{2m-1}}$. Hence we can replace $\beta$ by $\beta^{p^u}$ and assume without loss of generality that $\beta$ has order $q$.

It follows that $\Phi(H)$ is a $\mathrm{GF}(p)T$-module, where $T = \langle \beta \rangle$ is a cyclic group of order $q$. By Maschke's Theorem $\Phi(H)$ is a direct sum of irreducible $T$-modules. Let $N$ be the sum of all non-trivial irreducible submodules in the direct decomposition and write $P = H/N$. It is clear that $N$ is $\beta$-invariant and therefore $\beta$ induces an automorphism $\gamma$ of order $q$ in $P$. Let $Q = \langle z \rangle$ be a cyclic group of order $q^r$ acting on $P$ via $z \mapsto \gamma$. Let $G = [P]Q$ be the corresponding semidirect product.

It is easily checked that $G$ is a Schmidt group. Next we show that $P$ has order $p^{3m}$. From Theorem 3 we see that $\Phi(P)$ has order at most $p^m$, where $|P/\Phi(P)| = p^{2m}$. On the other hand, $|\Phi(H)| = p^{m(2m+1)}$, and $N$ has order a power of $p^{2m}$ because every faithful irreducible $\langle\beta\rangle$-module over $\mathrm{GF}(p)$ has dimension $2m$. Therefore $|\Phi(P)| = p^m$.

*Remark* 8. In the group of Construction 7, we may assume that $\bar g_{2m-1}^z = \bar g_0^{-c_0} \bar g_1^{-c_1} \cdots \bar g_{2m-1}^{-c_{2m-1}}$, where $\bar g_i = g_i N$.

*Proof.* We know that $\bar g_{2m-1}^z = \bar g_0^{-c_0} \bar g_1^{-c_1} \cdots \bar g_{2m-1}^{-c_{2m-1}} \bar w$, where $\bar w \in \Phi(P)$. Since $f(t)$ is irreducible, 1 is not a root of $f(t)$ and it follows that $c = c_0 + c_1 + \cdots + c_{2m-1} + 1 \not\equiv 0$

(mod $p$). Consequently there exists an integer $d$ such that $cd \equiv -1 \pmod{p}$. Put $\bar{w}_0 = \bar{w}^d$ and consider the automorphism $\delta$ of $P$ defined by $\bar{g}_i^{\delta} = \bar{g}_i \bar{w}_0$ for $0 \le i \le 2m-1$. If we write $\gamma_0 = \delta\gamma\delta^{-1}$, it is easily checked by an elementary calculation that $\bar{g}_i^{\gamma_0} = \bar{g}_{i+1}$ for $0 \le i \le 2m-2$, and $\bar{g}_{2m-1}^{\gamma_0} = \bar{g}_0^{-c_0}\bar{g}_1^{-c_1}\cdots\bar{g}_{2m-1}^{-c_{2m-1}}$. Let $\langle z_0 \rangle$ be a cyclic group of order $q^r$, with $z_0$ acting on $P$ via $z_0 \mapsto \gamma_0$. Since $\langle z_0 \rangle$ and $\langle z \rangle$ are conjugate in $\mathrm{Aut}(P)$, it follows by [3, B, 12.1] that the groups $P\langle z \rangle$ and $P\langle z_0 \rangle$ are isomorphic. □

*Remark* 9. The group in Construction 7 does not depend on the choice of irreducible factor $f(t)$.

*Proof.* Assume that the group $G_1 = [P_1]\langle z_1 \rangle$ has been constructed by using another irreducible factor $g(t)$ of the cyclotomic polynomial of order $q$ over $\mathrm{GF}(p)$. Since $G$ and $G_1$ have the same order, it will be enough to find a set of generators of $G_1$ for which the relations of $G$ hold. Since $z$ centralizes $\Phi(P)$ and $z_1$ centralizes $\Phi(P_1)$, we have $G/\Phi(P) \cong [P/\Phi(P)]\langle z \rangle$ and $G_1/\Phi(P_1) \cong [P_1/\Phi(P_1)]\langle z_1 \rangle$. But $P/\Phi(P)$ and $P_1/\Phi(P_1)$ are faithful irreducible modules for a cyclic group of order $q$. Therefore $[P/\Phi(P)](\langle z \rangle/\langle z^q \rangle)$ is isomorphic to $[P_1/\Phi(P_1)](\langle z_1 \rangle/\langle z_1^q \rangle)$ by [3, B, 12.4]. Let $\phi$ be an isomorphism between these groups. Then it is clear that $\phi$ induces an isomorphism $\psi$ between $G/\Phi(P)$ and $G_1/\Phi(P_1)$.

Let $\bar{h}_i = h_i\Phi(P)$, $0 \le i \le 2m-1$. Put $\bar{k}_i = \bar{h}_i^{\psi}$ and $\bar{u} = \bar{z}^{\psi}$. We show how to extend the isomorphism $\psi$ to an isomorphism between $G$ and $G_1$. In order to do so, we choose representatives $k_i$ of $\bar{k}_i$ and $u$ of $\bar{u}$ such that the order of $u$ is $q^r$. There is no loss of generality in assuming that $k_i^u = k_{i+1}$ for $0 \le i \le 2m-2$. Indeed, if $k_i^u = k_{i+1}w_{i+1}$ with $w_{i+1} \in \Phi(P_1)$, then $k_i' = k_i w_i \cdots w_1$ for $1 \le i \le 2m-1$, $k_0' = k_0$ are representatives of $\bar{k}_i$ and ${k_i'}^u = k_{i+1}'$ for $1 \le i \le 2m-1$ because $u$ centralizes $\Phi(P_1)$. By using the same argument as in Remark 8, we may also assume that $k_{2m-1}^u = k_0^{-c_0}k_1^{-c_1}\cdots k_{2m-1}^{-c_{2m-1}}$. Therefore $G$ and $G_1$ satisfy the same relations and by Von Dyck's theorem they are isomorphic. □

*Remark* 10. In Construction 7, it is not necessary to assume that $\beta$ has order $q$. Indeed, it can be proved that $\beta^q$ fixes all elements of $\Phi(H)$ and that the automorphism $\gamma$ induced by $\beta$ in $H/N$ has order $q$.

Gol'fand's result (Theorem 5) can be recovered with the help of Construction 7 and Theorem 3.

*Proof of Theorem* 5. Let $p$ and $q$ be distinct primes and let $a$ be the order of $p$ modulo $q$. Then $a$ is the dimension of each non-trivial irreducible module for a cyclic group of order $q$ over $\mathrm{GF}(p)$. Assume that $a$ is odd. Then every Schmidt group $G$ with a normal Sylow $p$-subgroup $P$ such that $|P/\Phi(P)| = p^a$ is of Type II or Type V. Then the theorem holds in this case because all Schmidt groups of the same type with isomorphic Sylow $q$-subgroups are actually isomorphic.

Assume now that $a$ is even, with say $a = 2m$. Then we are dealing with Schmidt groups of Type II or Type IV. Let $G_0$ be the group of Construction 7. Then $|G_0| = p^{3m}q^r$ and $|P_0/\Phi(P_0)| = p^{2m}$, where $P_0$ is a normal Sylow $p$-subgroup of $G_0$. It is clear that $G_0/\Phi(P_0)$ is a Schmidt group of Type II. Therefore, if $G$ is a Schmidt group of Type II with order $p^tq^r$ and a normal Sylow $p$-subgroup, then $G \cong G_0/\Phi(P_0)$ and $\Phi(P_0) \le \mathrm{Z}(G_0)$. Consequently, we need only show that all Schmidt groups of Type IV and order $p^tq^r$, $t \le 3m$, which have a normal Sylow $p$-subgroup are isomorphic to quotients of $G_0$ by central subgroups.

Let $\overline{G}$ be a Schmidt group of Type IV and order $p^t q^r$ with a normal Sylow $p$-subgroup $\bar{P}$. Then $G_0/\Phi(P_0)$ and $\overline{G}/\Phi(\overline{P})$ are isomorphic. Let us choose generators $z$ and $\bar{z}$ of Sylow $q$-subgroups $Q$ of $G_0$ and $\overline{Q}$ of $\overline{G}$ such that the minimum polynomials of the actions of $z$ on $P_0/\Phi(P_0)$ and $\bar{z}$ on $\overline{P}/\Phi(\overline{P})$ coincide. Also choose generators $g_0, g_1, \ldots, g_{2m-1}$ of the Sylow $p$-subgroup $P_0$ of $G_0$ and generators $\bar{g}_0, \bar{g}_1, \ldots, \bar{g}_{2m-1}$ of the Sylow $p$-subgroup $\overline{P}$ of $\overline{G}$ such that $g_j^z = g_{j+1}$ and $\bar{g}_j^{\bar{z}} = \bar{g}_{j+1}$ for $0 \leq j \leq 2m-2$. Since $\Phi(P_0) = P_0'$ and $\Phi(\overline{P}) = \overline{P}'$, and both $P_0$ and $\overline{P}$ have class 2, the subgroup $\Phi(P_0)$ can be generated by the commutators $[g_i, g_j]$, while $\Phi(\overline{P})$ is generated by the commutators $[\bar{g}_i, \bar{g}_j]$. On the other hand, if $u_i = [g_0, g_0^{z^i}]$, we have $u_i = u_i^{z^k} = [g_k, g_k^{z^i}]$. It is easy to see that $u_i = [g_0, g_0^{z^i}] = [g_0^{z^q}, g_0^{z^i}] = u_{q-i}^{-1}$.

Observe that $q$ is odd since $2m$ divides $q-1$: write $q = 2s+1$. By definition of the $g_i$ and $u_i$, and use of the minimum polynomial of the action of $z$ on $P_0/\Phi(P_0)$, it may be shown that for $l \geq 1$,

$$u_{s+m+l} = u_{s-m+l}^{-c_0}\, u_{s-m+l+1}^{-c_1} \cdots u_{s+m+l-2}^{-c_{2m-2}}\, u_{s+m+l-1}^{-c_{2m-1}}.$$

Now this formula and the relations $u_i = u_{q-i}^{-1}$ allow us to show by induction that each $u_{s+m+l}$ can be expressed in terms of elements of the set $B = \{u_{s-m+l}, u_{s-m+2}, \ldots, u_s\}$. Since $\Phi(P_0)$ has dimension $m$ over $\mathrm{GF}(p)$, this expression is unique. It follows that each $u_j$ can be uniquely expressed in terms of the elements of $B$, and so this is also true for each generator of $\Phi(P_0)$. The same argument shows that the generators of $\Phi(\overline{P})$ have a similar unique expression subject to the same relations.

The arguments of Remark 9 allow us to assume that

$$g_{2m-1}^z = g_0^{-c_0} g_1^{-c_1} \cdots g_{2m-1}^{-c_{2m-1}} \text{ and } \bar{g}_{2m-1}^{\bar{z}} = \bar{g}_0^{-c_0} \bar{g}_1^{-c_1} \cdots \bar{g}_{2m-1}^{-c_{2m-1}}.$$

Consequently, all relations of $G_0$ are satisfied by $\overline{G}$. By Von Dyck's theorem, it follows that $\overline{G}$ is an epimorphic image of $G_0$ by a central subgroup of $G_0$. □

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